Review Article

An Anisotropic Constitutive Equation for the Stress Tensor of Blood Based on Mixture Theory

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Based on ideas proposed by Massoudi and Rajagopal (M-R), we develop a model for blood using the theory of interacting continua, that is, the mixture theory. We first provide a brief review of mixture theory, and then discuss certain issues in constitutive modeling of a two-component mixture. In the present formulation, we ignore the biochemistry of blood and assume that blood is composed of red blood cells (RBCs) suspended in plasma, where the plasma behaves as a linearly viscous fluid and the RBCs are modeled as an anisotropic nonlinear density-gradient-type fluid. We obtain a constitutive relation for blood, based on the simplified constitutive relations derived for plasma and RBCs. A simple shear flow is discussed, and an exact solution is obtained for a very special case; for more general cases, it is necessary to solve the nonlinear coupled equations numerically.

1. Introduction

Amongst the multiphase (more accurately multicomponent) flows occurring in nature, one can identify blood, mud slides, avalanches, and so forth, and amongst the many flows found in the chemical industries, one can name fluidization and gasification, and in agricultural and pharmaceutical applications, one can name the transport, storage, drying of grains, and so forth. In most applications, the phases are not of the same material and thus it is more appropriate and accurate to refer to these cases as "multicomponent" problems. Historically, two distinct approaches have been used in modeling multicomponent flows (for simplicity in the remainder of the paper, we assume a two-component flow). In the *first* case, the amount of the dispersed component is so small that the motion of this component (measured by a field variable such as concentration) does not greatly affect the motion of the continuous component (the other component). This is generally known as the "dilute phase approach," sometimes also called the *Lagrangean* approach. This method is used extensively in applications such as atomization, sprays, and in flows where bubbles, droplets, and particles

are treated as the dispersed rather than a continuous component. In the *second* approach, known as the "dense phase approach," sometimes also called the *Eulerian* (or the two-fluid) formulation, the two components interact with each other to such an extent that each component directly influences the motion and the behavior of the other component. This method is used extensively in fluidization, gas-solid flows, pneumatic conveying, suspensions, polymeric solutions, and so forth, (see Kaloni et al. [1], Massoudi [2] and references therein).

The large number of articles published concerning two-component flows typically employs one of the two continuum theories developed to describe such situations: mixture theory (the theory of interacting continua) (Rajagopal and Tao [3]) or Averaging Method(s) (Ishii [4]). Both approaches are based on the underlying assumption that each component may be mathematically described as a continuum. The averaging method directly modifies the classical transport equations to account for the discontinuities or "jump" conditions at moving boundaries between the components (cf., Anderson and Jackson [5], Drew and Segal [6], Gidaspow [7], Jung et al. [8]). The modified balance equations must then be averaged in either space or time (hence the name averaging) to arrive at an acceptable local form. In this approach point-wise equations of motion, valid for a single fluid or a single particle, are modified to account for the presence of the other component and the interactions between components. These equations are then averaged over time or some suitable volume that is large compared with a characteristic dimension (e.g., particle spacing or the diameter of the particles) but small compared to the dimensions of the whole system. From the mathematical manipulation of the averaged quantities, a number of terms, some of unknown physical origin, arise. These terms are usually interpreted as some form of interaction between the constituents. Although the two methods seem similar, the way they approach the formulation of constitutive models is very different. In fact, as shown in Massoudi [2], many of the interaction models used by researchers in the averaging community are not frame-indifferent, thus violating basic principles in physics. Other differences between the two approaches are explained in Johnson et al. [9, 10], and Massoudi et al. [11, 12].

