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Research Article

A Simple Cocyclic Jacket Matrices

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We present a new class of cocyclic Jacket matrices over complex number field with any size. We also construct cocyclic Jacket matrices over the finite field. Such kind of matrices has close relation with unitary matrices which are a first hand tool in solving many problems in mathematical and theoretical physics. Based on the analysis of the relation between cocyclic Jacket matrices and unitary matrices, the common method for factorizing these two kinds of matrices is presented.

1. Introduction

The Walsh-Hadamard matrix is widely used for the Walsh representation of the data sequence in image coding and for Hadamard transform orthogonal code design for spread spectrum communications and quantum computation [1–4]. Their basic functions are sampled Walsh functions which can be expressed in terms of the Hadamard $[H]_N$ matrices. Using the orthogonality of Hadamard matrices, more general matrices have been developed [5]. These matrices are called as Jacket matrices and denoted by J_k. From [6], we have the following definition of Jacket matrix (http://en.wikipedia.org/wiki/Category:Matrices; http://en.wikipedia.org/wiki/user:Jacket Matrix).

Definition 1.1. If a matrix of size $m \times m$ has nonzero elements, and an inverse form which is only from the element-wise inverse and then transpose, such as

$$[J]_{m\times m} = \begin{bmatrix} j_{0,0} & j_{0,1} & \cdots & j_{0,m-1} \\ j_{1,0} & j_{1,1} & \cdots & j_{1,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ j_{m-1,0} & j_{m-1,1} & \cdots & j_{m-1,m-1} \end{bmatrix}, \tag{1.1}$$

and its inverse is

$$[J]_{m\times m}^{-1} \triangleq \frac{1}{C}\begin{bmatrix} j_{0,0}^{-1} & j_{0,1}^{-1} & \cdots & j_{0,m-1}^{-1} \\ j_{1,0}^{-1} & j_{1,1}^{-1} & \cdots & j_{1,m-1}^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ j_{m-1,0}^{-1} & j_{m-1,1}^{-1} & \cdots & j_{m-1,m-1}^{-1} \end{bmatrix}^{T}, \tag{1.2}$$

where C is the normalized value for this matrix, and T is the transpose, then this matrix is called as Jacket matrix.

Many interesting matrices, such as Hadamard, DFT, and Haar, belong to the Jacket family [6, 7]. In many applications, cocyclic matrices are very useful. The definition cocyclic matrix is as follows [8–10].

Definition 1.2. If G is a finite group of order v with operation "$\circ$," and C is a finite abelian group of order w, a "two-dimensional" cocycle is a mapping $\varphi : G \times G \rightarrow C$, satisfying

$$\varphi(g,h)\varphi(g\circ h,k) = \varphi(g,h\circ k)\varphi(h,k), \tag{1.3}$$

where $g,h,k \in G$. A square matrix M_{φ}, whose rows and columns are indexed by the elements of G, with entry $\varphi(g,h)$ in the position (g,h), that is, $M_{\varphi} = [\varphi(g,h)]$ where $g,h \in G$ and $\varphi(g,h) \in C$, can be called as a cocyclic matrix.

In [11], it is demonstrated that many well-known binary, quaternary, and q-ary codes are cocyclic Hadamard codes, that is, derived from a cocyclic generalized Hadamard matrix or its equivalents. In [9, 12, 13], Lee et al. proved that many Jacket matrices derived in [12, 14–16] are all cocyclic matrices and they are called cocyclic Jacket matrices. Hence, the Jacket matrices have many applications [9, 10, 17]. However, the derived Jacket matrices have only the sizes $N = 2^l, 2^l p$, where p is an odd prime. In this paper, we present an explicit construction of cocyclic Jacket matrices over complex field and finite field with any sizes. As a byproduct, a factorization of unitary matrices is given, which can be useful in many domains of mathematical and theoretical physics [18].

This paper is organized as follows: in Section 2, we present a class of cocyclic Jacket matrices over complex number field. The known Jacket matrices belong to this class of matrices. A class of cocyclic Jacket matrices over finite field is presented in Section 3. In Section 4, factorization of cocyclic Jacket matrices and unitary matrices is presented. Finally, conclusions are drawn in Section 5.

2. Cocyclic Jacket matrices over complex number field

In this section, we present a class of cocyclic Jacket matrices over complex number field.

2.1. Basic notations and results

Let p be an odd prime integer and $\alpha = e^{\sqrt{-1}(2\pi/p)}$. Thus, we have $\alpha^p = 1$, and $F_p = \{0, 1, 2, \ldots, p-1\}$ with the operations for $\langle a \cdot b\rangle$ are the finite field, where

$$\langle a\rangle \triangleq a \bmod p. \tag{2.1}$$

Let $a \in F_p$, we define a function

$$f_a(x) \triangleq \langle a \times x \rangle. \tag{2.2}$$

Let $\vec{V} = (\alpha^{v_0}, \alpha^{v_1}, \ldots, \alpha^{v_{p-1}})$ be a vector, where $v_i \in F_p$ for $i = 0, 1, \ldots, p-1$. We define a vector

$$\vec{V}_a = \left(\alpha^{f_a(v_0)}, \alpha^{f_a(v_1)}, \ldots, \alpha^{f_a(v_{p-1})}\right). \tag{2.3}$$

We have the following lemma.

Lemma 2.1. *Let* $\vec{V} = (1, \alpha^1, \alpha^2, \ldots, \alpha^{p-1})$, *then*

$$\begin{aligned} \vec{V}_0 \times \vec{V}_0^{\,T} &= p, \\ \vec{V}_a \times \vec{V}_b^{\,T} &= p, \quad \textit{for} \ \ \langle a + b \rangle = 0, \\ \vec{V}_a \times \vec{V}_b^{\,T} &= 0, \quad \textit{for} \ \ \langle a + b \rangle \neq 0 \ . \end{aligned} \tag{2.4}$$

Proof. The first equation can be easily proved because $\vec{V}_0 = (1, 1, \ldots, 1)$. For the second equation, since $\langle a + b \rangle = 0$, we have

$$\left(f_a(v_i) + f_b(v_i)\right) \bmod p = (a + b)v_i \bmod p = 0. \tag{2.5}$$

Thus the second equation is also true. Now we consider the last equation since p is an odd prime, we know that for any $0 < c < p$,

$$\{0, 1, 2, \ldots, p-1\} = \left\{0, \langle c \rangle, \langle 2c \rangle, \ldots, \langle (p-1)c \rangle\right\}. \tag{2.6}$$

