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Research Article

Marginal Distributions of Random Vectors Generated by Affine Transformations of Independent Two-Piece Normal Variables

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Marginal probability density and cumulative distribution functions are presented for multidimensional variables defined by nonsingular affine transformations of vectors of independent two-piece normal variables, the most important subclass of Ferreira and Steel's general multivariate skewed distributions. The marginal functions are obtained by first expressing the joint density as a mixture of Arellano-Valle and Azzalini's unified skew-normal densities and then using the property of closure under marginalization of the latter class.

1. Introduction

In the literature on probability distributions, there are several approaches for extending the multivariate normal distribution with the introduction of some sort of skewness. Arellano-Valle et al. [1] provide a unified view of this literature. The largest group of contributions was initiated by Azzalini and Dalla Valle [2] and Azzalini and Capitanio [3] and generalizes the univariate skew-normal (*SN*) distribution studied by Azzalini [4, 5]. These "multivariate skew-normal distributions" are generated from a normal distribution either by conditioning on a truncated variable or by a convolution mechanism.

An alternative approach was proposed by Ferreira and Steel [6–8] and is based on nonsingular affine transformations of random vectors with independent components, each having a skewed distribution with probability density function (pdf) constructed from a symmetric distribution using the inverse scaling factor method introduced by Fernández and Steel [9]. (Arellano-Valle et al. [10] consider a general class of asymmetric univariate distributions that includes the distributions generated according to the procedure proposed by Fernández and Steel [9] as a special case.) If the univariate symmetric distribution is the standard normal, then the corresponding univariate skewed distribution becomes (with a different parameterization) the two-piece normal (*tpn*) analyzed by John [11] (see also Johnson et al. [12]). To overcome an issue of overparameterization, Ferreira and Steel [7, 8] pay particular attention to the subclass associated with transformation matrices that can be factorized as the product of an orthogonal matrix and a diagonal positive definite matrix. Villani and Larsson [13] studied this subclass when the basic univariate skewed distribution is the *tpn* and named these distributions "multivariate split normal."

Under the acronym *SUN* (standing for "unified skew-normal"), Arellano-Valle and Azzalini [14] suggested a formulation for the first approach that encompasses the most relevant coexisting variants of multivariate skew-normal distributions. Like the multivariate normal and *SN* distributions, the class of *SUN* distributions is closed under affine transformations, marginalization, and conditioning to given values of some components. Besides these important properties, the *SUN* class is also closed under sums of independent components. However, one limitation of the *SUN* distributions is that the vector of location parameters does not have a direct interpretation as the mean or the mode of the distribution, which are rather complicated functions of all the parameters. Even in the simplest case of the basic *SN*, both the mean and the mode (for which there is no closed expression) depend on the parameters regulating dispersion and skewness.

Ferreira and Steel's independent components approach to the construction of multivariate skewed normal distributions (henceforth *FS-SN*) provides an alternative to the *SUN* class in applications for which it is important to have some location measure that does not depend on the dispersion and skewness parameters. Indeed, the *FS-SN* distributions have the convenient feature that the mode is part of the distribution parameters and therefore is invariant to dispersion and skewness. In addition, the *FS-SN* distributions are closed under nonsingular affine transformations. However, unlike the *SUN* class, the *FS-SN* distributions are not closed under marginalization (neither under conditioning) and, to my knowledge, general closed expressions of their marginal pdf and cumulative distribution function (cdf) are not available in the literature.

This paper aims at filling the gap and proposing expressions for the marginal density and cumulative distribution functions of an *FS-SN* distribution. Obviously, the expressions will also apply to the subclass of multivariate split normal distributions studied by Villani and Larsson [13]. The technique used to derive the marginal distributions is simple and consists of expressing the joint *FS-SN* distribution as a finite mixture of singular *SUN* distributions and then making use of their property of closure under marginalization.

An area of application of the results presented in this paper is macroeconomic density forecasting. Many institutions that publish macroeconomic forecasts complement their point forecasts with information on the dispersion and skewness of the probability distributions of the forecasting errors. Fan charts are one of the most popular tools to convey the predictive densities, and they gained prominence through their use in inflation reports released by many central banks, with the Bank of England and the Sveriges Riksbank (the Swedish central bank) featuring as pioneers in this respect [15, 16] (see also Wallis [17, 18] and Tay and Wallis [19]). The characterization of the forecast densities is complicated by the fact that typically institutional forecasts are not based on a single model but stem from different competing models combined with judgements by experts (the latter regarding, in particular, the skewness, i.e., the balance of upward and downward risks to the forecasts). Most of the procedures used to generate the fan charts take the point baseline forecasts as given and assume that the sources of uncertainty and asymmetry have univariate *tpn* distributions. These sources of forecasting error are then aggregated according to a linear mapping, envisaged as an approximation around the baseline to the underlying unknown data generating process. In the absence of closed expressions for the exact distribution of a linear combination of *tpn* variables, some aggregation procedures resort to informal approximations based on the first moments, while other procedures are based on numerical simulation. Examples of the first approach are Blix and Sellin [16, 20, 21] and Elekdag and Kannan [22], while Pinheiro and Esteves [23] opted to simulate the distribution. The results presented in Section 3 allow to overcome this aggregation difficulty.