Mixture theory, or the Theory of Interacting Continua, traces its origins to the work of Fick (1855) (see Rajagopal [13]) and was first presented within the framework of continuum mechanics by Truesdell [14]. It is a means of generalizing the equations and principles of the mechanics of a single continuum to include any number of superimposed continua. In an important paper, Rajagopal [13], within the confines of mixture theory, outlines a procedure to obtain a hierarchy of approximate models, used in multicomponent situations, including those of Fick and Darcy. mixture theory is in a sense a homogenization approach in which each component is regarded as a single continuum and at each instant of time, every point in space is considered to be occupied by a particle belonging to each component of the mixture (cf., Truesdell [15]). It provides a means for studying the motions of bodies made up of several constituents by generalizing the equations and principles of the mechanics of a single continuum and in recent years it has been applied to a variety of applications such as fluid-solid particles, lubrication with binary-mixtures of bubbly oil, viscoelastic porous mixtures, swelling porous media with microstructure, reacting immiscible mixtures, polymeric solutions, growth and remodeling of soft tissues, and ionized fluid mixtures (see Massoudi [16]). More detailed information, including an account of the historical development, is available in the articles by Atkin and Craine [17, 18], Bowen [19], Bedford and Drumheller [20], and in the books by Truesdell [15], Samohyl [21], and Rajagopal and Tao [3]. mixture theory has also been used in a variety of biomechanics applications (see, e.g., Ateshian et al. [22], Garikipati et al. [23], Humphrey and Rajagopal [24], Klisch and Lotz [25], Lemon et al. [26], Tao et al. [27], Ramtani [28]).

It is known that in large vessels (whole) blood behaves as a Navier-Stokes (Newtonian) fluid (see Fung [29, Chapter 3]); however, in a vessel whose characteristic dimension (diameter, e.g.) is about the same size as the characteristic size of blood cells, blood behaves as a non-Newtonian fluid, exhibiting shear-thinning and stress relaxation. Thurston [30, 31] pointed out the viscoelastic behavior of blood while stating that the stress relaxation is more significant for cases where the shear rate is low. With respect to modeling blood, it has been observed that as the geometry of the flow changes, the rheological characteristics of blood also change and as pointed out by Humphrey and Delange [32, page 357], when blood flows in capillaries (5–8 μm in diameter), "the red blood cells go through one at a time, with plasma in between." Flow in larger vessels, in the range of 500–50 μm (see Fung [29, page 173]), exhibits a unique phenomenon known as the Fahraeus-Lindqvist effect, whereby red blood cells migrate towards the centerline, creating a thin layer of cell-depleted plasma near the vessel wall. As they indicate, in such a case, the blood should be treated as a solid and fluid mixture.

In this paper, we first provide a brief review of mixture theory, and then discuss certain issues in constitutive modeling of blood. In the present formulation, we assume blood to form a mixture consisting of RBCs suspended in plasma, while ignoring the platelets, the white blood cells (WBCs), and the proteins in the sample. No biochemical effects or interconversion of mass are considered in this model. The volume fraction (or the concentration of the RBCs) is treated as a field variable in this model. We further assume that the plasma behaves as a linearly viscous fluid and the RBCs as an anisotropic nonlinear density-gradient-type fluid. We obtain a constitutive relation for blood, based on the simplified constitutive relations derived for plasma and RBCs. It is noted that we have only discussed the development a model and that specific boundary value problems need to be solved to test the efficacy of the model. A simple shear flow is studied and an exact solution is obtained for a very special case; for more general cases, it is necessary to solve the nonlinear coupled equations numerically.

2. A brief review of (a two-component) mixture theory

In this section, we will provide a brief review of the essential ideas and equations of a two-component mixture. At each instant of time, t, it is assumed that each point in space is occupied by particles belonging to both S_1 and S_2. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ denote the positions of particles of S_1 and S_2 in the reference configuration, where (see Bowen [19], Truesdell [15])

$$\mathbf{x}_1 = \widetilde{\chi}_1(\mathbf{X}_1, t), \qquad \mathbf{x}_2 = \widetilde{\chi}_2(\mathbf{X}_2, t). \tag{2.1}$$

