

Research Article

Bayesian Inference on the Shape Parameter and Future Observation of Exponentiated Family of Distributions

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The Bayes estimators of the shape parameter of exponentiated family of distributions have been derived by considering extension of Jeffreys' noninformative as well as conjugate priors under different scale-invariant loss functions, namely, weighted quadratic loss function, squared-log error loss function and general entropy loss function. The risk functions of these estimators have been studied. We have also considered the highest posterior density (HPD) intervals for the parameter and the equal-tail and HPD prediction intervals for future observation. Finally, we analyze one data set for illustration.

1. Introduction

Let X be a random variable whose cumulative distribution function (cdf) and probability density function (pdf) are given by

$$G(x; \alpha, \theta) = F^\alpha(x; \theta), \quad (1.1)$$

$$g(x; \alpha, \theta) = \alpha F^{\alpha-1}(x; \theta) f(x; \theta), \quad (1.2)$$

respectively. Here $F(\cdot, \theta)$ is the continuous baseline distribution function with the corresponding probability density function $f(x; \theta)$, and θ may be vector valued, and α is the positive shape parameter. Then, X is said to be belonging to the exponentiated family of distributions (abbreviated as EFD) or the proportional reversed hazard family.

If the baseline distribution is exponential, then it is called the generalized exponential (GE) distribution in the literature. In recent years, an impressive array of papers has been

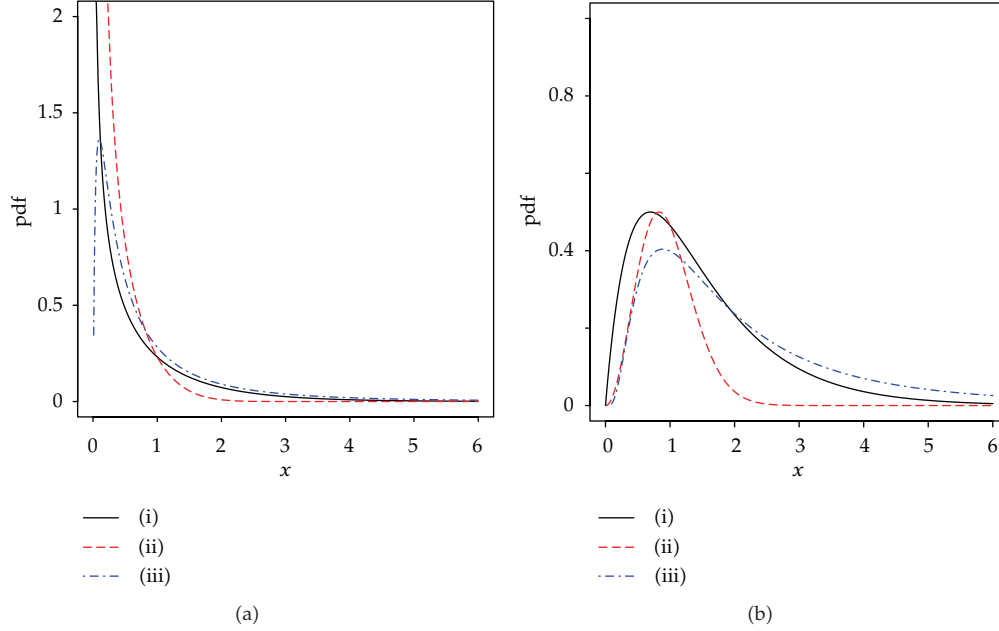


Figure 1: Exponentiated distributions for $\alpha = 0.5$ and $\alpha = 2$.

devoted to study the behavioral patterns of the parameters of the generalized exponential distribution using both classical and Bayesian framework, and a very good summary of this work can be found in Gupta and Kundu [1–4], Raqab [5], Raqab and Ahsanullah [6], Zheng [7], Raqab and Madi [8], Alamm et al. [9], Singh et al. [10], Dey [11], and the references cited there for some recent developments on GE distribution. If the baseline distribution is Weibull, then it is called the exponentiated Weibull distribution. Mudholkar and Srivastava [12], Nassar and Eissa [13], and Singh et al. [14] have studied this distribution.

In this paper, we assume that $F(x, \theta) = F(x)$ is known, but the shape parameter α is unknown. Then, the cdf and pdf become

$$G(x; \alpha) = F^\alpha(x), \quad (1.3)$$

$$g(x; \alpha) = \alpha F^{\alpha-1}(x) f(x), \quad (1.4)$$

respectively. If $F(x)$ is symmetric, then $G(x; \alpha)$ will be skewed distribution for different values of $\alpha (\neq 1)$. Hence α can be considered as a skewness parameter. Gupta and Gupta [15] have shown that positively skewed data can be analyzed very well for normal baseline distribution. Again α is the parameter of the proportional reversed hazard model in lifetime data analysis. For its various important roles, we are interested to find out the Bayes estimators and their performances under different loss functions using different priors. In Figure 1, the shape of (i) exponentiated distribution with $F(x) = 1 - e^{-x}$, (ii) exponentiated Rayleigh, distribution with $F(x) = 1 - e^{-x^2}$, and (iii) exponentiated lognormal distribution with $F(x) = \Phi(\ln x)$ has been shown for $\alpha = 0.5$ and $\alpha = 2$, respectively.

The paper is categorized into the following sections. Section 2 has a brief description of the prior distributions and loss functions. The Bayes estimators and associated risk functions

are provided in Sections 3 and 4, respectively. Section 5 presents the highest posterior density (HPD) interval for α . Section 6 is devoted to finding out the predictive distributions and equal-tail Bayesian predictive interval for the future observation. Section 7 deals with the Bayes predictive estimator and HPD prediction interval for a future observation. Section 8 presents the data application based on a real life data set. The paper ends with a concluding remark in Section 9.