2. The *SUN* and the *FS-SN* Distributions

If the M-dimensional random vector $\mathbf{Y} \sim SUN_{M,N}(\boldsymbol{\xi}, \boldsymbol{\gamma}, \overline{\boldsymbol{\omega}}, \boldsymbol{\Omega}^*)$, then its pdf and cdf are, respectively, for any point $\mathbf{y} \in R^M$

$$g_{\mathbf{Y}}(\mathbf{y} \mid \boldsymbol{\xi}, \boldsymbol{\gamma}, \overline{\boldsymbol{\omega}}, \boldsymbol{\Omega}^*) = \varphi_M(\mathbf{y} - \boldsymbol{\xi} \mid \boldsymbol{\Omega}) \frac{\boldsymbol{\Phi}_N\left(\boldsymbol{\gamma} + \boldsymbol{\Delta}'\overline{\boldsymbol{\Omega}}^{-1}\boldsymbol{\omega}^{-1}(\mathbf{y} - \boldsymbol{\xi}) \mid \boldsymbol{\Gamma} - \boldsymbol{\Delta}'\overline{\boldsymbol{\Omega}}^{-1}\boldsymbol{\Delta}\right)}{\boldsymbol{\Phi}_N(\boldsymbol{\gamma} \mid \boldsymbol{\Gamma})}, \tag{2.1}$$

$$G_{\mathbf{Y}}(\mathbf{y} \mid \boldsymbol{\xi}, \boldsymbol{\gamma}, \overline{\boldsymbol{\omega}}, \boldsymbol{\Omega}^*) = \frac{\boldsymbol{\Phi}_{N+M}\left(\begin{bmatrix} \boldsymbol{\gamma} \\ \boldsymbol{\omega}^{-1}(\mathbf{y}-\boldsymbol{\xi}) \end{bmatrix} \mid \boldsymbol{\Omega}^*\right)}{\boldsymbol{\Phi}_N(\boldsymbol{\gamma} \mid \boldsymbol{\Gamma})}, \tag{2.2}$$

where $\varphi_M(\mathbf{y}-\boldsymbol{\xi} \mid \boldsymbol{\Omega})$ and $\boldsymbol{\Phi}_M(\mathbf{y}-\boldsymbol{\xi} \mid \boldsymbol{\Omega})$ denote, respectively, the pdf and the cdf at point $\mathbf{y}$ of a normal distribution $N_M(\boldsymbol{\xi}, \boldsymbol{\Omega})$, $\boldsymbol{\xi}(M \times 1)$ and $\boldsymbol{\gamma}(N \times 1)$ are vectors of parameters, $\boldsymbol{\Omega}(M \times M)$ is a positive definite covariance matrix, $\boldsymbol{\omega}(M \times M)$ is the diagonal matrix formed by the standard deviations of $\boldsymbol{\Omega}$, $\overline{\boldsymbol{\Omega}}(M \times M)$ is the correlation matrix associated with $\boldsymbol{\Omega}$ (hence $\boldsymbol{\Omega} = \boldsymbol{\omega}\overline{\boldsymbol{\Omega}}\boldsymbol{\omega}$), $\overline{\boldsymbol{\omega}} = \boldsymbol{\omega}\boldsymbol{\iota}_N$ with $\boldsymbol{\iota}_N = [1 \cdots 1]'(N \times 1)$, $\boldsymbol{\Gamma}(N \times N)$ is a positive definite correlation matrix, and $\boldsymbol{\Delta}(M \times N)$ is such that

$$\boldsymbol{\Omega}^* = \begin{bmatrix} \boldsymbol{\Gamma} & \boldsymbol{\Delta}' \\ \boldsymbol{\Delta} & \overline{\boldsymbol{\Omega}} \end{bmatrix} ((N+M) \times (N+M)) \tag{2.3}$$

is also a (semi-definite positive) correlation matrix. (Arellano-Valle and Azzalini [14, Appendix C] consider three cases of singular *SUN* distributions: (i) $\boldsymbol{\Omega}$ singular; (ii) $\boldsymbol{\Gamma}$ singular; (iii) $\boldsymbol{\Omega}^*$ singular with nonsingular $\boldsymbol{\Omega}$ and $\boldsymbol{\Gamma}$. For our purposes, only the latter case is relevant.) The *SUN* distribution collapses to the multivariate normal when $\boldsymbol{\Delta} = \mathbf{0}$, $\boldsymbol{\Delta}$ being the matrix of parameters that regulate skewness. It collapses to the basic multivariate *SN* distribution suggested by Azzalini and Dalla Valle [2] when $N = 1$ and $\boldsymbol{\gamma} = \mathbf{0}$ (implying that $\boldsymbol{\Gamma} = 1$).

Now let the scalar random variable U_n be *tpn* distributed with zero mode. Its pdf may be parameterized as follows:

$$f_{U_n}(u_n \mid \omega_n, \theta_n) = \begin{cases} 2\omega_n^{-1}\left(\theta_n + \theta_n^{-1}\right)^{-1}\phi\left(\omega_n^{-1}\theta_n u_n\right) & (u_n \le 0), \\ 2\omega_n^{-1}\left(\theta_n + \theta_n^{-1}\right)^{-1}\phi\left(\omega_n^{-1}\theta_n^{-1} u_n\right) & (u_n > 0), \end{cases} \tag{2.4}$$

where $\phi(\,)$ denotes the $N(0,1)$ pdf, $\omega_n(>0)$ is a scale parameter, and $\theta_n(>0)$ is a shape parameter. When $\theta_n = 1$, the density becomes the normal pdf with zero mean and standard deviation ω_n (so that when the latter parameter is 1 the pdf collapses to $\phi(u_n)$). Values of θ_n above (below) unity correspond to densities skewed to the right (left). Let $\mathbf{U}$ be an N-dimensional random vector of independent *tpn* components u_n with zero mode and unitary scale $\omega_n = 1$. Its pdf is