These motions are assumed to be one-to-one, continuous, and invertible. The kinematical quantities associated with these motions are

$$\mathbf{v}_1 = \frac{D_1 \widetilde{\chi}_1}{Dt}, \qquad \mathbf{v}_2 = \frac{D_2 \widetilde{\chi}_2}{Dt},$$

$$\mathbf{a_1} = \frac{D_1 \mathbf{v_1}}{Dt}, \qquad \mathbf{a}_2 = \frac{D_2 \mathbf{v}_2}{Dt},$$

$$\mathbf{L}_1 = \frac{\partial \mathbf{v}_1}{\partial \mathbf{x}_1}, \qquad \mathbf{L}_2 = \frac{\partial \mathbf{v}_2}{\partial \mathbf{x}_2}, \tag{2.2}$$

$$\mathbf{W}_1 = \frac{1}{2}(\mathbf{L}_1 - \mathbf{L}_1^T), \qquad \mathbf{W}_2 = \frac{1}{2}(\mathbf{L}_2 - \mathbf{L}_2^T),$$

$$\mathbf{D}_1 = \frac{1}{2}(\mathbf{L}_1 + \mathbf{L}_1^T), \qquad \mathbf{D}_2 = \frac{1}{2}(\mathbf{L}_2 + \mathbf{L}_2^T),$$

where $\mathbf{v}$ denotes the velocity vector, $\mathbf{a}$ the acceleration vector, $\mathbf{L}$ the velocity gradient, $\mathbf{D}$ denotes the symmetric part of the velocity gradient, and $\mathbf{W}$ the spin tensor. D_1/Dt denotes differentiation with respect to t, holding $\mathbf{X}_1$ fixed, and D_2/Dt denotes the same operation holding $\mathbf{X}_2$ fixed; in general for any scalar β, $D^{(\alpha)}\beta/Dt = \partial\beta/\partial t + \mathbf{v}^{(\alpha)} \cdot \operatorname{grad}\beta$, $\alpha = 1, 2$, and (for any vector $\mathbf{w}$), $D^{(\alpha)}\mathbf{w}/Dt = \partial\mathbf{w}/\partial t + (\operatorname{grad}\mathbf{w})\mathbf{v}^{(\alpha)}$. Also, ρ_1 and ρ_2 are the bulk densities of the mixture components given by

$$\rho_1 = \gamma\rho_{10}, \qquad \rho_2 = \phi\rho_{20}, \tag{2.3}$$

where $\rho_{10} = \rho_f$ is the density of the first component (e.g., a fluid) in the reference configuration, $\rho_{20} = \rho_s$ is the density of the second component (e.g., solid particles) in its reference configuration, γ is the volume fraction of the fluid component, and ϕ is the volume fraction of the solid. For a saturated mixture $\gamma = 1 - \phi$. The mixture density ρ_m is given by

$$\rho_m = \rho_1 + \rho_2, \tag{2.4}$$

and the mean velocity $\mathbf{w_m}$ of the mixture is defined by

$$\rho_m\mathbf{w_m} = \rho_1\mathbf{v_1} + \rho_2\mathbf{v_2}. \tag{2.5}$$

Once the individual stress tensors are derived (or proposed), a mixture stress tensor can be defined as

$$\mathbf{T}_m = \mathbf{T}_1 + \mathbf{T}_2, \tag{2.6}$$

where

$$\mathbf{T}_1 = (1 - \phi)\,\mathbf{T}_f, \qquad \mathbf{T}_2 = \mathbf{T}_s, \tag{2.7}$$

so that the mixture stress tensor reduces to that of a pure fluid as $\phi \rightarrow 0$ and to that of the solid particles (e.g., a densely packed granular material) as $\gamma \rightarrow 0$. $\mathbf{T}_2$ may also be written as $\mathbf{T}_2 = \phi\widehat{\mathbf{T}}_s$, where $\widehat{\mathbf{T}}_s$ may be thought of as representing the stress tensor in the reference configuration of the granular material. In the absence of any thermochemical and electromagnetic effects, the governing equations are those of the conservation of mass and linear momentum.