2. Prior and Loss Functions

The Bayesian inference requires appropriate choice of prior(s) for the parameter(s). From the Bayesian viewpoint, there is no clear cut way from which one can conclude that one prior is better than the other. Nevertheless, very often priors are chosen according to one's subjective knowledge and beliefs. However, if one has adequate information about the parameter(s), it is better to choose informative prior(s); otherwise, it is preferable to use noninformative prior(s). In this paper we consider both type of priors: the extended Jeffreys' prior and the natural conjugate prior.

The extended Jeffreys' prior proposed by Al-Kutubi [16] is given as

$$\pi_2(\alpha) \propto \frac{1}{\alpha^{2c_1}}, \quad \alpha > 0, \quad c_1 > 0. \quad (2.1)$$

The conjugate prior in this case will be the gamma prior, and the probability density function is taken as

$$\pi_1(\alpha) = \frac{b^a}{\Gamma(a)} \alpha^{a-1} e^{-b\alpha}, \quad \alpha, a, b > 0. \quad (2.2)$$

With the above priors, we use three different loss functions for the model (1.1).

- (1) The first loss function considered is called weighted quadratic loss function and is given by

$$L_1(\alpha, \delta) = \left(\frac{\alpha - \delta}{\alpha} \right)^2, \quad (2.3)$$

where δ is a decision rule to estimate α . δ is to be chosen such that

$$\int_0^\infty \left(\frac{\alpha - \delta}{\alpha} \right)^2 \pi(\alpha | \underline{x}) d\alpha \quad (2.4)$$

is minimum. This can be equivalently written as

$$\int_0^\infty (\alpha - \delta)^2 q(\alpha | \underline{x}) d\alpha, \quad \text{with } q(\alpha | \underline{x}) = \frac{(1/\alpha^2)\pi(\alpha | \underline{x})}{\int_0^\infty (1/\alpha^2)\pi(\alpha | \underline{x}) d\alpha} \quad (2.5)$$

being minimum. Hence

$$\tilde{\alpha}_{\text{bq}} = \delta = E_q(\alpha | \underline{x}). \quad (2.6)$$

- (2) The second one is the squared-log error loss function proposed by Brown [17] and is defined as

$$L_2(\alpha, \delta) = (\ln \delta - \ln \alpha)^2 = \left(\ln \frac{\delta}{\alpha} \right)^2. \quad (2.7)$$

This loss function is balanced with $\lim L_2(\alpha, \delta) \rightarrow \infty$ as $\delta \rightarrow 0$ or ∞ . A balanced loss function takes both error of estimation and goodness of fit into account, but the unbalanced loss function only considers error of estimation. This loss function is convex for $\delta/\alpha \leq e$ and concave otherwise, but its risk function has a unique minimum with respect to δ . The Bayes estimator for the parameter α under squared-log error loss function may be given as

$$\tilde{\alpha}_{\text{bsl}} = \exp[E(\ln \alpha | \underline{x})], \quad (2.8)$$

where $E(\cdot)$ denotes the posterior expectation.

- (3) The third loss function is a particular type of asymmetric loss functions called the general entropy loss function proposed by Calabria and Pulcini [18] (Podder and Roy [19] called it the modified linear exponential (MLINEX) loss function) and is given by

$$L_3(\alpha, \delta) = w \left[\left(\frac{\delta}{\alpha} \right)^\gamma - \gamma \ln \left(\frac{\delta}{\alpha} \right) - 1 \right], \quad \gamma \neq 0, \quad w > 0. \quad (2.9)$$

If we replace $\delta - \alpha$ in place of $\ln(\delta/\alpha)$, that is, $\ln \delta - \ln \alpha$, we get the linear exponential (LINEX) loss function, $w[e^{\gamma(\delta-\alpha)} - \gamma(\delta-\alpha) - 1]$. Without loss of generality, we assume that $w = 1$. If $\gamma = 1$, it is the entropy loss function. Under this general entropy loss function, the Bayes estimator of α is obtained as follows:

$$\tilde{\alpha}_{\text{bge}} = [E_\alpha(\alpha^{-\gamma})]^{-1/\gamma}. \quad (2.10)$$

3. Estimation of Parameter

Let us consider a random sample $\underline{x} = (x_1, x_2, \dots, x_n)$ of size n from the exponentiated family of distributions. Then the likelihood function of α for the given sample observation is

$$\begin{aligned} L(\alpha | \underline{x}) &= \alpha^n \prod_{i=1}^n F^{\alpha-1}(x_i) f(x_i) \\ &= \alpha^n e^{\alpha \sum_{i=1}^n \ln F(x_i)} \prod_{i=1}^n \frac{f(x_i)}{F(x_i)}. \end{aligned} \quad (3.1)$$

Here, the maximum likelihood estimator (MLE) of α is $\hat{\alpha}_{\text{mle}} = n/T$, with $T = -\sum_{i=1}^n \ln F(x_i)$.

3.1. Estimation under the Assumption of Extended Jeffreys' Prior

Combining the prior distribution in (2.1) and the likelihood function, the posterior density of α is derived as follows:

$$\pi_1(\alpha | \underline{x}) = \frac{T^{n-2c_1+1}}{\Gamma(n-2c_1+1)} e^{-\alpha T} \alpha^{n-2c_1}, \quad \alpha > 0, \quad (3.2)$$

which is a gamma distribution $G(n-2c_1+1, T)$.

For differnt derivations in this section and subsequent sections, we use the expressions $\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx$, $\Gamma'(p) = \int_0^\infty \ln x x^{p-1} e^{-x} dx$, $\Gamma''(p) = \int_0^\infty (\ln x)^2 x^{p-1} e^{-x} dx$, $\psi(p) = d \ln \Gamma(p) / dp = \Gamma'(p) / \Gamma(p)$, the digamma function, and $\psi'(p) = d^2 \ln \Gamma(p) / dp^2 = ((\Gamma''(p) \cdot \Gamma(p) - [\Gamma'(p)]^2) / \Gamma^2(p))$, the trigamma function.