$$f_{\mathbf{U}}(\mathbf{u} \mid \boldsymbol{\theta}) = \prod_{n=1}^{N} f_{U_n}(u_n \mid 1, \theta_n), \tag{2.5}$$

where $f_{U_n}(\,)$ is as in (2.4) (with $\omega_n = 1$) and $\boldsymbol{\theta} = [\theta_1 \cdots \theta_N]'$. An N-dimensional random vector $\mathbf{X}$ is said to be $FS\text{-}SN_N(\boldsymbol{\mu}, \mathbf{A}, \boldsymbol{\theta})$ distributed if there is a random vector $\mathbf{U}$ with density (2.5) and two vectors $\boldsymbol{\mu}$ (the joint mode) and $\boldsymbol{\theta}$ (the "shape vector") and a nonsingular matrix $\mathbf{A}$ (the "scale matrix") such that $\mathbf{X} = \boldsymbol{\mu} + \mathbf{A}\mathbf{U}$. Vector $\mathbf{X}$ has pdf

$$f_{\mathbf{X}}(\mathbf{x} \mid \boldsymbol{\mu}, \mathbf{A}, \boldsymbol{\theta}) = |\det(\mathbf{A})|^{-1} f_{\mathbf{U}}\left(\mathbf{A}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \mid \boldsymbol{\theta}\right). \tag{2.6}$$

It is straightforward to confirm that (i) when $\boldsymbol{\theta} = \mathbf{0}$, this density collapses to the pdf of a $N_N(\boldsymbol{\mu}, \mathbf{A}\mathbf{A}')$ distribution, (ii) the $FS\text{-}SN_N(\boldsymbol{\mu}, \mathbf{A}, \boldsymbol{\theta})$ distribution is unimodal with mode $\boldsymbol{\mu}$, invariant with respect to $\mathbf{A}$ and $\boldsymbol{\theta}$, and (iii) by construction, the *FS-SN* class is closed under nonsingular affine transformations.

3. The Marginal *FS-SN* Distributions

To establish additional notations, let $\mathbf{I}_N$ denote the identity matrix of order N, and let $\eta(\mathbf{z})$ the number of zero elements in vector $\mathbf{z}$, $\psi(\mathbf{z})$ one if all elements of vector $\mathbf{z}$ are nonnegative and zero otherwise, and $\mathbf{k}(i) = (k_1(i), \ldots, k_n(i), \ldots, k_N(i))$ the generic element of the Nth Cartesian power of $\{-1;\ 1\}$(with cardinal 2^N), $\mathbf{K}(i) = \operatorname{diag}_n(k_n(i))(N \times N)$, $\boldsymbol{\Theta}(i) = \operatorname{diag}_n(\theta_n^{k_n(i)})(N \times N)$, $\boldsymbol{\Omega}(i) = [\mathbf{A}\boldsymbol{\Theta}(i)\ \mathbf{K}(i)][\mathbf{A}\boldsymbol{\Theta}(i)\ \mathbf{K}(i)]' = \mathbf{A}\boldsymbol{\Theta}^2(i)\mathbf{A}'$, $\boldsymbol{\omega}(i) = [\operatorname{diag}(\boldsymbol{\Omega}(i))]^{1/2}$, $\overline{\boldsymbol{\varpi}}(i) = \boldsymbol{\omega}(i)\boldsymbol{\iota}_N$, $\boldsymbol{\Delta}(i) = \boldsymbol{\omega}^{-1}(i)\mathbf{A}\boldsymbol{\Theta}(i)\mathbf{K}(i)$ and $\overline{\boldsymbol{\Omega}}(i) = \boldsymbol{\omega}^{-1}(i)\boldsymbol{\Omega}(i)\boldsymbol{\omega}^{-1}(i) = \boldsymbol{\Delta}(i)\boldsymbol{\Delta}(i)'$.

Proposition 3.1. *The pdf and the cdf of the N-dimensional random vector*

$$\mathbf{X} \sim FS\text{-}SN_N(\boldsymbol{\mu}, \mathbf{A}, \boldsymbol{\theta}) \tag{3.1}$$

with nonsingular scale matrix $\mathbf{A}$ *can be expressed, respectively, as*

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{x} \mid \boldsymbol{\mu}, \mathbf{A}, \boldsymbol{\theta}) &= \sum_{i=1}^{2^N} \left[\prod_{n=1}^{N} \left(1 + \theta_n^{-2k_n(i)}\right)\right]^{-1} g_{\mathbf{X}}(\mathbf{x} \mid \boldsymbol{\mu}, \mathbf{0}, \overline{\boldsymbol{\varpi}}(i), \boldsymbol{\Omega}^*(i)), \\ F_{\mathbf{X}}(\mathbf{x} \mid \boldsymbol{\mu}, \mathbf{A}, \boldsymbol{\theta}) &= \sum_{i=1}^{2^N} \left[\prod_{n=1}^{N} \left(1 + \theta_n^{-2k_n(i)}\right)\right]^{-1} G_{\mathbf{X}}(\mathbf{x} \mid \boldsymbol{\mu}, \mathbf{0}, \overline{\boldsymbol{\varpi}}(i), \boldsymbol{\Omega}^*(i)), \end{aligned} \tag{3.2}$$

where $g_{\mathbf{X}}(\ )$ *and* $G_{\mathbf{X}}(\ )$ *are pdfs and cdfs of singular* $SUN_{N,N}(\boldsymbol{\mu}, \mathbf{0}, \overline{\boldsymbol{\omega}}(i), \boldsymbol{\Omega}^*(i))$ *distributions, with* $\boldsymbol{\Omega}^*(i) = \begin{bmatrix} \mathbf{I}_N & \boldsymbol{\Delta}(i)' \\ \boldsymbol{\Delta}(i) & \overline{\boldsymbol{\Omega}}(i) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_N \\ \boldsymbol{\Delta}(i) \end{bmatrix} [\mathbf{I}_N \ \ \boldsymbol{\Delta}(i)']$. *The latter functions may be written as*