2.1. Conservation of mass

Assuming no interconversion of mass between the two constituents, the equations for the conservation of mass for the two components are

$$\frac{\partial\rho_1}{\partial t} + \operatorname{div}(\rho_1\mathbf{v}_1) = 0, \tag{2.8}$$

$$\frac{\partial\rho_2}{\partial t} + \operatorname{div}(\rho_2\mathbf{v}_2) = 0. \tag{2.9}$$

2.2. Conservation of linear momentum

Let $\mathbf{T}_1$ and $\mathbf{T}_2$ denote the partial stress tensors. Then, the balance of linear momentum equations for the two components is given by

$$\rho_1\frac{D_1\mathbf{v}_1}{Dt} = \operatorname{div}\mathbf{T}_1 + \rho_1\mathbf{b}_1 + \mathbf{f}_I, \tag{2.10}$$

$$\rho_2\frac{D_2\mathbf{v}_2}{Dt} = \operatorname{div}\mathbf{T}_2 + \rho_2\mathbf{b}_2 - \mathbf{f}_I, \tag{2.11}$$

where $\mathbf{b}$ represents the body force and $\mathbf{f}_I$ represents the mechanical interaction (local exchange of momentum) between the components.

2.3. Conservation of angular momentum

The balance of moment of momentum implies that

$$\mathbf{T}_1 + \mathbf{T}_2 = \mathbf{T}_1^T + \mathbf{T}_2^T. \tag{2.12}$$

The partial stresses however do not need to be symmetric. Equations (2.8)–(2.11) represent the basic governing equations for two-component flows, where the effects of electromagnetic and temperature fields are ignored. These equations have to be supplemented with constitutive relations for $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{f}_I$. This is known as the "closure" problem. In addition to the above balance equations (note that we have ignored the balance of energy and the entropy equation), we need to discuss certain constraints which might have to be considered in mixture theory.

2.4. Volume additivity constraint

Mills [33] derived the volume additivity constraint by defining an incompressible mixture as one:

$$\frac{1}{\rho_{10}}\frac{dm_1}{dV_1} + \frac{1}{\rho_{20}}\frac{dm_2}{dV_2} = 1, \tag{2.13}$$

where "m" denotes mass and "V" the volume, and where $dV_1 + dV_2 = dV$, $dm_1 + dm_2 = dm$, $\rho_{10} = dm_1/dV_1$, and $\rho_{20} = dm_2/dV_2$. Now, the densities in the current configuration, that is, in the mixture are given by $\rho_1 = dm_1/dV$, $\rho_2 = dm_2/dV$. Thus,

$$\frac{\rho_1}{\rho_{10}} + \frac{\rho_2}{\rho_{20}} = 1. \tag{2.14}$$

Other types of constraints such as incompressibility and inextensibility can also be incorporated in mixture theory.

2.5. Surface (boundary) conditions

In continuum mechanics, in addition to the balance (governing) equations, possibly subjected to some constraints, one must also specify meaningful (physical) boundary conditions, in order to have a well-posed problem. If the equations are nonlinear, the multiple solutions and stability of those solutions are very often of interest. The typical boundary conditions used in analytical/numerical procedures to solve any differential equations are (i) Dirichlet boundary conditions where the value of the unknowns is prescribed on the boundary, (ii) Neumann condition where the normal gradient of the unknowns is specified, and (iii) boundary conditions where a combination of the unknown quantities and their normal gradients is specified. The need for additional boundary conditions arises in many areas of mechanics, whenever nonlinear or microstructural theories are used. For example, in higher grade fluids in the mechanics of non-Newtonian fluids (see Rajagopal and Kaloni [34]), in polar fluids (see Atkin et al. [35]), in liquid crystals (see Leslie [36] and Ericksen [37]), or in mixture theory (see Rajagopal et al. [38]), the need is discussed and suggestions are offered.

One of the fundamental questions in mixture theory is concerned with the boundary conditions and how to split the (total) traction vector, related to the (total) stress tensor, or the (total) velocity vector (see Ramtani [39]). In an important paper, Rajagopal et al.