Using extended Jeffreys' prior of the form (2.1), the Bayes estimators of α under weighted quadratic, squared-log error and general entropy loss functions are derived as follows:

$$\begin{aligned} \tilde{\alpha}_{\text{bq}}^E &= \frac{n-2c_1-1}{T}, \\ \tilde{\alpha}_{\text{bsl}}^E &= \frac{e^{\psi(n-2c_1+1)}}{T}, \\ \tilde{\alpha}_{\text{bge}}^E &= \left[\frac{\Gamma(n-2c_1+1)}{\Gamma(n-2c_1+1-\gamma)} \right]^{1/\gamma} \cdot \frac{1}{T} = \frac{k}{T}, \end{aligned} \quad (3.3)$$

with $k = [\Gamma(n-2c_1+1) / \Gamma(n-2c_1+1-\gamma)]^{1/\gamma}$, respectively.

Remark 3.1. We get the Jeffreys' noninformative prior for $c_1 = 1/2$ and the Hartigan's noninformative prior for $c_1 = 3/2$.

3.2. Estimation under the Assumption of Conjugate Prior

Combining the prior distribution in (2.2) and the likelihood function, the posterior density of α is derived as follows:

$$\pi_2(\alpha | \underline{x}) = \frac{T_b^{n+a}}{\Gamma(n+a)} e^{-\alpha T_b} \alpha^{n+a-1}, \quad \alpha > 0, \quad (3.4)$$

which is a gamma distribution $G(n+a, T_b)$ with $T_b = T + b$.

Using a conjugate prior of the form (2.2), the Bayes estimators under weighted quadratic, squared-log error and general entropy loss functions are derived as follows:

$$\tilde{\alpha}_{\text{bq}}^c = \frac{n-2+a}{T_b}, \quad (3.5)$$

$$\tilde{\alpha}_{\text{bsl}}^c = \frac{e^{\psi(n+a)}}{T_b}, \quad (3.6)$$

$$\tilde{\alpha}_{\text{bge}}^c = \left[\frac{\Gamma(n+a)}{\Gamma(n+a-\gamma)} \right]^{1/\gamma} \cdot \frac{1}{T_b}, \quad (3.7)$$

respectively. It is to be noted that Bayes' estimators given in (3.5), (3.6), and (3.7) depend on a and b which are the parameters of the prior distribution of α . These parameters, that is, a and b could be estimated by means of empirical Bayes' procedure (see Lindley [20] and Awad and Gharraf [21]). Given the random sample $\underline{x} = (x_1, x_2, \dots, x_n)$, the likelihood function of α is gamma density with parameter $(n + 1, T)$. Hence it is proposed to estimate the prior parameters a and b from the sample by $n + 1$ and T , respectively. Therefore, (3.5), (3.6), and (3.7) will become

$$\tilde{\alpha}_{\text{bq}}^c = \frac{2n - 1}{2T}, \quad (3.8)$$

$$\tilde{\alpha}_{\text{bsl}}^c = \frac{e^{\psi(2n+1)}}{2T}, \quad (3.9)$$

$$\tilde{\alpha}_{\text{bge}}^c = \left[\frac{\Gamma(2n + 1)}{\Gamma(2n + 1 - \gamma)} \right]^{1/\gamma} \cdot \frac{1}{2T} = \frac{K_1}{2T}, \quad \text{where } K_1 = \left[\frac{\Gamma(2n + 1)}{\Gamma(2n + 1 - \gamma)} \right]^{1/\gamma}, \quad (3.10)$$

respectively.

4. Risks of the Bayes Estimators

Since X follows the exponentiated family of distributions with parameter α , then $T = -\sum_{i=1}^n \ln F(x_i)$ is distributed as $G(n, \alpha)$. Then the probability density function of T is

$$h_T(t) = \frac{\alpha^n}{\Gamma(n)} e^{-\alpha t} t^{n-1}, \quad t > 0. \quad (4.1)$$

Therefore,

$$E_h(T^{-\gamma}) = \int_0^{\infty} t^{-\gamma} h_T(t) dt = \frac{\alpha^n}{\Gamma(n)} \int_0^{\infty} e^{-\alpha t} t^{n-\gamma-1} dt = \frac{\Gamma(n-\gamma)}{\Gamma(n)} \alpha^\gamma. \quad (4.2)$$

The risk function of $\tilde{\alpha}_{\text{bq}}^E$ is

$$\begin{aligned} R(\tilde{\alpha}_{\text{bq}}^E) &= E_h \left[L(\tilde{\alpha}_{\text{bq}}^E, \delta) \right] \\ &= \frac{1}{\alpha^2} \left[\alpha^2 - 2\alpha(n - 2c_1 - 1) E_h \left(\frac{1}{T} \right) + (n - 2c_1 - 1)^2 E_h \left(\frac{1}{T^2} \right) \right] \\ &= \frac{1}{\alpha^2} \left[\alpha^2 - 2\alpha(n - 2c_1 - 1) \frac{\alpha}{n-1} + (n - 2c_1 - 1)^2 \frac{\alpha^2}{(n-1)(n-2)} \right] \\ &= \left[1 - \frac{2(n - 2c_1 - 1)}{n-1} + \frac{(n - 2c_1 - 1)^2}{(n-1)(n-2)} \right]. \end{aligned} \quad (4.3)$$