Furthermore, for $\langle a + b \rangle \neq 0$, we have

$$\{0, 1, 2, \ldots, p-1\} = \left\{0, \langle a + b \rangle, \langle 2(a + b) \rangle, \ldots, \langle (p-1)(a + b) \rangle\right\}, \tag{2.7}$$

that is,

$$\vec{V}_a \times \vec{V}_b = \sum_{i=0}^{p-1} \alpha^i. \tag{2.8}$$

On the other hand, from $\alpha^p = 1$, we have

$$0 = \alpha^p - 1 = (\alpha - 1) \sum_{i=0}^{p-1} \alpha^i. \tag{2.9}$$

Since $\alpha - 1 \neq 0$, $\sum_{i=0}^{p-1} \alpha^i$ should be zero. Thus the last equation is also true. The proof is completed. □

Example 2.2. Let us consider $p = 5$ and $\alpha = e^{\sqrt{-1}(2\pi/5)}$. We have $\alpha^5 = 1$ and

$$F_5 = \{0, 1, 2, 3, 4\}. \tag{2.10}$$

Let $\vec{V} = \{1, \alpha, \alpha^2, \alpha^3, \alpha^4\}$, then we have

$$\begin{aligned}
\vec{V}_0 &= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \\
\vec{V}_1 &= \begin{pmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \end{pmatrix}, \\
\vec{V}_2 &= \begin{pmatrix} 1 & \alpha^2 & \alpha^4 & \alpha & \alpha^3 \end{pmatrix}, \\
\vec{V}_3 &= \begin{pmatrix} 1 & \alpha^3 & \alpha & \alpha^4 & \alpha^2 \end{pmatrix}, \\
\vec{V}_4 &= \begin{pmatrix} 1 & \alpha^4 & \alpha^3 & \alpha^2 & \alpha \end{pmatrix}.
\end{aligned} \tag{2.11}$$

It can be seen that

$$\begin{gathered}
\vec{V}_0 \times \vec{V}_0^{\,T} = 5, \\
\vec{V}_a \times \vec{V}_b^{\,T} = 5, \quad \text{for } \langle a + b \rangle = 0, \\
\vec{V}_a \times \vec{V}_b^{\,T} = 0, \quad \text{for } \langle a + b \rangle \neq 0 \ .
\end{gathered} \tag{2.12}$$

2.2. Cocyclic Jacket matrix with size p

Now we are going to construct $p \times p$ cocyclic Jacket matrix over complex number field. For a given odd prime p, let $\alpha = e^{\sqrt{-1}(2\pi/p)}$, and

$$\vec{V} = \left\{1, \alpha, \alpha^2, \ldots, \alpha^{p-2}, \alpha^{p-1}\right\}. \tag{2.13}$$

Definition 2.3. One has the following equation:

$$[J]_p \triangleq \begin{bmatrix} \vec{V}_0 \\ \vec{V}_1 \\ \vdots \\ \vec{V}_{p-1} \end{bmatrix}. \tag{2.14}$$

The inverse of $[J]_p$ is denoted by $[J]_p^{-1}$. From Lemma 2.1, it can be easily checked that if [6]

$$[J]_p^{-1} = \frac{1}{p}\left[\vec{V}_0^{\,T} \ \ \vec{V}_1^{\,T} \ \ \cdots \ \ \vec{V}_{p-1}^{\,T}\right] = \frac{1}{p}[J]_p^T, \tag{2.15}$$

then

$$[J]_p \times [J]_p^{-1} = [I]_p = [J]_p^{-1} \times [J]_p,$$
$$[J]_p \times [J]_p^T = p[I]_p. \tag{2.16}$$

According to the Definition 1.1, from (2.13)–(2.15), $[J]_p$ is a Jacket matrix over complex number of field. The following lemma shows that $[J]_p$ is acocyclic Jacket matrix [10].

Lemma 2.4. *Let $G = F_p$ with the operation $i \circ j \overset{\Delta}{=} \langle i + j \rangle$, $C = \{1, \alpha, \ldots, \alpha^{p-1}\}$ with traditional multiplication, the rows and columns are indexed by the elements of F_p under the increasing order (i.e., $0, 1, \ldots, (p-1)$), and the entry of (q, h) is $\varphi(g, h)$. Then, the Jacket matrix $[J]_p$ is a symmetric normalized cocyclic matrix.*

Proof. Let $g, h, j, i \in G = F_p$. Based on the above increasing order and from (1.3), we have

$$\varphi(g, 0) = \varphi(0, g) = \varphi(0, 0) = 1,$$
$$\varphi(g, h) = \alpha^{\langle gh \rangle},$$
$$\varphi(g, h \circ k) = \alpha^{\langle g(h+k) \rangle},$$
$$\varphi(g, h)\varphi(i, k) = \alpha^{\langle gh+ik \rangle}. \tag{2.17}$$

Therefore, for any $g, h, k \in G$, we have

$$\varphi(g, h)\varphi(g \circ h, k) = \alpha^{\langle g,h \rangle} \times \alpha^{\langle (g+h)k \rangle} = \alpha^{\langle gh+(g+h)k \rangle},$$
$$\varphi(g, h \circ k)\varphi(h, k) = \alpha^{\langle g(h+k) \rangle} \times \alpha^{\langle hk \rangle} = \alpha^{\langle g(h+k)+hk \rangle}. \tag{2.18}$$

Since

$$\langle gh + (g + h)k \rangle = \langle g(h + k) + hk \rangle, \tag{2.19}$$

we have

$$\varphi(g, h)\varphi(g \circ h, k) = \varphi(g, h \circ k)\varphi(h, k). \tag{2.20}$$

Therefore, $[J]_p$ is a cocyclic matrix. □

Hence, we have the following theorem.

Theorem 2.5. *The matrix $[J]_p$ is a cocyclic Jacket matrix with size p over complex number field.*

Table 1

$g\setminus h$	0	1	2	3	4
0	1	1	1	1	1
1	1	α	α^2	α^3	α^4
2	1	α^2	α^4	α	α^3
3	1	α^3	α	α^4	α^2
4	1	α^4	α^3	α^2	α

Example 2.6. Let us consider $p = 5$. From Example 2.2, we have

$$\begin{gathered}
\vec{V}_0 = \begin{pmatrix}1 & 1 & 1 & 1 & 1\end{pmatrix}, \\
\vec{V}_1 = \begin{pmatrix}1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4\end{pmatrix}, \\
\vec{V}_2 = \begin{pmatrix}1 & \alpha^2 & \alpha^4 & \alpha & \alpha^3\end{pmatrix}, \\
\vec{V}_3 = \begin{pmatrix}1 & \alpha^3 & \alpha & \alpha^4 & \alpha^2\end{pmatrix}, \\
\vec{V}_4 = \begin{pmatrix}1 & \alpha^4 & \alpha^3 & \alpha^2 & \alpha\end{pmatrix}, \\
[J]_5 = \begin{bmatrix}1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 1 & \alpha^2 & \alpha^4 & \alpha & \alpha^3 \\ 1 & \alpha^3 & \alpha & \alpha^4 & \alpha^2 \\ 1 & \alpha^4 & \alpha^3 & \alpha^2 & \alpha\end{bmatrix}, \\
([J]_5)^{-1} = \frac{1}{5}\begin{bmatrix}1 & 1 & 1 & 1 & 1 \\ 1 & \alpha^4 & \alpha^3 & \alpha^2 & \alpha \\ 1 & \alpha^3 & \alpha & \alpha^4 & \alpha^2 \\ 1 & \alpha^2 & \alpha^4 & \alpha & \alpha^3 \\ 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4\end{bmatrix} = \frac{1}{5}\left[j^{-1}\right]_5^T.
\end{gathered} \tag{2.21}$$

Moreover, the Jacket matrix $[J]_5$ can be mapped as shown in Table 1. It can be verified that $[J]_5$ is a cocyclic matrix.