$$\begin{aligned} g_{\mathbf{X}}(\mathbf{x} \mid \boldsymbol{\mu}, \mathbf{0}, \overline{\boldsymbol{\omega}}(i), \boldsymbol{\Omega}^*(i)) &= 2^{N-\eta(\mathbf{x}-\boldsymbol{\mu})} \varphi_N(\mathbf{x} - \boldsymbol{\mu} \mid \boldsymbol{\Omega}(i)) \psi\left(\mathbf{K}(i)\boldsymbol{\Theta}^{-1}(i)\mathbf{A}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right), \\ G_{\mathbf{X}}(\mathbf{x} \mid \boldsymbol{\mu}, \mathbf{0}, \overline{\boldsymbol{\omega}}(i), \boldsymbol{\Omega}^*(i)) &= 2^N \Phi_{2N}\left(\begin{bmatrix} \mathbf{0} \\ \boldsymbol{\omega}^{-1}(i)(\mathbf{x}-\boldsymbol{\mu}) \end{bmatrix} \mid \boldsymbol{\Omega}^*(i) \right) \\ &= 2^N \int_{\{\mathbf{z} \mid \mathbf{z} \le \mathbf{0},\ \mathbf{A}\boldsymbol{\Theta}(i)\mathbf{K}(i)\mathbf{z} \le \mathbf{x}-\boldsymbol{\mu}\}} \varphi_N(\mathbf{z} \mid \mathbf{I}_N) d\mathbf{z}. \end{aligned} \tag{3.3}$$

Note that

$$\sum_{i=1}^{2^N} \left[\prod_{n=1}^{N} \left(1 + \theta_n^{-2k_n(i)}\right) \right]^{-1} = \left[\prod_{n=1}^{N} \left(\theta_n + \theta_n^{-1}\right) \right]^{-1} \sum_{i=1}^{2^N} \left[\prod_{n=1}^{N} \theta_n^{k_n(i)} \right] = 1. \tag{3.4}$$

Hence, the distribution $FS\text{-}SN_N$ can be envisaged as a finite mixture of singular $SUN_{N,N}$ distributions.

As pointed out by Arellano-Valle and Azzalini [14, Appendix C], the rank deficiency of $\boldsymbol{\Omega}^*(i)$ does not affect the properties of the *SUN* distributions and its only impact is of a computational nature. In our case, it actually simplifies the computation of the pdf values because the evaluation of a normal cdf is not required anymore, unlike when computing (2.1), the general expression of a *SUN* pdf.

In order to derive the marginal pdfs and cdfs of $\mathbf{X}$, one needs to consider its partition $\mathbf{X} = [\mathbf{X}_1' \ \ \mathbf{X}_2']'$ with $\mathbf{X}_1$ and $\mathbf{X}_2$ of dimensions N_1 and N_2, respectively, and the corresponding partitions

$$\begin{gathered} \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}, \quad \boldsymbol{\Delta}(i) = \begin{bmatrix} \boldsymbol{\Delta}_1(i) \\ \boldsymbol{\Delta}_2(i) \end{bmatrix}, \quad \boldsymbol{\Omega}(i) = \begin{bmatrix} \boldsymbol{\Omega}_{11}(i) & \boldsymbol{\Omega}_{12}(i) \\ \boldsymbol{\Omega}_{12}(i)' & \boldsymbol{\Omega}_{22}(i) \end{bmatrix}, \\ \boldsymbol{\omega}(i) = \begin{bmatrix} \boldsymbol{\omega}_1(i) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\omega}_2(i) \end{bmatrix}, \quad \overline{\boldsymbol{\omega}}(i) = \begin{bmatrix} \overline{\boldsymbol{\omega}}_1(i) \\ \overline{\boldsymbol{\omega}}_2(i) \end{bmatrix} \end{gathered} \tag{3.5}$$

with $\mathbf{A}_1(N_1 \times N)$, $\boldsymbol{\Delta}_1(i) = \boldsymbol{\omega}_1^{-1}(i)\mathbf{A}_1\boldsymbol{\Theta}(i)\mathbf{K}(i)(N_1 \times N)$, $\boldsymbol{\Omega}_{11}(i) = \mathbf{A}_1\boldsymbol{\Theta}^2(i)\mathbf{A}_1'(N_1 \times N_1)$, $\boldsymbol{\omega}_1(i) = [\operatorname{diag}(\boldsymbol{\Omega}_{11}(i))]^{1/2}$, $\overline{\boldsymbol{\Omega}}_{11}(i) = \boldsymbol{\omega}_1^{-1}(i)\boldsymbol{\Omega}_{11}(i)\boldsymbol{\omega}_1^{-1}(i) = \boldsymbol{\Delta}_1(i)\boldsymbol{\Delta}_1(i)'$, and $\overline{\boldsymbol{\omega}}_1(i) = \boldsymbol{\omega}_1(i)\boldsymbol{\iota}_{N_1}$. Proposition 3.2 follows directly from Proposition 3.1 and from the result of Arellano-Valle and Azzalini [14, Appendix A] on the marginal distributions of members of the *SUN* class.