Similarly, the risk functions of $\tilde{\alpha}_{\text{bsl}}^E$ and $\tilde{\alpha}_{\text{bge}}^E$ under squared-log error loss and general entropy loss functions are

$$R(\tilde{\alpha}_{\text{bsl}}^E) = \psi'(n) + [\psi(n - 2c_1 + 1) - \psi(n)]^2, \quad (4.4)$$

$$R(\tilde{\alpha}_{\text{bge}}^E) = k^\gamma \frac{\Gamma(n - \gamma)}{\Gamma(n)} + \gamma[\psi(n) - \ln k] - 1, \quad (4.5)$$

respectively. The risk functions of $\tilde{\alpha}_{\text{bq}}^c$, $\tilde{\alpha}_{\text{bsl}}^c$, and $\tilde{\alpha}_{\text{bge}}^c$, assuming conjugate prior are

$$R(\tilde{\alpha}_{\text{bq}}^c) = \left[1 - \frac{2n - 1}{n - 1} + \frac{(2n - 1)^2}{4(n - 1)(n - 2)} \right], \quad (4.6)$$

$$R(\tilde{\alpha}_{\text{bsl}}^c) = \psi'(n) + [\psi(2n + 1) - \psi(n) - \ln 2]^2, \quad (4.7)$$

$$R(\tilde{\alpha}_{\text{bge}}^c) = k_1^\gamma \frac{\Gamma(n - \gamma)}{\Gamma(n)} + \gamma[\psi(n) - \ln k_1 + \ln 2] - 1, \quad (4.8)$$

respectively.

The risk functions of $\hat{\alpha}_{\text{mle}}$ under weighted quadratic, squared-log error and general entropy loss functions are

$$R_q(\hat{\alpha}_{\text{mle}}) = \left[1 - \frac{2n}{n - 1} + \frac{n^2}{(n - 1)(n - 2)} \right], \quad (4.9)$$

$$R_{sl}(\hat{\alpha}_{\text{mle}}) = \psi'(n) + [\psi(n) - \ln n]^2, \quad (4.10)$$

$$R_{ge}(\hat{\alpha}_{\text{mle}}) = n^\gamma \frac{\Gamma(n - \gamma)}{\Gamma(n)} + \gamma[\psi(n) - \ln n] - 1, \quad (4.11)$$

respectively.

The estimators, developed in Section 3, are studied here on the basis of their risks obtained under three different loss functions, namely, (a) weighted quadratic loss, (b) squared-log error loss, (c) general entropy loss for $\gamma = 0.5$, (d) general entropy loss for $\gamma = 1$, and (e) general entropy loss for $\gamma = 1.5$. Risk functions of the proposed estimators are shown in Figures 2–5. The thick lines in each figure show the risk of the Bayes estimators under extended Jeffreys' prior and conjugate prior, and dotted lines show the risk of MLE under different loss functions. In Figure 2, risk functions have been plotted for all the loss functions under extended Jeffreys' prior for different values of c_1 and for $n = 30$. It is observed that risks are increasing with the increase in c_1 . Risks under general entropy loss for $\gamma < 1$ are ordinarily less than those of weighted quadratic and squared-log error losses. For small values of c_1 , risks of Bayes estimators are lower than those of maximum likelihood estimators for each loss function considered. The Bayes estimators perform better for some values of c_1 and loss functions under consideration; for example, in the figure, risk of $\tilde{\alpha}_{\text{bq}}^E$ is less for $c_1 < 1.5$ (approximately) whereas that of $\tilde{\alpha}_{\text{bsl}}^E$ is less for $c_1 < 0.8$ (approximately). Figures 3 and 4 show the risks for different values of n for $c_1 = 0.5$ and 2, respectively. We find that the risks are decreasing with the increase in n for all values of c_1 .

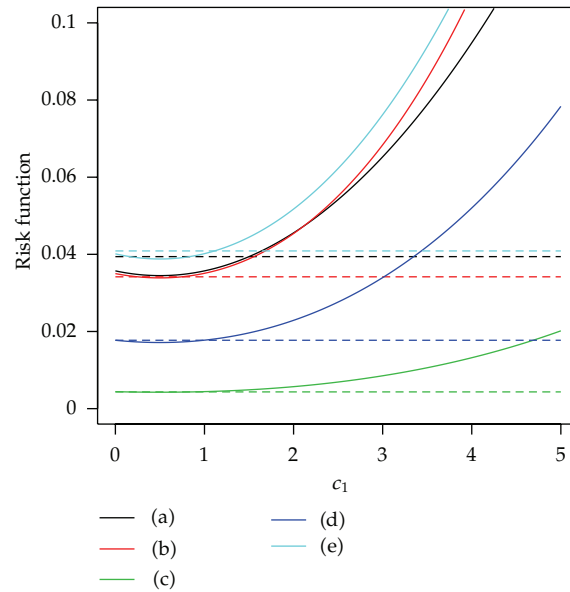


Figure 2: Risk function of estimators under the extended Jeffreys' prior for different values of c_1 with $n = 30$.

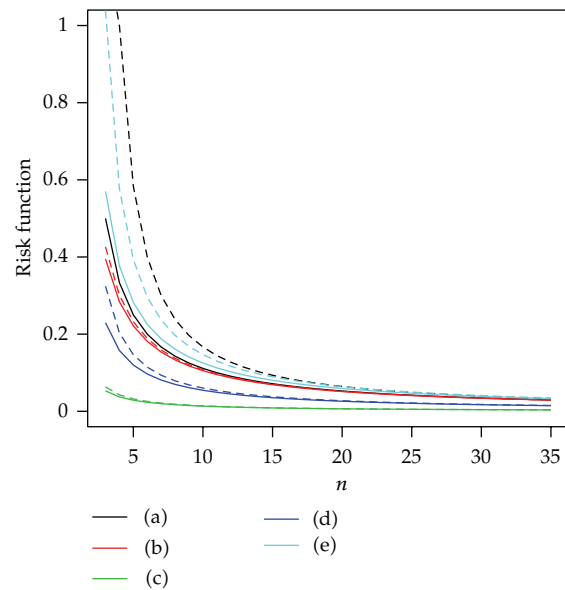


Figure 3: Risk function of estimators under the extended Jeffreys' prior for different values of n with $c_1 = 0.5$.

