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Research Article

Improved Estimators of the Mean of a Normal Distribution with a Known Coefficient of Variation

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This paper is to find the estimators of the mean θ for a normal distribution with mean θ and variance $a\theta^2$, $a > 0$, $\theta > 0$. These estimators are proposed when the coefficient of variation is known. A mean square error (MSE) is a criterion to evaluate the estimators. The results show that the proposed estimators have preference for asymptotic comparisons. Moreover, the estimator based on jackknife technique has preference over others proposed estimators with some simulations studies.

1. Introduction

For the population that is distributed as normal with mean (θ) and variance (σ^2), the sample mean ($\overline{X}$) is the unbiased and minimum variance estimator. In the situation that coefficient of variation (β) is known where $\beta^2 = a = \sigma^2/\theta^2$ for $a > 0$ and $\theta > 0$, Khan [1] proposed the unbiased estimator (d^*) and the asymptotic variance of d^* is $a\theta^2/n(1+2a)$. This estimator is the linear combination between $\overline{X}$ and sample variance (S) and the asymptotic variance of estimator d^* is the Cramer-Rao bound.

Arnholt and Hebert [2] improved the estimator $\delta_k^* = kT$ where T is an unbiased estimator of θ, $k = (c\beta^2+1)^{-1}$, and constant c are known. They found that δ_k^* has smaller mean square error (MSE) than the estimator T. They also gave the example for $T = \overline{X}$ and obtained the estimator $\delta_k^* = n(\beta^2+n)^{-1}\overline{X}$ and $\text{MSE}(\delta_k^*) = a\theta^2/(a+n)$. Then, δ_k^* has MSE smaller than the estimator $\overline{X}$.

This paper focuses on improving the estimators of θ when the coefficient of variation is known. MSE is a criterion for evaluating the estimators. The estimators are proposed by using the method of Khan [1] and Arnholt and Hebert [2]. Also, the jackknife technique [3] is used to reduce the bias of estimator. Moreover, The Bayesian estimator [4] is proposed based on noninformative prior distribution by using Jeffreys' prior distribution.

The paper is organized as follows. The improved estimators are proposed in Section 2. Asymptotic comparison and simulation study results are presented in Section 3. Finally, Section 4 contains conclusions.

2. Improved Estimators

Let $X_1, X_2, \ldots, X_n$ be independent and distributed as normal with mean (θ) and variance (σ^2), and coefficient of variation (β) is known where $\beta^2 = a = \sigma^2/\theta^2$ for $a > 0$ and $\theta > 0$. There are three estimators proposed as follows.

(1) Let T_1 be the proposed estimator of θ based on Khan [1], and Arnholt and Hebert [2], $T_1 = kd^*$ where k is a constant. T_1 is a biased estimator of θ with $\text{Bias}(T_1) = \theta(k-1)$ and $\text{MSE}(T_1) = k^2 \text{Var}(d^*) + \theta^2(k-1)^2$ where the asymptotic variance of d^* is $a\theta^2/n(1+2a)$. The minimum MSE of T_1 is obtained by

$$\text{MSE}\left(T_1^*\right) = \frac{a\theta^2}{(a+n+2an)}, \tag{2.1}$$

since $k = (n+2an)/(a+n+2an)$.

(2) Let T_2 be the proposed estimator of θ based on the jackknife technique. The estimator T_1^* is used to construct the jackknife estimator T_2 as follows. Let $T_{1,-i}^*$ be an estimator T_1^* based on the sample size $n-1$ by deleting the ith sample. Denote

$$T_{1,i}^* = nT_1^* - (n-1)T_{1,-i}^*, \quad i = 1, 2, \ldots, n. \tag{2.2}$$

The estimator T_2 is given by

$$T_2 = \sum_{i=1}^{n} \frac{T_{1,i}^*}{n}. \tag{2.3}$$

The MSE of T_2 is shown in the simulation study in Section 3.

(3) Bayes estimator T_3 is obtained as follows.

The likelihood function of θ given data is

$$L(\theta \mid \text{data}) = \frac{1}{(2\pi a\theta^2)^{n/2}} \exp\left\{-\frac{1}{2a\theta^2}\sum_{i=1}^{n}(x_i-\theta)^2\right\}. \tag{2.4}$$

The log likelihood function is

$$\log L\left(\sigma^2, \rho \mid \text{data}\right) = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln a\theta^2 - \frac{1}{2a\theta^2}\sum_{i=1}^{n}(x_i-\theta)^2. \tag{2.5}$$

The Jeffreys prior distribution is

$$\pi(\theta) \propto I^{1/2}(\theta), \tag{2.6}$$

where $I(\theta)$ is Fisher's information.

Then, the prior distribution is

$$\pi(\theta) \propto \sqrt{\frac{1+2a}{a\theta^2}}. \tag{2.7}$$

The posterior distribution, the distribution of θ given data is

$$\pi(\theta \mid \text{data}) = \frac{\sqrt{(1+2a)/a\theta^2}\left(1/(2\pi a\theta^2)^{n/2}\right)\exp\left\{-(1/2a\theta^2)\sum_{i=1}^{n}(x_i-\theta)^2\right\}}{\int_0^\infty \sqrt{(1+2a)/a\theta^2}\left(1/(2\pi a\theta^2)^{n/2}\right)\exp\left\{-(1/2a\theta^2)\sum_{i=1}^{n}(x_i-\theta)^2\right\}d\theta}. \tag{2.8}$$

Therefore, The Bayes estimator of θ, T_3 is given as

$$E(\theta \mid \text{data}) = \int_0^\infty \theta\pi(\theta \mid \text{data})d\theta. \tag{2.9}$$

The MSE of T_3 is shown in the simulation study in Section 3.