[38] developed a novel scheme to split the total stress for a class of problems in which the boundary of the mixture is assumed to be in a state of saturation. These problems deal with diffusion of fluids through nonlinear elastic materials, such as rubber. This additional boundary condition is obtained by prescribing the variation of the Gibbs free energy of dilution being zero. As a result of this thermodynamic restriction, a relationship between the total stress tensor, the stretch tensor, and the volume fraction of the solid component is obtained. This saturation condition is explained by Rajagopal and Tao [3, page 31] as "...a state in which a small element of the solid adjacent to the fluid is in a state in which it cannot absorb any more fluid, that is whatever fluid enters the elemental volume along the boundary has to exit through the elemental volume so that there is no accumulation of the fluid." Later, Tao and Rajagopal [3] suggested a purely mechanical method for the splitting of the traction. If $\mathbf{x}$ is a point on the boundary surface of the mixture region and $\mathbf{t}^s$ and $\mathbf{t}^f$ are the surface tractions at $\mathbf{x}$ associated with the solid and the fluid components respectively, then

$$
\begin{aligned}
\mathbf{t}^s &= (\mathbf{T}^s)^T \mathbf{n}, \\
\mathbf{t}^f &= (\mathbf{T}^f)^T \mathbf{n},
\end{aligned}
\tag{2.15}
$$

where $\mathbf{n}$ is the unit outward normal vector at $\mathbf{x}$. They assumed that the boundary surface fraction $\delta S^f / \delta S$ occupied by the fluid component at $\mathbf{x}$ is given by $\delta S^f / \delta S = \delta V^f / \delta V$, where δV^f is the volume occupied by the fluid component in δV. Then, the traction vector $\mathbf{t}^f$ at $\mathbf{x}$ is given by

$$
\mathbf{t^f} = \frac{\delta S^f}{\delta S} \mathbf{t},
\tag{2.16a}
$$

where $\mathbf{t}$ is the total applied surface traction at $\mathbf{x}$. They also showed that

$$
(\mathbf{T}^f)^T \mathbf{n} = \frac{\rho^f}{\rho_R^f} \mathbf{t},
\tag{2.16b}
$$

$$
(\mathbf{T}^s)^T \mathbf{n} = \frac{\rho^s}{\rho_R^s} \mathbf{t},
\tag{2.16c}
$$

where ρ_R^s and ρ_R^f are the mass densities of the solid component and the fluid component in their reference states, and ρ^s and ρ^f denote their mass densities in the current state. In Rajagopal and Massoudi's model (see Massoudi et al. [12] for details), it is not the tractions which are of interest in general but the velocities. In their approach, the solid particles and fluid component are assumed to both move and diffuse through each other, thereby transforming the issue of traction boundary conditions to one of velocities at the boundary. For free surface flows, on the other hand, the splitting of the traction vector remains the main difficulty (see Ravindran et al. [40]). Sometimes these additional boundary conditions can be provided from experimental data, and sometimes they can be based on other theories or physical insights.

In Rajagopal and Massoudi's approach no couple stresses are allowed. Nevertheless, due to the higher order gradients of volume fraction, they also find it necessary to provide additional boundary conditions for solving practical and simple boundary value problems (see Massoudi [41] for a discussion of boundary conditions). For most practical applications, these can sometimes be satisfied by certain symmetry conditions; in certain cases, the values

of the unknowns or their derivatives have to be specified as surface conditions at the solid walls or at the free surface.

For problems with bounded domains and with a certain degree of symmetry, such as fully developed flow in a vertical pipe (see Gudhe et al. [42]), one can prescribe at the center of the pipe the following condition:

$$\frac{d\phi}{dr} = 0, \tag{2.17a}$$

$$\frac{dv}{dr} = \frac{du}{dr} = 0, \tag{2.17b}$$

where u and v are the velocities. It is much more difficult to specify the value of the volume fraction (or concentration) at the solid surfaces. For example, based on experimental evidence, or artificially and manually gluing particles (or a thin layer) to the wall, one can prescribe a value such as

$$\phi = \phi_w \quad \text{(at the wall).} \tag{2.18a}$$

Other helpful conditions, though strictly speaking not a boundary condition, are integral conditions such as a mass flux, which provide an average value. For example, for the case of pipe flow, we can use

$$N = 2\pi \int_0^r \phi r \, dr, \tag{2.18b}$$

where the value of N is an input to the problem. In most cases, it can be assumed that the no-slip boundary condition applies to the velocity components. Therefore,

$$u = v = 0 \quad \text{at the boundaries.} \tag{2.19a}$$

However, in many situations slip may occur at the wall, and therefore the classical assumption of adherence boundary condition at the wall no longer applies. For most applications, a generalization of Navier's hypothesis can be used (see Rajagopal [43])