Example 2.7. Let us consider $p = 2$, this p is not an odd prime, but it is a prime. Let $\alpha = e^{\sqrt{-1}(\pi/2)}$, we have $\alpha^2 = -1$. We have $\vec{V} = (1, \alpha^2)$ and

$$\begin{gathered}
\vec{V}_0 = (1, 1), \\
\vec{V}_1 = (1, \alpha^2) = (1, -1).
\end{gathered} \tag{2.22}$$

Thus, we have

$$[J]_2 = \begin{bmatrix} 1 & 1 \\ 1\alpha^2 & \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = [H]_2,$$
$$[J]_2^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & \alpha^2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = [H]_2^{-1} = \frac{1}{2}[J]_2^T, \tag{2.23}$$

where $[H]_2$ is Walsh-Hadamard matrix.

2.3. Cocyclic Jacket matrix with size $p_1^{e_1}p_2^{e_2}\cdots p_s^{e_s}$

First we introduce some lemmas which are useful to derive the construction of the cocyclic Jacket matrix with size $p_1^{e_1}p_2^{e_2}\cdots p_s^{e_s}$.

Lemma 2.8. *One has the following equation:*

$$(A_{i\times j} \otimes B_{h\times k})(C_{j\times s} \otimes D_{k\times t}) = (A_{i\times j} \times C_{j\times s}) \otimes (B_{h\times k} \times D_{k\times t}), \tag{2.24}$$

where ⊗ denotes the Kronecker product [1–3, 5, 6].

Lemma 2.9. *One has the following equation:*

$$(A_h \otimes B_k)^{-1} = A_h^{-1} \otimes B_k^{-1},$$
$$(A_h \otimes B_k)^T = A_h^T \otimes B_k^T. \tag{2.25}$$

Now we are going to prove the following theorem.

Theorem 2.10. *If $A_{u\times u}$ and $B_{v\times v}$ are cocyclic Jacket matrices, then $A_{u\times u} \otimes B_{v\times v}$ is also a cocyclic Jacket matrix with size uv.*

Proof. Since $A_u = [a_{i,j}]_{u\times u}$ and $B_v = [b_{s,t}]_{v\times v}$ are cocyclic Jacket matrices, according to the property of Jacket matrix, we have

$$A_{u\times u}{}^{-1} = \frac{1}{C_a}[a_{i,j}{}^{-1}]_{u\times u}{}^T,$$
$$B_{v\times v}{}^{-1} = \frac{1}{C_b}[b_{s,t}{}^{-1}]_{v\times v}{}^T. \tag{2.26}$$

Let

$$A_{u\times u} \otimes B_{v\times v} = [m_{iu+s,ju+t}]_{uv\times uv}, \tag{2.27}$$

where $m_{iu+s,ju+t} = a_{i,j}b_{s,t}$. On the other hand, from (2.25) and (2.26), we have

$$(A_{u\times u} \otimes B_{v\times v})^{-1} = \frac{1}{C_aC_b}[(a_{i,j}b_{s,t})^{-1}]_{uv\times uv}{}^T = \frac{1}{C_aC_b}[(m_{iu+s,ju+t})^{-1}]_{uv\times uv}{}^T. \tag{2.28}$$

From (2.27), (2.28), and Definition 1.1, $A_{u\times u} \otimes B_{v\times v}$ is a Jacket matrix. Next, we will prove that $A_{u\times u} \otimes B_{v\times v}$ is also a cocyclic matrix.

Table 2

(a)

$g \setminus h$	$\cdots$	$g_{cj}^{(A)}$	$\cdots$
$\vdots$	$\ddots$	$\vdots$	$\ddots$
$g_{ri}^{(A)}$	$\cdots$	$\varphi_A(g_{ri}^{(A)}, g_{cj}^{(A)})$	$\cdots$
$\vdots$	$\ddots$	$\vdots$	$\ddots$

(b)

$g \setminus h$	$\cdots$	$g_{ck}^{(B)}$	$\cdots$
$\vdots$	$\ddots$	$\vdots$	$\ddots$
$g_{rh}^{(B)}$	$\cdots$	$\varphi_B(g_{rh}^{(B)}, g_{ck}^{(B)})$	$\cdots$
$\vdots$	$\ddots$	$\vdots$	$\ddots$

(c)

$g \setminus h$	$\cdots$	$g_{cj}^{(A)} g_{ck}^{(B)}$	$\cdots$
$\vdots$	$\ddots$	$\vdots$	$\ddots$
$g_{ri}^{(A)} g_{rh}^{(B)}$	$\cdots$	$\varphi_{AB}(g_{ri}^{(A)} g_{rh}^{(B)}, g_{cj}^{(A)} g_{ck}^{(B)})$	$\cdots$
$\vdots$	$\ddots$	$\vdots$	$\ddots$

Assume that $A_{u\times u}$ and $B_{v\times v}$ are cocyclic under the following row and column index orders:

$$\begin{gathered} g_{s1}^{(A)} \prec g_{s2}^{(A)} \prec \cdots \prec g_{su}^{(A)}, \quad \text{for } g_{sj}^{(A)} \in G_A, \\ g_{s1}^{(B)} \prec g_{s2}^{(B)} \prec \cdots \prec g_{sv}^{(B)}, \quad \text{for } g_{sk}^{(B)} \in G_B, \end{gathered} \tag{2.29}$$

where $s = r$ or c, $g_{rj}^{(A)}$ and $g_{cj}^{(A)}$ denote the jth row index and jth column index of matrix A. Similarly, $g_{rk}^{(B)}$ and $g_{ck}^{(B)}$ denote the kth row index and kth column index of matrix B. Then, for matrix $A_u \otimes B_v$, the row and column index orders are defined as follows:

$$g_{sj}^{(A)} g_{sk}^{(B)} \prec g_{si}^{(A)} g_{sh}^{(B)}, \tag{2.30}$$

$$\varphi_{AB}\left(g_{ri}^{(A)} g_{rh}^{(B)}, g_{cj}^{(A)} g_{ck}^{(B)}\right) \triangleq \varphi_A\left(g_{ri}^{(A)}, g_{cj}^{(A)}\right)\varphi_B\left(g_{rh}^{(B)}, g_{ck}^{(B)}\right). \tag{2.31}$$