When we consider conjugate prior, we see that risks are less for the squared-log error and weighted quadratic losses than the general entropy loss, and only for these two losses, the Bayes estimators are less than the MLE for small n (Figure 5). The risks under conjugate prior are generally higher than those under Jeffreys' prior.

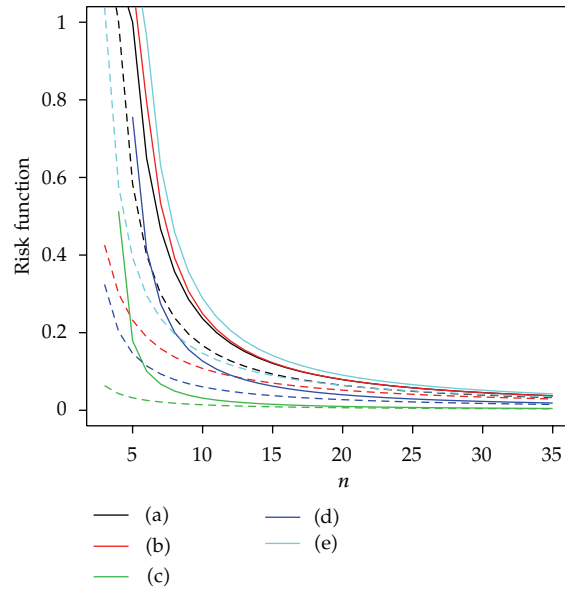


Figure 4: Risk function of estimators under the extended Jeffreys' prior for different n with $c_1 = 2$.

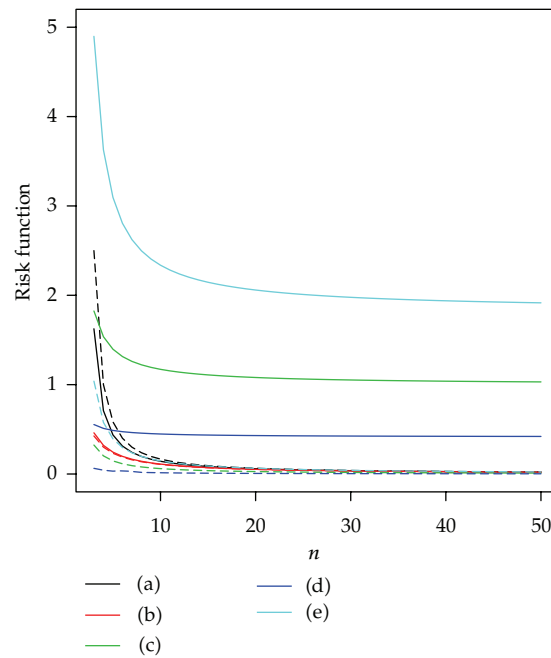


Figure 5: Risk function of estimators under conjugate prior for different n .

5. Highest Posterior Density Intervals for α

In this section our objective is to provide a highest posterior density (HPD) interval for the unknown parameter α of the model (1.2). HPD interval is one of the most useful tools to measure posterior uncertainty. This interval is such that it includes more probable values

of the parameter and excludes the less probable ones. Since the posterior density (3.2) is unimodal, the $100(1 - \eta)\%$ HPD interval $[H_1^E, H_2^E]$ for α must satisfy

$$\int_{H_1^E}^{H_2^E} \pi_1(\alpha | \underline{x}) d\alpha = 1 - \eta, \quad (5.1)$$

that is

$$\Gamma(n - 2c_1 + 1, H_2^E T) - \Gamma(n - 2c_1 + 1, H_1^E T) = 1 - \eta, \quad (5.2)$$

$$\pi_1(H_1^E | x) = \pi_1(H_2^E | x), \quad (5.3)$$

that is

$$\left[\frac{H_1^E}{H_2^E} \right]^{n-2c_1} = e^{T(H_1^E - H_2^E)}, \quad (5.4)$$

simultaneously. The HPD interval $[H_1^E, H_2^E]$ is the simultaneous solution of (5.2) and (5.4).

Similarly, the posterior density (3.4) is unimodal, and the $100(1 - \eta)\%$ HPD interval $[H_1^c, H_2^c]$ for α must satisfy

$$\int_{H_1^c}^{H_2^c} \pi_2(\alpha | T_a) d\alpha = 1 - \eta, \quad (5.5)$$

that is

$$\Gamma(n + b, H_2^c T_a) - \Gamma(n + b, H_1^c T_a) = 1 - \eta, \quad (5.6)$$

$$\pi_2(H_1^c | T_a) = \pi_2(H_2^c | T_a), \quad (5.7)$$

that is

$$\left[\frac{H_1^c}{H_2^c} \right]^{n+b-1} = e^{T_a(H_1^c - H_2^c)}, \quad (5.8)$$

simultaneously. Therefore, the HPD interval $[H_1^c, H_2^c]$ is the simultaneous solution of (5.6) and (5.8). If a and b are not known, then substituting the empirical Bayes estimate of a and b , we get the equations as follows:

$$\Gamma(2n + 1, 2H_2^c T) - \Gamma(2n + 1, 2H_1^c T) = 1 - \eta, \quad (5.9)$$

$$\left[\frac{H_1^c}{H_2^c} \right]^{2n} = e^{2T(H_1^c - H_2^c)}, \quad (5.10)$$

respectively.