Proposition 3.2. *Let* $\mathbf{X} = [\mathbf{X}_1' \ \ \mathbf{X}_2']' \sim FS\text{-}SN_N(\boldsymbol{\mu}, \mathbf{A}, \boldsymbol{\theta})$. *Then, the marginal pdf and the cdf of the* N_1*-dimensional subvector* $\mathbf{X}_1$ *are, respectively,*

$$\begin{aligned} f_{\mathbf{X}_1}(\mathbf{x}_1 \mid \boldsymbol{\mu}_1, \mathbf{A}_1, \boldsymbol{\theta}) &= \sum_{i=1}^{2^N} \left[\prod_{n=1}^{N} \left(1 + \theta_n^{-2k_n(i)}\right) \right]^{-1} g_{\mathbf{X}_1}(\mathbf{x}_1 \mid \boldsymbol{\mu}_1, \mathbf{0}, \overline{\boldsymbol{\omega}}_1(i), \boldsymbol{\Omega}_{11}^*(i)), \\ F_{\mathbf{X}_1}(\mathbf{x}_1 \mid \boldsymbol{\mu}_1, \mathbf{A}_1, \boldsymbol{\theta}) &= \sum_{i=1}^{2^N} \left[\prod_{n=1}^{N} \left(1 + \theta_n^{-2k_n(i)}\right) \right]^{-1} G_{\mathbf{X}_1}(\mathbf{x}_1 \mid \boldsymbol{\mu}_1, \mathbf{0}, \overline{\boldsymbol{\omega}}_1(i), \boldsymbol{\Omega}_{11}^*(i)), \end{aligned} \tag{3.6}$$

where $g_{\mathbf{X}_1}(\ )$ *and* $G_{\mathbf{X}_1}(\ )$ *are pdfs and cdfs of singular* $SUN_{N_1,N}(\boldsymbol{\mu}_1, \mathbf{0}, \overline{\boldsymbol{\omega}}_1(i), \boldsymbol{\Omega}_{11}^*(i))$ *distributions, with* $\boldsymbol{\Omega}_{11}^*(i) = \begin{bmatrix} \mathbf{I}_N & \boldsymbol{\Delta}_1(i)' \\ \boldsymbol{\Delta}_1(i) & \overline{\boldsymbol{\Omega}}_{11}(i) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_N \\ \boldsymbol{\Delta}_1(i) \end{bmatrix} [\mathbf{I}_N \quad \boldsymbol{\Delta}_1(i)']$. *The latter functions may be written as*

$$\begin{aligned} & g_{\mathbf{X}_1}(\mathbf{x}_1 \mid \boldsymbol{\mu}_1, \mathbf{0}, \overline{\boldsymbol{\omega}}_1(i), \boldsymbol{\Omega}_{11}^*(i)) \\ & \quad = 2^{N-\eta(\mathbf{x}_1-\boldsymbol{\mu}_1)} \varphi_{N_1}(\mathbf{x}_1 - \boldsymbol{\mu}_1 \mid \boldsymbol{\Omega}_{11}(i)) \psi\left(\mathbf{K}(i) \boldsymbol{\Theta}(i) \mathbf{A}_1' \left[\mathbf{A}_1 \boldsymbol{\Theta}^2(i) \mathbf{A}_1' \right]^{-1} (\mathbf{x}_1 - \boldsymbol{\mu}_1) \right), \\ & G_{\mathbf{X}_1}(\mathbf{x}_1 \mid \boldsymbol{\mu}_1, \mathbf{0}, \overline{\boldsymbol{\omega}}_1(i), \boldsymbol{\Omega}_{11}^*(i)) = 2^N \Phi_{N+N_1}\left(\begin{bmatrix} \mathbf{0} \\ \boldsymbol{\omega}_1^{-1}(i)(\mathbf{x}_1 - \boldsymbol{\mu}_1) \end{bmatrix} \mid \boldsymbol{\Omega}_{11}^*(i) \right) \\ & \quad = 2^N \int_{\{\mathbf{z} \mid \mathbf{z} \le \mathbf{0},\ \mathbf{A}_1 \boldsymbol{\Theta}(i) \mathbf{K}(i) \mathbf{z} \le \mathbf{x}_1 - \boldsymbol{\mu}_1\}} \varphi_N(\mathbf{z} \mid \mathbf{I}_N) d\mathbf{z}. \end{aligned} \tag{3.7}$$

Appendix

Proof of Proposition 3.1

When $\omega_n = 1$, the pdf of the univariate *tpn* (2.4) can be written as

$$f_{U_n}(u_n \mid 1, \theta_n) = \left(1 + \theta_n^2\right)^{-1} h\left(-u_n \mid \theta_n^{-1}\right) + \left(1 + \theta_n^{-2}\right)^{-1} h(u_n \mid \theta_n), \tag{A.1}$$

where

$$h(z \mid \sigma) = \begin{cases} 0 & (z < 0), \\ \left(\sigma\sqrt{2\pi}\right)^{-1} & (z = 0), \\ \left(\dfrac{2}{\sigma}\right) \phi\left(\dfrac{z}{\sigma}\right) & (z > 0). \end{cases} \tag{A.2}$$

Hence, from (2.5),

$$\begin{aligned} f_{\mathbf{U}}(\mathbf{u} \mid \boldsymbol{\theta}) &= \prod_{n=1}^{N} \left[\left(1 + \theta_n^2\right)^{-1} h\left(-u_n \mid \theta_n^{-1}\right) + \left(1 + \theta_n^{-2}\right)^{-1} h(u_n \mid \theta_n) \right] \\ &= \sum_{i=1}^{2^N} \prod_{n=1}^{N} \left(1 + \theta_n^{-2k_n(i)}\right)^{-1} h\left(k_n(i) u_n \mid \theta_n^{k_n(i)}\right). \end{aligned} \tag{A.3}$$