In order to understand (2.29), (2.30), and (2.31) better, we interpret matrices $A_{u\times u}, B_{v\times v}$, and $A_u \otimes B_v$ as the following three forms shown in Table 2. Since $A_{u\times u}$ and $B_{v\times v}$ are cocyclic matrices, thus their elements $\varphi_A(g_{ri}^{(A)}, g_{cj}^{(A)})$ and $\varphi_B(g_{rh}^{(B)}, g_{ck}^{(B)})$ should satisfy (1.3). From (2.31), and the above fact, it can be verified that $\varphi_{AB}(g_{ri}^{(A)} g_{rh}^{(B)}, g_{cj}^{(A)} g_{ck}^{(B)}) \triangleq \varphi_A(g_{ri}^{(A)}, g_{cj}^{(A)})\varphi_B(g_{rh}^{(B)}, g_{ck}^{(B)})$ is also satisfied (1.3) under the index orders (2.30). Hence, $A_{u\times u} \otimes B_{v\times v}$ is a cocyclic matrix. □

Table 3

$g\setminus h$	00	01	02	03	04	10	11	12	13	14	20	21	22	23	24
00	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
01	1	β	β^2	β^3	β^4	1	β	β^2	β^3	β^4	1	β	β^2	β^3	β^4
02	1	β^2	β^4	β	β^3	1	β^2	β^4	β	β^3	1	β^2	β^4	β	β^3
03	1	β^3	β	β^4	β^2	1	β^3	β	β^4	β^2	1	β^3	β	β^4	β^2
04	1	β^4	β^3	β^2	β	1	β^4	β^3	β^2	β	1	β^4	β^3	β^2	β
10	1	1	1	1	1	μ	μ	μ	μ	μ	μ^2	μ^2	μ^2	μ^2	μ^2
11	1	β	β^2	β^3	β^4	μ	$\beta\mu$	$\beta^2\mu$	$\beta^3\mu$	$\beta^4\mu$	μ^2	$\beta\mu^2$	$\beta^2\mu^2$	$\beta^3\mu^2$	$\beta^4\mu^2$
12	1	β^2	β^4	β	β^3	μ	$\beta^2\mu$	$\beta^4\mu$	$\beta\mu$	$\beta^3\mu$	μ^2	$\beta^2\mu^2$	$\beta^4\mu^2$	$\beta\mu^2$	$\beta^3\mu^2$
13	1	β^3	β	β^4	β^2	μ	$\beta^3\mu$	$\beta\mu$	$\beta^4\mu$	$\beta^2\mu$	μ^2	$\beta^3\mu^2$	$\beta\mu^2$	$\beta^4\mu^2$	$\beta^2\mu^2$
14	1	β^4	β^3	β^2	β	μ	$\beta^4\mu$	$\beta^3\mu$	$\beta^2\mu$	$\beta\mu$	μ^2	$\beta^4\mu^2$	$\beta^3\mu^2$	$\beta^2\mu^2$	$\beta\mu^2$
20	1	1	1	1	1	μ^2	μ^2	μ^2	μ^2	μ^2	μ	μ	μ	μ	μ
21	1	β	β^2	β^3	β^4	μ^2	$\beta\mu^2$	$\beta^2\mu^2$	$\beta^3\mu^2$	$\beta^4\mu^2$	μ	$\beta\mu$	$\beta^2\mu$	$\beta^3\mu$	$\beta^4\mu$
22	1	β^2	β^4	β	β^3	μ^2	$\beta^2\mu^2$	$\beta^4\mu^2$	$\beta\mu^2$	$\beta^3\mu^2$	μ	$\beta^2\mu$	$\beta^4\mu$	$\beta\mu$	$\beta^3\mu$
23	1	β^3	β	β^4	β^2	μ^2	$\beta^3\mu^2$	$\beta\mu^2$	$\beta^4\mu^2$	$\beta^2\mu^2$	μ	$\beta^3\mu$	$\beta\mu$	$\beta^4\mu$	$\beta^2\mu$
24	1	β^4	β^3	β^2	β	μ^2	$\beta^4\mu^2$	$\beta^3\mu^2$	$\beta^2\mu^2$	$\beta\mu^2$	μ	$\beta^4\mu$	$\beta^3\mu$	$\beta^2\mu$	$\beta\mu$

Example 2.11. Let us consider $[J]_3 \otimes [J]_5$, let $\beta = e^{\sqrt{-1}(2\pi/5)}$, and let $\mu = e^{\sqrt{-1}(2\pi/3)}$. Then we have