3. Asymptotic Comparison and Simulation Study Results

(1) For asymptotic comparison, the estimators are compared based on the relative efficiency (RE) of MSEs. The RE of d^* with respect to T_1^* is obtained by

$$\text{RE} = \frac{\text{MSE}(d^*)}{\text{MSE}(T_1^*)} = \frac{a\theta^2/(n+2an)}{a\theta^2/(a+n+2an)} = \frac{a+n+2an}{n+2an} > 1. \tag{3.1}$$

It shows that $\text{MSE}(T_1^*)$ is smaller than $\text{MSE}(d^*)$.

The RE of δ_k^* with respect to T_1^* is obtained by

$$\text{RE} = \frac{\text{MSE}(\delta_k^*)}{\text{MSE}(T_1^*)} = \frac{a\theta^2/(a+n)}{a\theta^2/(a+n+2an)} = \frac{a+n+2an}{a+n} > 1. \tag{3.2}$$

It shows that $\text{MSE}(T_1^*)$ is smaller than $\text{MSE}(\delta_k^*)$.

Therefore, from (3.1) and (3.2), the proposed estimator T_1^* has smaller MSE than d^* and δ_k^*.

(2) The simulation results are shown for the comparison MSEs among the three proposed estimators, T_1^*, T_2, and T_3. Let parameters $\theta = 5$, 10, and 15, and $a = 0.01$, 0.09, and 0.25 with small sample size $n = 10$, 20, and 30. The results are shown in Tables 1, 2, and 3.

Table 1: MSEs of the proposed estimators T_1^*, T_2, and T_3 when $n = 10$.

θ	a	MSE (T_1^*)	MSE (T_2)	MSE (T_3)
	0.01	$1.061e-5$	$1.039e-5$	$5.362e-4$
5	0.09	$5.831e-5$	$4.921e-5$	$5.183e-4$
	0.25	$9.867e-3$	$7.116e-3$	$1.383e-2$
	0.01	$1.277e-4$	$1.244e-4$	$1.785e-3$
10	0.09	$2.016e-3$	$1.302e-3$	$2.026e-2$
	0.25	$5.127e-2$	$3.014e-2$	$3.691e-1$
	0.01	$1.446e-5$	$1.307e-5$	$2.748e-4$
15	0.09	$1.260e-2$	$1.068e-2$	$1.598e-1$
	0.25	1.31905	1.08686	3.91847

Table 2: MSEs of the proposed estimators T_1^*, T_2, and T_3 when $n = 20$.

θ	a	MSE (T_1^*)	MSE (T_2)	MSE (T_3)
	0.01	$2.061e-5$	$2.601e-5$	$5.362e-4$
5	0.09	$1.068e-4$	$5.353e-5$	$5.362e-4$
	0.25	$1.303e-2$	$8.556e-3$	$1.439e-2$
	0.01	$2.366e-5$	$2.315e-5$	$1.785e-3$
10	0.09	$1.573e-2$	$1.342e-2$	$2.003e-2$
	0.25	$2.511e-1$	$1.921e-1$	$3.459e-1$
	0.01	$1.358e-5$	$1.286e-5$	$2.748e-4$
15	0.09	$8.374e-2$	$7.339e-2$	$1.849e-1$
	0.25	$7.765e-1$	$4.851e-1$	3.91847

Table 3: MSEs of the proposed estimators T_1^*, T_2, and T_3 when $n = 30$.

θ	a	MSE (T_1^*)	MSE (T_2)	MSE (T_3)
	0.01	$3.303e-8$	$3.106e-8$	$5.362e-4$
5	0.09	$9.253e-4$	$7.898e-4$	$5.362e-4$
	0.25	$3.522e-3$	$1.705e-3$	$1.342e-2$
	0.01	$7.008e-8$	$6.171e-8$	$1.785e-3$
10	0.09	$5.700e-4$	$2.060e-4$	$2.003e-2$
	0.25	$1.649e-1$	$1.034e-1$	$3.480e-1$
	0.01	$2.259e-5$	$2.246e-5$	$2.748e-4$
15	0.09	$1.606e-1$	$1.421e-1$	$1.849e-1$
	0.25	1.40001	1.02217	3.91846

From Tables 1–3, the results show that, for small sample size n, the estimator T_2 has smaller MSEs than the estimator T_3. We also see that, the estimator T_2 has smaller MSEs than the estimator T_1^*, since T_2 is constructed by using the jackknife technique to reduce bias of the biased estimator T_1^*. Therefore, the estimator T_2 is better than the estimators T_1^* and T_3 within the intervals of θ and a.

4. Conclusions

These estimators T_1^*, T_2, and T_3 are proposed. The estimator T_1^* is improved based on the methods of Khan [1] and Arnholt and Hebert [2]. The estimator T_2 is obtained by reducing bias of T_1^*. The estimator T_3 is a Bayesian estimator for the noninformative prior distribution by using the Jeffreys prior distribution. The estimator T_1^* is better than the estimators d^* and δ_k^* in the asymptotic comparison. Moreover, the estimator T_2 is better than the estimators T_1^* and T_3 with some simulation studies.

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