Note that $h(k_n(i)u_n \mid \theta_n^{k_n(i)}) = 0$ whenever $k_n(i)u_n < 0$. Hence, the nonzero terms in the latter summation are those associated with N-tuples $\mathbf{k}(i)$ for which $k_n(i)u_n \ge 0$ $(n = 1, \ldots, N)$. If $u_n \neq 0$ $(n = 1, \ldots, N)$ there is only one such term. If $\mathbf{u}$ includes $\eta(\mathbf{u})$ zero elements, there are

$2^{\eta(\mathbf{u})}$ nonzero identical terms in the previous summation. In both cases, the density of $\mathbf{U}$ may be expressed as follows:

$$\begin{aligned}
f_{\mathbf{U}}(\mathbf{u} \mid \boldsymbol{\theta}) &= 2^N \prod_{n=1}^{N} \left(\theta_n + \theta_n^{-1}\right)^{-1} \varphi\left(\theta_n^{-\operatorname{sgn}(u_n)} u_n\right) \\
&= \left[\prod_{n=1}^{N}\left(\theta_n + \theta_n^{-1}\right)\right]^{-1} \prod_{n=1}^{N}\left[2\varphi(\theta_n u_n)\Phi(-\lambda\theta_n u_n) + 2\varphi\left(\theta_n^{-1} u_n\right)\Phi\left(\lambda\theta_n^{-1} u_n\right)\right] \\
&= \left[\prod_{n=1}^{N}\left(\theta_n + \theta_n^{-1}\right)\right]^{-1} \prod_{n=1}^{N}\left[\theta_n^{-1} s\left(u_n \mid 0, \theta_n^{-2}, -\lambda\right) + \theta_n s\left(u_n \mid 0, \theta_n^{2}, \lambda\right)\right] \\
&= \left[\prod_{n=1}^{N}\left(\theta_n + \theta_n^{-1}\right)\right]^{-1} \sum_{i=1}^{2^N}\left[\prod_{n=1}^{N} \theta_n^{k_n(i)} s\left(u_n \mid 0, \theta_n^{2k_n(i)}, k_n(i)\lambda\right)\right] \\
&= 2^N \left[\prod_{n=1}^{N}\left(\theta_n + \theta_n^{-1}\right)\right]^{-1} \sum_{i=1}^{2^N} \varphi_N\left(\boldsymbol{\Theta}^{-1}(i)\mathbf{u} \mid \mathbf{I}_N\right)\Phi_N\left(\lambda\mathbf{K}(i)\boldsymbol{\Theta}^{-1}(i)\mathbf{u} \mid \mathbf{I}_N\right),
\end{aligned} \tag{A.4}$$

where sgn( ) is the sign function and $s(v \mid 0, \sigma^2, \alpha)$ is the pdf of the univariate *SN* distribution with zero location parameter, scale parameter σ, and shape parameter α:

$$s\left(v \mid 0, \sigma^2, \alpha\right) = \frac{2}{\sigma}\varphi\left(\frac{v}{\sigma}\right)\Phi\left(\frac{\alpha v}{\sigma}\right). \tag{A.5}$$

From the above expression of $f_{\mathbf{U}}(\mathbf{u} \mid \boldsymbol{\theta})$, by considering the change of variable $\mathbf{X} = \boldsymbol{\mu} + \mathbf{A}\mathbf{U}$ with $\mathbf{A}$ nonsingular, one obtains the pdf of $\mathbf{X}$:

$$\begin{aligned}
f_{\mathbf{X}}(\mathbf{x} \mid \boldsymbol{\mu}, \mathbf{A}, \boldsymbol{\theta}) &= 2^N \left[\prod_{n=1}^{N}\left(\theta_n + \theta_n^{-1}\right)\right]^{-1} \sum_{i=1}^{2^N} \varphi_N(\mathbf{x} - \boldsymbol{\mu} \mid \boldsymbol{\Omega}(i)) \\
&\quad \times \Phi_N\left(\lambda\mathbf{K}(i)\boldsymbol{\Theta}^{-1}(i)\mathbf{A}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \mid \mathbf{I}_N\right)\prod_{n=1}^{N}\theta_n^{k_n(i)} \\
&= \sum_{i=1}^{2^N}\left[\prod_{n=1}^{N}\left(1 + \theta_n^{-2k_n(i)}\right)\right]^{-1} h(\mathbf{x})
\end{aligned} \tag{A.6}$$

with

$$h(\mathbf{x}) = 2^N \varphi_N(\mathbf{x} - \boldsymbol{\mu} \mid \boldsymbol{\Omega}(i))\Phi_N\left(\lambda\mathbf{K}(i)\boldsymbol{\Theta}^{-1}(i)\mathbf{A}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \mid \mathbf{I}_N\right). \tag{A.7}$$

In order to show that $h(\mathbf{x})$ is the pdf of a $SUN_{N,N}(\boldsymbol{\mu}, \mathbf{0}, \overline{\boldsymbol{\omega}}(i), \boldsymbol{\Omega}^{**}(\lambda, i))$, note that