6. Predictive Distribution

In this section our objective is to obtain the posterior predictive density of future observation, based on current observations. Another objective is to attain equal-tail Bayesian prediction interval for the future observation and then compare this interval with frequentist predictive interval. The posterior predictive distribution for $y = x_{n+1}$ given $\underline{x} = (x_1, x_2, \dots, x_n)$ under (3.2) is defined by

$$\begin{aligned}\xi^E(y | \underline{x}) &= \int_0^\infty \pi_1(\alpha | \underline{x}) g(y; \alpha) d\alpha \\ &= \frac{n - 2c_1 + 1}{T} \frac{1}{[1 - (\ln F(y)/T)]^{n-2c_1+2}} \frac{f(y)}{F(y)}.\end{aligned}\quad (6.1)$$

A $100(1 - \eta)\%$ equal-tail prediction interval $[y_1^E, y_2^E]$ is the solution of

$$\int_0^{y_1^E} \xi^E(y | \underline{x}) dy = \int_{y_2^E}^\infty \xi^E(y | \underline{x}) dy = \frac{\eta}{2}.\quad (6.2)$$

Using (6.1), we get (after simplification)

$$\begin{aligned}y_1^E &= F^{-1}\left[e^{T\{1-(1-(\eta/2))^{-1/(n-2c_1+1)}\}}\right], \\ y_2^E &= F^{-1}\left[e^{T\{1-(\eta/2)^{-1/(n-2c_1+1)}\}}\right].\end{aligned}\quad (6.3)$$

The posterior predictive distribution for $y = x_{n+1}$ given $\underline{x} = (x_1, x_2, \dots, x_n)$ under (3.4) is defined by

$$\begin{aligned}\xi^c(y | \underline{x}) &= \int_0^\infty \pi_2(\alpha | \underline{x}) g(y; \alpha) d\alpha \\ &= \frac{n + b}{T_a} \frac{1}{[1 - (\ln F(y)/T_a)]^{n+b+1}} \frac{f(y)}{F(y)}.\end{aligned}\quad (6.4)$$

A $100(1 - \eta)\%$ equal-tail prediction interval $[y_1^c, y_2^c]$ is the solution of

$$\int_0^{y_1^c} \xi^c(y | \underline{x}) dy = \int_{y_2^c}^\infty \xi^c(y | \underline{x}) dy = \frac{\eta}{2}.\quad (6.5)$$

Using (6.4), we get (after simplification)

$$\begin{aligned}y_1^c &= F^{-1}\left[e^{T_a\{1-(1-(\eta/2))^{-1/(n+b)}\}}\right], \\ y_2^c &= F^{-1}\left[e^{T_a\{1-(\eta/2)^{-1/(n+b)}\}}\right].\end{aligned}\quad (6.6)$$

If a and b are not known, then substituting the empirical Bayes estimate of a and b , we get the prediction limits as follows:

$$\begin{aligned} y_1^c &= F^{-1} \left[e^{2T \{1 - (1 - (\eta/2))^{-1/(2n+1)}\}} \right], \\ y_2^c &= F^{-1} \left[e^{2T \{1 - (\eta/2)^{-1/2n+1}\}} \right]. \end{aligned} \quad (6.7)$$

For deriving classical intervals, we notice that $Z = -\ln F(\mathbf{y})/T$ is distributed as a beta-variate of the second kind with parameters 1 and n . The pdf of Z has the following form:

$$h(z) = \frac{1}{B(1, n)} \frac{1}{(1+z)^{n+1}}, \quad z > 0. \quad (6.8)$$

Solving for (z_1, z_2) in

$$\int_0^{z_1} h(z) dz = \int_{z_2}^{\infty} h(z) dz = \frac{\eta}{2}, \quad (6.9)$$

and using (6.8), we get (after simplification)

$$\begin{aligned} z_1 &= F^{-1} \left[e^{T \{1 - (1 - (\eta/2))^{-1/n}\}} \right], \\ z_2 &= F^{-1} \left[e^{T \{1 - (\eta/2)^{-1/n}\}} \right]. \end{aligned} \quad (6.10)$$

It is to be noted that if we take $c_1 = 0.5$ in (6.3), we get classical $100(1 - \eta)\%$ equal-tail prediction interval.

7. Bayes Predictive Estimator and HPD Prediction Interval for a Future Observation

In this section, we introduce the Bayes predictive estimator for different priors under the above-mentioned loss functions, and later we obtain HPD predictive intervals for the future observation. The Bayes predictive estimators of y under a weighted quadratic error loss function assuming the extended Jeffreys' prior and the conjugate prior are

$$y_1^{*E} = \frac{\int_0^{\infty} y \cdot (1/y^2) \xi^E(\mathbf{y} | \underline{x}) dy}{\int_0^{\infty} (1/y^2) \xi^E(\mathbf{y} | \underline{x}) dy}, \quad (7.1)$$

$$y_1^{*c} = \frac{\int_0^{\infty} y \cdot (1/y^2) \xi^c(\mathbf{y} | \underline{x}) dy}{\int_0^{\infty} (1/y^2) \xi^c(\mathbf{y} | \underline{x}) dy}, \quad (7.2)$$

respectively. The Bayes predictive estimators of y under the squared-log error loss function assuming the extended Jeffreys' prior and the conjugate prior are

$$y_2^{*E} = \exp[E(\ln Y | \underline{x})], \quad \text{with } E(\ln Y | \underline{x}) = \int_0^{\infty} \ln y \cdot \xi^E(y | \underline{x}) dy, \quad (7.3)$$

$$y_2^{*c} = \exp[E(\ln Y | \underline{x})], \quad \text{with } E(\ln Y | \underline{x}) = \int_0^{\infty} \ln y \cdot \xi^c(y | \underline{x}) dy, \quad (7.4)$$

respectively. The Bayes predictive estimators of y under the general entropy loss function assuming the extended Jeffreys' prior and the conjugate prior are

$$y_3^{*E} = [J_g^E]^{-1/\gamma} \quad \text{with } J_g^E = \int_0^{\infty} y^{-\gamma} \xi^E(y | \underline{x}) dy, \quad (7.5)$$

$$y_3^{*c} = [J_g^c]^{-1/\gamma} \quad \text{with } J_g^c = \int_0^{\infty} y^{-\gamma} \xi^c(y | \underline{x}) dy, \quad (7.6)$$

respectively. The closed form expressions of (7.1)–(7.6) seem to be intractable, and calculations are to be made using numerical method.