$$[J]_3 \otimes [J]_5 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \beta & \beta^2 & \beta^3 & \beta^4 & 1 & \beta & \beta^2 & \beta^3 & \beta^4 & 1 & \beta & \beta^2 & \beta^3 & \beta^4 \\
1 & \beta^2 & \beta^4 & \beta & \beta^3 & 1 & \beta^2 & \beta^4 & \beta & \beta^3 & 1 & \beta^2 & \beta^4 & \beta & \beta^3 \\
1 & \beta^3 & \beta & \beta^4 & \beta^2 & 1 & \beta^3 & \beta & \beta^4 & \beta^2 & 1 & \beta^3 & \beta & \beta^4 & \beta^2 \\
1 & \beta^4 & \beta^3 & \beta^2 & \beta & 1 & \beta^4 & \beta^3 & \beta^2 & \beta & 1 & \beta^4 & \beta^3 & \beta^2 & \beta \\
1 & 1 & 1 & 1 & 1 & \mu & \mu & \mu & \mu & \mu & \mu^2 & \mu^2 & \mu^2 & \mu^2 & \mu^2 \\
1 & \beta & \beta^2 & \beta^3 & \beta^4 & \mu & \beta\mu & \beta^2\mu & \beta^3\mu & \beta^4\mu & \mu^2 & \beta\mu^2 & \beta^2\mu^2 & \beta^3\mu^2 & \beta^4\mu^2 \\
1 & \beta^2 & \beta^4 & \beta & \beta^3 & \mu & \beta^2\mu & \beta^4\mu & \beta\mu & \beta^3\mu & \mu^2 & \beta^2\mu^2 & \beta^4\mu^2 & \beta\mu^2 & \beta^3\mu^2 \\
1 & \beta^3 & \beta & \beta^4 & \beta^2 & \mu & \beta^3\mu & \beta\mu & \beta^4\mu & \beta^2\mu & \mu^2 & \beta^3\mu^2 & \beta\mu^2 & \beta^4\mu^2 & \beta^2\mu^2 \\
1 & \beta^4 & \beta^3 & \beta^2 & \beta & \mu & \beta^4\mu & \beta^3\mu & \beta^2\mu & \beta\mu & \mu^2 & \beta^4\mu^2 & \beta^3\mu^2 & \beta^2\mu^2 & \beta\mu^2 \\
1 & 1 & 1 & 1 & 1 & \mu^2 & \mu^2 & \mu^2 & \mu^2 & \mu^2 & \mu & \mu & \mu & \mu & \mu \\
1 & \beta & \beta^2 & \beta^3 & \beta^4 & \mu^2 & \beta\mu^2 & \beta^2\mu^2 & \beta^3\mu^2 & \beta^4\mu^2 & \mu & \beta\mu & \beta^2\mu & \beta^3\mu & \beta^4\mu \\
1 & \beta^2 & \beta^4 & \beta & \beta^3 & \mu^2 & \beta^2\mu^2 & \beta^4\mu^2 & \beta\mu^2 & \beta^3\mu^2 & \mu & \beta^2\mu & \beta^4\mu & \beta\mu & \beta^3\mu \\
1 & \beta^3 & \beta & \beta^4 & \beta^2 & \mu^2 & \beta^3\mu^2 & \beta\mu^2 & \beta^4\mu^2 & \beta^2\mu^2 & \mu & \beta^3\mu & \beta\mu & \beta^4\mu & \beta^2\mu \\
1 & \beta^4 & \beta^3 & \beta^2 & \beta & \mu^2 & \beta^4\mu^2 & \beta^3\mu^2 & \beta^2\mu^2 & \beta\mu^2 & \mu & \beta^4\mu & \beta^3\mu & \beta^2\mu & \beta\mu
\end{bmatrix}. \tag{2.32}$$

It can be easily verified that $[J]_3 \otimes [J]_5$ is a Jacket matrix. We also present its index order matrix as shown in Table 3, where the row and column index orders are

$$00 \prec 01 \prec 02 \prec 03 \prec 04 \prec 10 \prec 11 \prec 12 \prec 13 \prec 14 \prec 20 \prec 21 \prec 22 \prec 23 \prec 24,$$
$$ij \circ hk \triangleq \langle i+h \rangle_3 \langle j+k \rangle_5. \tag{2.33}$$

For example, $23 \circ 14 = 02$. It can be easily verified that $[J]_5 \otimes [J]_3$ is a cocyclic matrix.

Next, we are going to construct a cocyclic Jacket matrix using the complex number field with size $p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$, where p_i, for $i = 1, 2, \ldots, s$, are primes.

Definition 2.12. One has the following equation:

$$[J]_{p_1^{e1}p_2^{e2}\cdots p_s^{es}} \triangleq [J]_{p_1^{e1}} \otimes [J]_{p_2^{e2}} \otimes \cdots \otimes [J]_{p_s^{es}}, \tag{2.34}$$

where

$$[J]_{p_i^{e_i}} \triangleq [J]_{p_i} \otimes [J]_{p_i} \otimes \cdots \otimes [J]_{p_i} \quad \text{for } i = 1, 2, \ldots, s. \tag{2.35}$$

From Lemma 2.8 and Theorem 2.10, we have the following theorem.

Theorem 2.13. *The matrix from Definition 2.12 is a cocyclic Jacket matrix over the complex number field.*

Example 2.14. Let us consider $p_1 = 3$, $p_2 = 2$, and $e_1 = e_2 = 1$. Thus, $N = 2 \times 3 = 6$. Let $\beta = e^{\sqrt{-1}(\pi/3)}$ and $\alpha = e^{\sqrt{-1}(2\pi/3)}$, that is, $\alpha = \beta^2$. We have

$$[J]_6 = [J]_3 \otimes [J]_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & \alpha & \alpha & \alpha^2 & \alpha^2 \\ 1 & -1 & -\alpha & \alpha & \alpha^2 & -\alpha^2 \\ 1 & 1 & \alpha^2 & \alpha^2 & \alpha & \alpha \\ 1 & -1 & \alpha^2 & -\alpha^2 & \alpha & -\alpha \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & \beta^2 & \beta^2 & \beta^4 & \beta^4 \\ 1 & -1 & \beta^2 & \beta^5 & \beta^4 & \beta \\ 1 & 1 & \beta^4 & \beta^4 & \beta^2 & \beta^2 \\ 1 & -1 & \beta^4 & \beta & \beta^2 & \beta^5 \end{bmatrix}. \tag{2.36}$$

From [19], we know that

$$[JM]_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \beta & \beta^2 & \beta^5 & \beta^4 & -1 \\ 1 & \beta^2 & \beta^4 & \beta^4 & \beta^2 & 1 \\ 1 & \beta^5 & \beta^4 & \beta & \beta^2 & -1 \\ 1 & \beta^4 & \beta^2 & \beta^2 & \beta^4 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}. \tag{2.37}$$

It can be seen that

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \times [JM]_6 \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} = [J]_6, \tag{2.38}$$

where $[JM]_6$ is the generalized Jacket matrix of order 6.

From Lemma 2.9 and the definition of $J_{p_1^{e1}p_2^{e2}\cdots p_s^{es}}$, it can be verified that $J_{p_1^{e1}p_2^{e2}\cdots p_s^{es}}$ is an orthogonal matrix and its inverse matrix can be determined as

$$J^{-1}_{p_1^{e1}p_2^{e2}\cdots p_s^{es}} = \frac{\left(J_{p_1^{e1}} \otimes J_{p_2^{e2}} \otimes \cdots \otimes J_{p_s^{es}}\right)^{-1}}{p_1^{e1}p_2^{e2}\cdots p_s^{es}} = \frac{1}{p_1^{e1}p_2^{e2}\cdots p_s^{es}}\left(J^{-1}_{p_1^{e1}} \otimes J^{-2}_{p_2^{e2}} \otimes \cdots \otimes J^{-1}_{p_s^{es}}\right), \tag{2.39}$$

where $J^{-1}_{p_i^{ei}} = \underbrace{J^{-1}_{p_i} \otimes J^{-1}_{p_i} \otimes \cdots \otimes J^{-1}_{p_i}}_{ei}$.