$$\begin{aligned}
&h(\mathbf{x}) \\
&= 2^N \varphi_N(\mathbf{x}-\boldsymbol{\mu} \mid \boldsymbol{\Omega}(i))\Phi_N\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\boldsymbol{\Delta}(i)'\overline{\boldsymbol{\Omega}}^{-1}(i)\boldsymbol{\omega}^{-1}(i)(\mathbf{x}-\boldsymbol{\mu}) \mid \frac{1}{1+\lambda^2}\mathbf{I}_N\right) \\
&= \varphi_N(\mathbf{x}-\boldsymbol{\mu} \mid \boldsymbol{\Omega}(i)) \\
&\quad\times \frac{\lim_{\lambda\to+\infty}\Phi_N\left(\left(\lambda/\sqrt{1+\lambda^2}\right)\boldsymbol{\Delta}(i)'\overline{\boldsymbol{\Omega}}^{-1}(i)\boldsymbol{\omega}^{-1}(i)(\mathbf{x}-\boldsymbol{\mu}) \mid \mathbf{I}_N - \left(\lambda^2/\left(1+\lambda^2\right)\right)\boldsymbol{\Delta}(i)'\overline{\boldsymbol{\Omega}}^{-1}(i)\boldsymbol{\Delta}(i)\right)}{\Phi_N(\mathbf{0} \mid \mathbf{I}_N)} \\
&= g_{\mathbf{X}}(\mathbf{x} \mid \boldsymbol{\mu}, \mathbf{0}, \overline{\boldsymbol{\omega}}(i), \boldsymbol{\Omega}^{**}(\lambda, i)),
\end{aligned} \tag{A.8}$$

where $g_{\mathbf{X}}(\ )$ is the density of a $SUN_{N,N}(\boldsymbol{\mu}, \mathbf{0}, \overline{\boldsymbol{\omega}}(i), \boldsymbol{\Omega}^{**}(\lambda, i))$ distribution with

$$\boldsymbol{\Omega}^{**}(\lambda, i) = \begin{bmatrix} \mathbf{I}_N & \dfrac{\lambda}{\sqrt{1+\lambda^2}}\boldsymbol{\Delta}(i)' \\ \dfrac{\lambda}{\sqrt{1+\lambda^2}}\boldsymbol{\Delta}(i) & \overline{\boldsymbol{\Omega}}(i) \end{bmatrix}. \tag{A.9}$$

Thus, as $\lim_{\lambda\to+\infty}\boldsymbol{\Omega}^{**}(\lambda, i) = \boldsymbol{\Omega}^{*}(i)$,

$$g_{\mathbf{X}}(\mathbf{x} \mid \boldsymbol{\mu}, \mathbf{0}, \overline{\boldsymbol{\omega}}(i), \boldsymbol{\Omega}^{**}(\lambda, i)) = g_{\mathbf{X}}(\mathbf{x} \mid \boldsymbol{\mu}, \mathbf{0}, \overline{\boldsymbol{\omega}}(i), \boldsymbol{\Omega}^{*}(i)). \tag{A.10}$$

The simplified expression for $g_{\mathbf{X}}(\mathbf{x} \mid \boldsymbol{\mu}, \mathbf{0}, \overline{\boldsymbol{\omega}}(i), \boldsymbol{\Omega}^{*}(i))$ presented in Proposition 3.1 is obtained from (A) simply by taking into account that

$$\Phi_N\left(\lambda\mathbf{K}(i)\boldsymbol{\Theta}^{-1}(i)\mathbf{A}^{-1}(\mathbf{x}-\boldsymbol{\mu}) \mid \mathbf{I}_N\right) = 2^{-\eta(\mathbf{x}-\boldsymbol{\mu})}\varphi\left(\mathbf{K}(i)\boldsymbol{\Theta}^{-1}(i)\mathbf{A}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right). \tag{A.11}$$

As regards the cdf of $\mathbf{X}$,

$$\begin{aligned}
F_{\mathbf{X}}(\mathbf{x} \mid \boldsymbol{\mu}, \mathbf{A}, \boldsymbol{\theta}) &= \int_{z\le x} f_{\mathbf{X}}(\mathbf{z} \mid \boldsymbol{\mu}, \mathbf{A}, \boldsymbol{\theta})\mathbf{dz} \\
&= \sum_{i=1}^{2^N}\left[\prod_{n=1}^{N}\left(1+\theta_n^{-2k_n(i)}\right)\right]^{-1}\int_{z\le x} g_{\mathbf{X}}(\mathbf{z} \mid \boldsymbol{\mu}, \mathbf{0}, \overline{\boldsymbol{\omega}}(i), \boldsymbol{\Omega}^{*}(i))\mathbf{dz} \\
&= \sum_{i=1}^{2^N}\left[\prod_{n=1}^{N}\left(1+\theta_n^{-2k_n(i)}\right)\right]^{-1} G_{\mathbf{X}}(\mathbf{x} \mid \boldsymbol{\mu}, \mathbf{0}, \overline{\boldsymbol{\omega}}(i), \boldsymbol{\Omega}^{*}(i)).
\end{aligned} \tag{A.12}$$

Moreover, one gets from(2.1)

$$G_{\mathbf{X}}(\mathbf{x} \mid \boldsymbol{\mu}, \mathbf{0}, \overline{\boldsymbol{\omega}}(i), \boldsymbol{\Omega}^*(i)) = \frac{\Phi_{2N}\left(\begin{bmatrix} \mathbf{0} \\ \boldsymbol{\omega}^{-1}(i)(\mathbf{x}-\boldsymbol{\mu}) \end{bmatrix} \mid \boldsymbol{\Omega}^*(i)\right)}{\Phi_N(\mathbf{0} \mid \mathbf{I}_N)} = 2^N \Phi_{2N}\left(\begin{bmatrix} \mathbf{0} \\ \boldsymbol{\omega}^{-1}(i)(\mathbf{x}-\boldsymbol{\mu}) \end{bmatrix} \mid \boldsymbol{\Omega}^*(i)\right)$$
$$= 2^N \int_{\{\mathbf{z} \mid \mathbf{z} \leq \mathbf{0},\ \mathbf{A}\boldsymbol{\Theta}(i)\mathbf{K}(i)\mathbf{z} \leq \mathbf{x}-\boldsymbol{\mu}\}} \varphi_N(\mathbf{z} \mid \mathbf{I}_N) d\mathbf{z}. \tag{A.13}$$