For the unimodal predictive density (6.1), the HPD-predictive interval $[h_1^E, h_2^E]$ with probability $1 - \eta$ for y is the simultaneous solution of the following:

$$P(h_1^E < Y < h_2^E) = 1 - \eta, \quad (7.7)$$

that is

$$\frac{1}{[1 - (\ln F(h_2^E)/T)]^{n-2c_1+1}} - \frac{1}{[1 - (\ln F(h_1^E)/T)]^{n-2c_1+1}} = 1 - \eta, \quad (7.8)$$

$$\xi^E(h_1^E | \underline{x}) = \xi^E(h_2^E | \underline{x}), \quad (7.9)$$

that is

$$\left[\frac{1 - (\ln F(h_2^E)/T)}{1 - (\ln F(h_1^E)/T)} \right]^{n-2c_1+2} = \frac{f(h_2^E)}{F(h_2^E)} \cdot \frac{F(h_1^E)}{f(h_1^E)}. \quad (7.10)$$

Similarly, for the unimodal predictive density (6.4), the HPD-predictive interval $[h_1^c, h_2^c]$ with probability $1 - \eta$ for y is the simultaneous solution of the following:

$$P(h_1^c < Y < h_2^c) = 1 - \eta, \quad (7.11)$$

that is

$$\frac{1}{[1 - (\ln F(h_2^c)/T_a)]^{n+b}} - \frac{1}{[1 - (\ln F(h_1^c)/T_a)]^{n+b}} = 1 - \eta, \quad (7.12)$$

$$\xi^c(h_1^c | \underline{x}) = \xi^c(h_2^c | \underline{x}), \quad (7.13)$$

that is

$$\left[\frac{1 - (\ln F(h_2^c)/T_a)}{1 - (\ln F(h_1^c)/T_a)} \right]^{n+b+1} = \frac{f(h_2^c)}{F(h_2^c)} \cdot \frac{F(h_1^c)}{f(h_1^c)}. \quad (7.14)$$

If a and b are not known, then substituting the empirical Bayes estimate of a and b , we get the HPD-prediction limits for future observation as follows:

$$\frac{1}{[1 - (\ln F(h_2^c)/2T)]^{2n+1}} - \frac{1}{[1 - (\ln F(h_1^c)/2T)]^{2n+1}} = 1 - \eta, \quad (7.15)$$

$$\left[\frac{1 - (\ln F(h_2^c)/2T)}{1 - (\ln F(h_1^c)/2T)} \right]^{2n+2} = \frac{f(h_2^c)}{F(h_2^c)} \cdot \frac{F(h_1^c)}{f(h_1^c)}.$$

8. Data Analysis

Consider the following data which arose in tests on endurance of deep groove ball bearings (Lawless [22, page 228]). The data are the number of million revolutions before failure for each of the 23 ball bearings in the life test:

$$17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, \quad (8.1)$$

$$68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.$$

To study the goodness of fit of the exponentiated exponential model, Gupta and Kundu [2] computed the χ^2 statistic as 0.783 with the corresponding P value being 0.376. The estimate of α , the shape parameter, is 5.2589 and that of θ , the rate of the exponential distribution, is 0.0314. Here our aim is to obtain the Bayes estimates of α for this data set under three different loss functions and for two priors by assuming that the base line distribution is exponential with $\theta = 0.0314$. At the same time, we are interested in studying the HPD intervals for the parameter α . Further, our intention is to obtain the future observation based on a given set of observations and the HPD predictive intervals for the future observation based on the current observations. Figure 6 shows the estimated predictive distribution.

Tables 1–4 summarize the result from the data analysis. Tables 1(a) and 1(b) represents the Bayes estimates of α and corresponding risks under the extended Jeffreys' prior the conjugate prior, respectively. It is evident from Table 1(a) that Bayes estimates under general entropy loss ($\gamma = 0.5$ and $\gamma = 1$) give better estimates than all other estimates. It is also evident from Table 1(a) that the estimates decrease and the corresponding risks increase with the increase in c_1 . In case of conjugate prior (Table 1(b)), the estimates under weighted

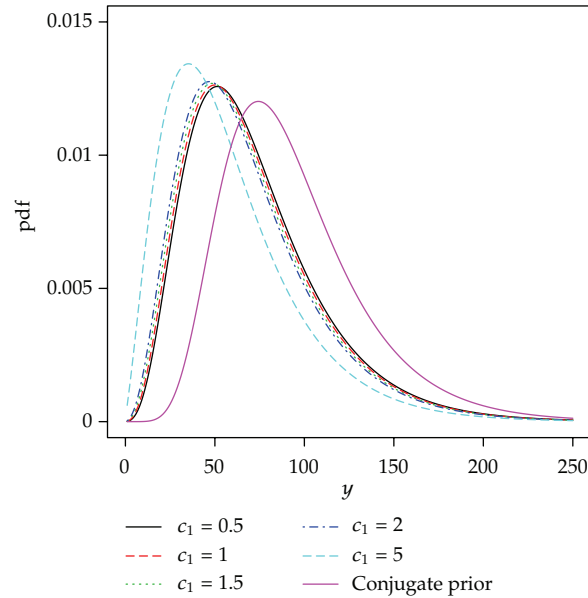


Figure 6: Estimate of the predictive distribution for the given data set for extended Jeffreys’ prior and conjugate prior.