Table 4

$g \backslash h$	0	1	2	3
0	1	1	1	1
1	1	−1	1	−1
2	1	1	−1	−1
3	1	−1	−1	1

Example 2.15. Let us consider J_4 $[J]_4 = [J]_2 \otimes [J]_2 = [H]_2 \otimes [H]_2$. Thus, we have

$$[J]_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}. \tag{2.40}$$

The Jacket matrix J_4 can be mapped as shown in Table 4. Then J_4 is also a cocyclic matrix.

3. Cocyclic Jacket matrices over finite field

In this section, we will construct the cocyclic Jacket matrices over $GF(2^m)$. Let α be a primitive element of $GF(2^m)$. Then,

$$GF(2^m) = \{0, 1, \alpha, \alpha^2, \ldots, \alpha^{2^m-2}\}, \tag{3.1}$$

and we have the following lemma.

Lemma 3.1. *One has the following equation:*

$$\sum_{i=0}^{2^m-2} \alpha^{ri} = \begin{cases} 1, & \textit{for } r = 0, \\ 0, & \textit{for } 1 \le r \le 2^m - 2. \end{cases} \tag{3.2}$$

Proof. It is evident that $\sum_{i=0}^{2^m-2} \alpha^{ri}$ contains 2^m-1 terms, that is, odd terms. If $r = 0$, then $\sum_{i=0}^{2^m-2} \alpha^{ri}$ is a sum of odd 1's and should be 1. Thus, the first equation is proved.

We now consider the case of $1 \le r \le 2^m - 2$. Since $\alpha^{r(2^m-2)=1}$ =1, we have

$$0 = \alpha^{r(2^m-1)} + 1 = (\alpha^r + 1)\left(\sum_{i=0}^{2^m-2} a^{ri}\right). \tag{3.3}$$

Since $1 \le r \le 2^m - 2$, that is, $\alpha^r + 1 \ne 0$, we have $\sum_{i=0}^{2^m-2} a^{ri} = 0$. The proof is completed. □

Let $[JF]_{2^m-1} = [m_{i,j}]_{(2^m-1)(2^m-1)}$, where

$$m_{ij} = \alpha^{ij} \quad \text{for } 0 \le i,\ j \le 2^m - 2, \tag{3.4}$$

then, we have the following theorem.

Theorem 3.2. $[JF]_{2^m-1}$ *is a cocyclic Jacket matrix.*

Table 5: Binary representation of $GF(2^3)$.

Elements	Binary representation
0	(0 0 0)
1	(1 0 0)
α	(0 1 0)
α^2	(0 0 1)
α^3	(1 1 0)
α^4	(0 1 1)
α^5	(1 1 1)
α^6	(1 0 1)

Proof. Let

$$[JF]_{2^m-1}^{-1} = \left[m_{i,j}^{-1}\right]_{(2^m-1)\times(2^m-1)}^{T}, \tag{3.5}$$

where $m_{i,j} = \alpha^{ij}$. From the definition of $[JF]_{2^m-1}$ and Lemma 3.1, we have

$$[JF]_{2^m-1} \times [JF]_{2^m-1}^{-1} = [JF]_{2^m-1}^{-1} \times [JF]_{2^m-1} = I_{(2^m-1)}. \tag{3.6}$$

Hence, $[JF]_{2^m-1}$ is a Jacket matrix. Next, we will prove that $[JF]_{2^m-1}$ is also a cocyclic matrix. Let $\varphi(i,j)$ be the entry of row i and column j, where the order of rows and columns is from 0 to 2^m-2. From (3.4), we have

$$\begin{gathered}\varphi(i,0) = \varphi(0,i) = \varphi(0,0) = \alpha^0 = 1,\\ \varphi(i,j) = \alpha^{ij},\\ \varphi(i,j\circ h) = \alpha^{i(j+k)},\\ \varphi(i,j)\varphi(h,k) = \alpha^{ij+hk}.\end{gathered} \tag{3.7}$$

Therefore, for any $g,h,k \in Z_{2^m-1}$, we have

$$\begin{gathered}\varphi(g,h)\varphi(g\circ h,k) = \alpha^{gh} \times \alpha^{(g+h)k} = \alpha^{gh+(g+h)k},\\ \varphi(g,h\circ k)\varphi(h,k) = \alpha^{g(h+k)} \times \alpha^{hk} = \alpha^{g(h+k)+hk}.\end{gathered} \tag{3.8}$$

Since $\langle gh + (g+h)k\rangle = \langle g(h+k) + hk\rangle$, we have

$$\varphi(g,h)\varphi(g\circ h,k) = \varphi(g,h\circ k)\varphi(h,k). \tag{3.9}$$

In terms of (1.3), $[JF]_{2^m-1}$ is a cocyclic matrix. The proof is completed. □

Example 3.3. Let us consider $[JF]_7 = [JF]_{2^3-1}$. Let α and $x^3+x+1=0$ be the primitive element and primitive polynomial of $GF(2^3)$, respectively. Thus, $GF(2^3) = \{\alpha,\alpha^2,\alpha^3,\alpha^4,\alpha^5,\alpha^6\}$ and $\alpha^7 = 1$. On the other hand, any element $\beta \in GF(2^3)$ can be represented as a binary vector (b_0,b_1,b_2), where $b_i \in \{0,1\}$ for $i = 0,1,2$ such that

$$\beta = b_0 + b_1\alpha + b_2\alpha^2, \tag{3.10}$$

as shown in Table 5.

Table 6: Index mapping of order-8 Cocyclic Jacket matrix.

$g \setminus h$	0	1	2	3	4	5	6
0	1	1	1	1	1	1	1
1	1	α	α^2	α^3	α^4	α^5	α^6
2	1	α^2	α^4	α^6	α	α^3	α^5
3	1	α^3	α^6	α^2	α^5	α	α^4
4	1	α^4	α	α^5	α^2	α^6	α^3
5	1	α^5	α^3	α	α^6	α^4	α^2
6	1	α^6	α^5	α^4	α^3	α^2	α

Table 7: Binary representation of $GF(3^2)$.