The latter equality follows from the singularity of $\boldsymbol{\Omega}^*(i)$, which for given $\mathbf{x}$ allows one to write the probability of

$$\mathbf{r} \leq \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\omega}^{-1}(i)(\mathbf{x}-\boldsymbol{\mu}) \end{bmatrix}, \tag{A.14}$$

where $\mathbf{r} \sim N(\mathbf{0}, \boldsymbol{\Omega}^*(i))$, as the probability of

$$\begin{bmatrix} \mathbf{I}_N \\ \boldsymbol{\Delta}(i) \end{bmatrix} \mathbf{z} \leq \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\omega}^{-1}(i)(\mathbf{x}-\boldsymbol{\mu}) \end{bmatrix} \Longleftrightarrow \{\mathbf{z} \mid \mathbf{z} \leq \mathbf{0},\ \mathbf{A}\boldsymbol{\Theta}(i)\mathbf{K}(i)\mathbf{z} \leq \mathbf{x}-\boldsymbol{\mu}\} \tag{A.15}$$

for $\mathbf{z} \sim N(\mathbf{0}, \mathbf{I}_N)$.

References

[1] R. B. Arellano-Valle, M. D. Branco, and M. G. Genton, “A unified view on skewed distributions arising from selections,” *Canadian Journal of Statistics*, vol. 34, no. 4, pp. 581–601, 2006.

[2] A. Azzalini and A. Dalla Valle, “The multivariate skew-normal distribution,” *Biometrika*, vol. 83, no. 4, pp. 715–726, 1996.

[3] A. Azzalini and A. Capitanio, “Statistical applications of the multivariate skew normal distribution,” *Journal of the Royal Statistical Society Series B*, vol. 61, no. 3, pp. 579–602, 1999.

[4] A. Azzalini, “A class of distributions which includes the normal ones,” *Scandinavian Journal of Statistics*, vol. 12, pp. 171–178, 1985.

[5] A. Azzalini, “Further results on a class of distributions which includes the normal ones,” *Statistica*, vol. 46, pp. 199–208, 1986.

[6] J. Ferreira and M. Steel, “Bayesian multivariate regression analysis with a new class of skewed distributions, Mimeo,” University of Warwick, 2004.

[7] J. T. A. S. Ferreira and M. F. J. Steel, “A new class of skewed multivariate distributions with applications to regression analysis,” *Statistica Sinica*, vol. 17, no. 2, pp. 505–529, 2007.

[8] J. T. A. S. Ferreira and M. F. J. Steel, “Model comparison of coordinate-free multivariate skewed distributions with an application to stochastic frontiers,” *Journal of Econometrics*, vol. 137, no. 2, pp. 641–673, 2007.

[9] C. Fernández and M. F. J. Steel, “On Bayesian modeling of fat tails and skewness,” *Journal of the American Statistical Association*, vol. 93, no. 441, pp. 359–371, 1998.

[10] R. B. Arellano-Valle, H. W. Gómez, and F. A. Quintana, “Statistical inference for a general class of asymmetric distributions,” *Journal of Statistical Planning and Inference*, vol. 128, no. 2, pp. 427–443, 2005.

[11] S. John, “The three parameter two-piece normal family of distributions and its fitting,” *Communications in Statistics*, vol. 11, pp. 879–885, 1982.

[12] N. Johnson, S. Kotz, and N. Balakrishnan, *Continuous Univariate Distributions*, vol. 1, Wiley, New York, NY, USA, 2nd edition, 1994.

[13] M. Villani and R. Larsson, "The multivariate split normal distribution and asymmetric principal components analysis," *Communications in Statistics*, vol. 35, no. 6, pp. 1123–1140, 2006.

[14] R. Arellano-Valle and A. Azzalini, "On the unification of families of skew-normal distributions," *Scandinavian Journal of Statistics*, vol. 33, pp. 561–574, 2006.

[15] E. Britton, P. Fisher, and J. Whitley, "The Inflation Report projections: understanding the fan chart," *Bank of England Quarterly Bulletin*, pp. 30–37, 1998.

[16] M. Blix and P. Sellin, "Uncertainty bands for inflation forecasts," Sveriges Riksbank Working Paper 65, 1998.

[17] K. Wallis, "Asymmetric density forecasts of inflation and the Bank of England's fan chart," *National Institute Economic Review*, vol. 167, pp. 106–112, 1999.

[18] K. Wallis, "An assessment of Bank of England and National Institute inflation forecast uncertainties," *National Institute Economic Review*, vol. 189, pp. 64–71, 2004.

[19] A. S. Tay and K. F. Wallis, "Density forecasting: a survey," *Journal of Forecasting*, vol. 19, no. 4, pp. 235–254, 2000.

[20] M. Blix and P. Sellin, "Inflation forecasts with uncertainty intervals," *Sveriges Riksbank Quarterly Review*, vol. 2, pp. 12–28, 1999.

[21] M. Blix and P. Sellin, "A bivariate distribution for inflation and output forecasts," Sveriges Riksbank Working Paper 102, 2000.

[22] S. Elekdag and P. Kannan, "Incorporating market information into the construction of the fan chart," IMF Working Paper 09/178, 2009.

[23] M. Pinheiro and P. Esteves, "On the uncertainty and risks of macroeconomic forecasts: combining judgements with sample and model information," *Empirical Economics*. In press, http://www.springerlink.com/content/v785651t74h0k252/.