Table 1

(a) The Bayes estimates of the shape parameter under extended prior and the corresponding risk function under three different loss functions for the data set

c_1	$\tilde{\alpha}_{bq}^E$	$R(\tilde{\alpha}_{bq}^E)$	$\tilde{\alpha}_{bsl}^E$	$R(\tilde{\alpha}_{bsl}^E)$	$\gamma = 0.5$		$\gamma = 1.0$		$\gamma = 1.5$	
					$\tilde{\alpha}_{bge}^E$	$R(\tilde{\alpha}_{bge}^E)$	$\tilde{\alpha}_{bge}^E$	$R(\tilde{\alpha}_{bge}^E)$	$\tilde{\alpha}_{bge}^E$	$R(\tilde{\alpha}_{bge}^E)$
0.5	4.607	0.0454	4.936	0.0444	4.881	0.0055	4.826	0.0225	4.771	0.0511
1.0	4.387	0.0476	4.717	0.0465	4.662	0.0058	4.607	0.0236	4.551	0.0535
1.5	4.168	0.0541	4.498	0.0531	4.443	0.0066	4.387	0.0269	4.332	0.0611
2.0	3.949	0.0649	4.278	0.0649	4.223	0.0081	4.168	0.0327	4.113	0.0741
5.0	2.632	0.2207	2.962	0.3052	2.907	0.0364	2.852	0.1395	2.796	0.3012

(b) The Bayes estimates of the shape parameter under conjugate prior and the corresponding risk function under three different loss functions for the data set

$\tilde{\alpha}_{bq}^c$	$R(\tilde{\alpha}_{bq}^c)$	$\tilde{\alpha}_{bsl}^c$	$R(\tilde{\alpha}_{bsl}^c)$	$\gamma = 0.5$		$\gamma = 1.0$		$\gamma = 1.5$	
				$\tilde{\alpha}_{bge}^c$	$R(\tilde{\alpha}_{bge}^c)$	$\tilde{\alpha}_{bge}^c$	$R(\tilde{\alpha}_{bge}^c)$	$\tilde{\alpha}_{bge}^c$	$R(\tilde{\alpha}_{bge}^c)$
4.936	0.0503	5.101	0.0455	5.073	0.4281	5.046	1.0691	5.018	2.0264

quadratic and squared-log error loss functions seem to be better. In Table 2, the HPD intervals under conjugate prior appear to be slightly better than the extended Jeffreys’ with respect to minimum length. Table 3 presents the estimates of future observation based on the data set. It is observed from the last column of the table that the general entropy loss function at $\gamma = 1.5$ gives a quite reasonable estimate. Table 4 shows the Bayesian predictive and HPD predictive intervals for future observation of the data set. The first row in each cell represents the Bayesian predictive intervals, and the second row represents the HPD intervals for future

Table 2: HPD intervals of the shape parameter for the data set.

c_1	(H_1^E, H_2^E)		
	$\eta = 0.01$	$\eta = 0.05$	$\eta = 0.10$
0.5	(2.631, 7.997)	(2.999, 7.079)	(3.019, 6.531)
1.0	(2.472, 7.718)	(2.902, 6.878)	(2.999, 6.361)
1.5	(2.314, 7.437)	(2.729, 6.612)	(2.961, 6.214)
2.0	(2.158, 7.154)	(2.558, 6.344)	(2.782, 5.954)
5.0	(2.061, 7.052)	(2.365, 6.131)	(2.532, 5.691)
(H_1^c, H_2^c)	(2.108, 7.065)	(2.236, 6.452)	(2.871, 6.141)

Table 3: Estimated future observation for the data set under three different loss functions.

c_1	y_1^*	y_2^*	y_3^*		
			$\gamma = 0.5$	$\gamma = 1.0$	$\gamma = 1.5$
0.5	33.359	62.788	57.977	53.098	48.026
1.0	30.572	61.338	56.432	51.431	46.169
1.5	26.726	59.822	54.812	49.673	44.173
2.0	24.755	58.233	53.111	47.813	42.017
5.0	18.285	46.701	40.504	33.421	27.315
conjugate prior	55.329	13.047	320.483	315.724	190.144

Table 4: Bayesian predictive and HPD predictive intervals for future observation of the data set.

η	Conjugate prior	$c_1 = 0.5$	$c_1 = 1.0$	$c_1 = 1.5$	$c_1 = 2.0$	$c_1 = 5.0$
0.01	(0.015, 17.32)	(0.031, 37.61)	(0.033, 39.53)	(0.034, 41.66)	(0.036, 44.03)	(0.051, 66.78)
	(7.819, 220.83)	(5.966, 199.82)	(5.257, 198.29)	(4.552, 196.67)	(3.856, 194.97)	(0.398, 182.53)
0.05	(0.078, 11.85)	(0.159, 25.25)	(0.167, 26.49)	(0.175, 27.87)	(0.183, 29.41)	(0.262, 43.76)
	(16.206, 168.96)	(10.712, 148.26)	(10.217, 147.45)	(9.654, 146.41)	(8.671, 144.69)	(2.635, 131.90)
0.10	(0.158, 9.55)	(0.324, 20.19)	(0.338, 21.17)	(0.354, 22.25)	(0.372, 23.45)	(0.532, 34.63)
	(26.138, 146.11)	(14.176, 124.48)	(13.812, 123.37)	(12.103, 122.28)	(10.916, 121.24)	(4.939, 110.22)

observation. It is apparent that the HPD intervals for future observation with respect to conjugate prior are reasonably good.

9. Concluding Remark

In this paper, we have derived the Bayes estimators of the shape parameter of the exponentiated family of distributions under the extended Jeffreys' prior as well as conjugate prior using three different loss functions. Though the extended Jeffreys' prior gives the opportunity of covering wide spectrum of priors, yet at times the conjugate prior also gives better Bayes estimates and HPD intervals of the parameter and of future observations.

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