Elements	Binary representation
0	(0 0)
1	(1 0)
α	(0 0)
α^2	(2 1)
α^3	(2 2)
α^4	(0 2)
α^5	(2 0)
α^6	(1 2)
α^7	(1 1)

Using Table 5, it can be easily checked that (3.9) is true for $GF(2^3)$. Thus, we have

$$[JF]_7 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 & \alpha^6 \\ 1 & \alpha^2 & \alpha^4 & \alpha^6 & \alpha & \alpha^3 & \alpha^5 \\ 1 & \alpha^3 & \alpha^6 & \alpha^2 & \alpha^5 & \alpha & \alpha^4 \\ 1 & \alpha^4 & \alpha & \alpha^5 & \alpha^2 & \alpha^6 & \alpha^3 \\ 1 & \alpha^5 & \alpha^3 & \alpha & \alpha^6 & \alpha^4 & \alpha^2 \\ 1 & \alpha^6 & \alpha^5 & \alpha^4 & \alpha^3 & \alpha^2 & \alpha \end{bmatrix},$$
$$[JF]_7^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha^6 & \alpha^5 & \alpha^4 & \alpha^3 & \alpha^2 & \alpha \\ 1 & \alpha^5 & \alpha^3 & \alpha & \alpha^6 & \alpha^4 & \alpha^2 \\ 1 & \alpha^4 & \alpha & \alpha^5 & \alpha^2 & \alpha^6 & \alpha^3 \\ 1 & \alpha^3 & \alpha^6 & \alpha^2 & \alpha^5 & \alpha & \alpha^4 \\ 1 & \alpha^2 & \alpha^4 & \alpha^6 & \alpha & \alpha^3 & \alpha^5 \\ 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 & \alpha^6 \end{bmatrix}, \tag{3.11}$$

and index mapping of order-8 Cocyclic Jacket matrix (see Table 6).

It can be verified that $[JF]_7$ is a cocyclic Jacket matrix over $GF(2^3)$.

Example 3.4. Let us consider $[JF]_{3^2-1} = [JF]_8$ over $GF(3^2)$. Let α and $x^2 + x + 2 = 0$ be the primitive and primitive polynomial of $GF(3^2)$, respectively. Thus, $GF(3^2) = \{0, 1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, \alpha^7\}$ and $\alpha^8 = 1$. Conversely, any element $\beta \in GF(3^2)$ can be represented as a vector over $GF(3) : \beta = b_0 + b_1\alpha$, where $b_0, b_1 \in \{0, 1, 2\}$ (see Table 7).

Table 8: Index mapping of order-9 Cocyclic Jacket matrix.

$g \setminus h$	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	α	α^2	α^3	α^4	α^5	α^6	α^7
2	1	α^2	α^4	α^6	1	α^2	α^4	α^6
3	1	α^3	α^6	α	α^4	α^7	α^2	α^5
4	1	α^4	1	α^4	1	α^4	1	α^4
5	1	α^5	α^2	α^7	α^4	α	α^6	α^3
6	1	α^6	α^4	α^2	1	α^6	α^4	α^2
7	1	α^7	α^6	α^5	α^4	α^3	α^2	α

Using this table, it is easy to deduce that (3.2) is true for $GF(3^2)$ (change 2^m to 3^m). Thus, we have

$$
\begin{aligned}
[JF]_8 &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 & \alpha^6 & \alpha^7 \\ 1 & \alpha^2 & \alpha^4 & \alpha^6 & 1 & \alpha^2 & \alpha^4 & \alpha^6 \\ 1 & \alpha^3 & \alpha^6 & \alpha & \alpha^4 & \alpha^7 & \alpha^2 & \alpha^5 \\ 1 & \alpha^4 & 1 & \alpha^4 & 1 & \alpha^4 & 1 & \alpha^4 \\ 1 & \alpha^5 & \alpha^2 & \alpha^7 & \alpha^4 & \alpha & \alpha^6 & \alpha^3 \\ 1 & \alpha^6 & \alpha^4 & \alpha^2 & 1 & \alpha^6 & \alpha^4 & \alpha^2 \\ 1 & \alpha^7 & \alpha^6 & \alpha^5 & \alpha^4 & \alpha^3 & \alpha^2 & \alpha \end{bmatrix}, \\
[JF]_8^{-1} &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha^7 & \alpha^6 & \alpha^5 & \alpha^4 & \alpha^3 & \alpha^2 & \alpha \\ 1 & \alpha^6 & \alpha^4 & \alpha^2 & 1 & \alpha^6 & \alpha^4 & \alpha^2 \\ 1 & \alpha^5 & \alpha^2 & \alpha^7 & \alpha^4 & \alpha & \alpha^6 & \alpha^3 \\ 1 & \alpha^4 & 1 & \alpha^4 & 1 & \alpha^4 & 1 & \alpha^4 \\ 1 & \alpha^3 & \alpha^6 & \alpha & \alpha^4 & \alpha^7 & \alpha^2 & \alpha^5 \\ 1 & \alpha^2 & \alpha^4 & \alpha^6 & 1 & \alpha^2 & \alpha^4 & \alpha^6 \\ 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 & \alpha^6 & \alpha^7 \end{bmatrix},
\end{aligned}
\tag{3.12}
$$

and index mapping of order-9 Cocyclic Jacket matrix (see Table 8).

It is easy to verify that $[JF]_8$ is a cocyclic Jacket matrix over $GF(3^2)$.

Remark 3.5. We can also construct cocyclic Jacket matrices based on additive characters of the finite field F_q and first-order q-ary Reed-Muller codes $RM_p(1,n)$ [20], where $F_q = \{\alpha_1 = 0, \alpha_2, \ldots, \alpha_q\}$ is a finite field of q elements, $q = p^m$, and p is a prime number. The way of construction is described by the following lemma.

Lemma 3.6. *The cocyclic Jacket matrix with order* $N = p^n$ *is* $J_{p^n} = [\omega^{\vec{i} \circ \vec{j}}]$, *where* $\omega = \exp(2\pi\sqrt{-1}/p)$, *and one defines*

$$
\begin{aligned}
\vec{i} \cdot \vec{j} &= (i_0, i_1, \ldots, i_{n-1}) \cdot (j_0, j_1, \ldots, j_{n-1}) \\
&= \langle i_0 \times j_0 \rangle_p + \langle i_1 \times j_1 \rangle_p + \cdots + \langle i_{n-1} \times j_{n-1} \rangle_p
\end{aligned}
\tag{3.13}
$$

for $0 \le i_k, j_k \le p-1$ $(0 \le k \le n-1)$.

Table 9: The correspondence between the indices and the entries of J_{3^2}.

$i/\vec{j}$		$j/\vec{j}$ 0 00	1 01	2 02	3 10	4 11	5 12	6 20	7 21	8 22
0	00	ω^0	ω^0	ω^0	ω^0	ω^0	ω^0	ω^0	ω^0	ω^0
1	01	ω^0	ω^1	ω^2	ω^0	ω^1	ω^2	ω^0	ω^1	ω^2
2	02	ω^0	ω^2	ω^1	ω^0	ω^2	ω^1	ω^0	ω^2	ω^1
3	10	ω^0	ω^0	ω^0	ω^1	ω^1	ω^1	ω^2	ω^2	ω^2
4	11	ω^0	ω^1	ω^2	ω^1	ω^2	ω^0	ω^2	ω^0	ω^1
5	12	ω^0	ω^2	ω^1	ω^1	ω^0	ω^2	ω^2	ω^1	ω^0
6	20	ω^0	ω^0	ω^0	ω^2	ω^2	ω^2	ω^1	ω^1	ω^1
7	21	ω^0	ω^1	ω^2	ω^2	ω^0	ω^1	ω^1	ω^2	ω^0
8	22	ω^0	ω^2	ω^1	ω^2	ω^1	ω^0	ω^1	ω^0	ω^2

Example 3.7. Let $p = 3$, $n = 2$, and $m = 1$, the finite field of $q = p^1 = 3$ elements $F_3 = \{0,1,2\}, RM_3(1,2)$ is as follows:

$$RM_3(1,2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 \\ 0 & 2 & 1 & 1 & 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 2 & 1 & 0 & 1 & 0 & 2 \end{pmatrix}. \tag{3.14}$$

The entries of J_{3^2} are shown in Table 9.

From Table 9, we can see when $i = 2$, $j = 4$, we have

$$\omega^{(02)\circ(11)} = \omega^{\langle 0\times 1\rangle_3 + \langle 2\times 1\rangle_3} = \omega^2. \tag{3.15}$$

The other entries can be obtained using the same fashion, perfectly.

4. The factorization of cocyclic Jacket matrices and unitary matrices

Definition 4.1. A square matrix U is a unitary matrix if $U^{-1} = U^H$, where U^H denote the conjugate transpose and U^{-1} is the matrix inverse.

Proposition 4.2. *The matrix $U_n = (1/\sqrt{c})J_n$ is a unitary matrix where J_n is the cocyclic Jacket matrix, c is the normalized value for J_n.*

Proof. From the definition of Jacket matrix, we have $J_n = [j]_n^{-1} = (1/c)[j^{-1}]_n^T$, and the entries in cocyclic Jacket matrices also satisfy $\|j\| = 1$, we have $j^{-1} = \bar{j}$, then $[j]_n^{-1} = (1/c)[\bar{j}]_n^T$, $[\bar{j}]_n^T = [j]_n^H = c[j]_n^{-1}$. Certainly,

$$U_n \cdot U_n^H = \frac{1}{\sqrt{c}}[j]_n \cdot \frac{1}{\sqrt{c}}[j]_n^H = \frac{1}{c}[j]_n \cdot [j]_n^H = \frac{1}{c}[j]_n \cdot c[j]_n^{-1} = I_n. \tag{4.1}$$

□

Example 4.3. Based on Example 2.14, we have

$$[J]_6 = [J]_3 \otimes [J]_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & \alpha & \alpha & \alpha^2 & \alpha^2 \\ 1 & -1 & -\alpha & \alpha & \alpha^2 & -\alpha^2 \\ 1 & 1 & \alpha^2 & \alpha^2 & \alpha & \alpha \\ 1 & -1 & \alpha^2 & -\alpha^2 & \alpha & -\alpha \end{bmatrix}, \tag{4.2}$$

where $\alpha = e^{\sqrt{-1}(2\pi/3)} = e^{i\theta}$, $i = \sqrt{1}$, $\theta = 2\pi/3$, then

$$[U]_6 \cdot [U]_6^H = \frac{1}{6}[J]_6 \cdot [J]_6^H$$
$$= \frac{1}{6}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & e^{i\theta} & e^{i\theta} & e^{i2\theta} & e^{i2\theta} \\ 1 & -1 & -e^{i\theta} & e^{i\theta} & e^{i2\theta} & -e^{i2\theta} \\ 1 & 1 & e^{i2\theta} & e^{i2\theta} & e^{i\theta} & e^{i\theta} \\ 1 & -1 & e^{i2\theta} & -e^{i2\theta} & e^{i\theta} & -e^{i\theta} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & e^{-i\theta} & -e^{-i\theta} & e^{-2i\theta} & e^{-2i\theta} \\ 1 & -1 & e^{-i\theta} & e^{-i\theta} & e^{-2i\theta} & -e^{-2i\theta} \\ 1 & 1 & e^{-2i\theta} & e^{-2i\theta} & e^{-i\theta} & e^{-i\theta} \\ 1 & -1 & e^{-2i\theta} & -e^{-2i\theta} & e^{-i\theta} & -e^{-i\theta} \end{bmatrix} = [I]_6. \tag{4.3}$$

A special feature of cocyclic Jacket matrices has been introduced in [21]. If the cocyclic Jacket matrices with order $N = p_1 p_2 \cdots p_n$, p_i is the prime number, then

$$J_N = J_{p_1} \otimes J_{p_2} \otimes \cdots \otimes J_{p_n} = A_{p_1}^1 A_{p_2}^2 \cdots A_{p_n}^n, \tag{4.4}$$

where

$$A_{p_m}^m = \underbrace{I_{p_1} \otimes I_{p_2} \otimes \cdots I_{p_{m-1}}}_{m-1} \otimes J_{p_m} \otimes \underbrace{I_{p_{m+1}} \otimes I_{p_{m+2}} \otimes \cdots \otimes I_{p_n}}_{n-m}, \tag{4.5}$$

and based on this characteristic of cocyclic Jacket matrices, we can easy decompose the unitary matrices with sparse matrices

$$U_N = \frac{1}{\sqrt{N}} J_N = \frac{1}{\sqrt{N}} A_{p_1}^1 A_{p_2}^2 \cdots A_{p_n}^n. \tag{4.6}$$

From (4.6), the U_6 can be decomposed as

$$U_6 = \frac{1}{\sqrt{6}}(I_3 \otimes J_2)(J_3 \otimes I_2) = \frac{1}{\sqrt{6}}(I_3 \otimes \sqrt{2}U_2)(\sqrt{3}U_3 \otimes I_2) = (I_3 \otimes U_2)(U_3 \otimes I_2)$$
$$= \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & e^{i\theta} & 0 & e^{i2\theta} & 0 \\ 0 & 1 & 0 & e^{i\theta} & 0 & e^{i2\theta} \\ 1 & 0 & e^{i2\theta} & 0 & e^{i\theta} & 0 \\ 0 & 1 & 0 & e^{i2\theta} & 0 & e^{i\theta} \end{bmatrix}. \tag{4.7}$$

Clearly, (4.7) is the new factorization matrix.

5. Conclusions

In this paper, we present a new class of cocyclic Jacket matrices over complex number field and finite field. Using this way, we can get such kind of matrix with order p^k directly, for the other orders $N = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$, they can be obtained from the Kronecker product with some matrices whose orders are $p_i^{k_i}$. The cocyclic Jacket matrices also have a close relation with unitary matrices. In particular, the factorizations of unitary matrices have the similar patterns with that of cocyclic Jacket matrices. Therefore, the door for using cocyclic Jacket matrices in signal processing [7], cryptography [9], mobile communication [4, 6], Jacket transform coding [13, 20], and quantum processing [17, 22] is opened.

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