$$\mathbf{u} \cdot \mathbf{t} = -k(\mathbf{Tn} \cdot \mathbf{t}), \quad k > 0, \tag{2.19b}$$

where $\mathbf{u}$ is the velocity vector of the fluid, $\mathbf{T}$ the stress tensor for the fluid, $\mathbf{t}$ and $\mathbf{n}$ are the unit tangent and normal vectors at the boundary, and k, in general, can depend on the normal stress as well as the shear rate. Massoudi and Phuoc [44] proposed that for granular materials the slip velocity is proportional to the stress vector at the wall, that is, $\mathbf{u}_s = \mathbf{g}[(\mathbf{T_s n})_x, (\mathbf{T_s n})_y]$, where $\mathbf{T_s}$ is the stress tensor for the granular component, $\mathbf{n}$ is the unit normal vector, and $\mathbf{g}$ in general could be a function of surface roughness, volume fraction (density), shear rate, and so forth.

For free surface flows, in general, the location of the free surface is not known and one must find this surface as part of the solution. For steady flows, one can use the kinematical constraint

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{0}. \tag{2.20}$$

For unsteady flows, the problem of specifying the boundary conditions is more complicated. On the free surface it is known that the stress is zero, that is, one has the traction-free boundary condition. For a mixture, however, it is not clear whether the total traction vector $\mathbf{t}$ is zero, or whether the individual traction vectors $\mathbf{t}_s$ and $\mathbf{t}_f$ are zero. For a fully developed flow, where the location of the free surface, that is, the constant height of the free surface is known a priori, and based on the work of Tao and Rajagopal [45], Ravindran et al. [40] assumed that the tangential components of the individual traction vectors are zero, while the normal components are weighted according to the volume fraction. That is, they assumed that the atmospheric pressure on the top surface is split between the two components in the ratio of their respective volume fractions:

$$T_s \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \phi P_{atm} \end{pmatrix}, \tag{2.21a}$$

and for the fluid, we have

$$T_f \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ (1-\phi)P_{atm} \end{pmatrix}. \tag{2.21b}$$

In general, if one attempts to eliminate the pressure term via cross-differentiation, the order of the equations increases, and one needs additional boundary condition. This can be obtained by specifying the flow rate of the mixture (see Beevers and Craine [46]):

$$Q_{\text{volumetric}} = \int_0^1 V_m \, dY = \int_0^1 [(1-\phi)u + \phi v] \, dY. \tag{2.22a}$$

Alternatively, the flow rate of the mixture can be prescribed. The mass flow rate for a two-component mixture is defined as

$$Q_m = \int_0^1 \rho_m V_m \, dY = \int_0^1 [(1-\phi)\rho_f + \phi\rho_s][(1-\phi)u + \phi v] \, dY. \tag{2.22b}$$

When the mixture is neutrally buoyant, that is, $\rho_f = \rho_s = \rho$, these two quantities are related by $Q_m = \rho Q$.

Finally, for a complete study of a thermomechanical problem, not only in mixture theory, but in continuum mechanics in general, the second law of thermodynamics has to be considered. In other words, in addition to other principles in continuum mechanics such as material symmetry, and frame indifference, the second law imposes important restrictions on the type of motion and/or the constitutive parameters (for a thorough discussion of important concepts in constitutive equations of mechanics, we refer the reader to the books by Antman [47] Maugin [48], Coussot [49], Liu [50], Batra [51]). Since, there is no general agreement on the functional form of the constitutive relation and since the Helmholtz free energy is not known, a complete thermodynamical treatment of the model used in our studies is lacking. In recent years, Rajagopal and colleagues (see, e.g., Rajagopal and Srinivasa [52, 53]) have devised a thermodynamic framework, the multiple natural configuration theory, by appealing to the maximization of the rate of entropy production to obtain a class of constitutive relations for many different types of materials. Unlike the traditional thermodynamic approach whereby a form for the stress is assumed (or derived)

and restriction on the material parameters is obtained by invoking the Clausius-Duhem inequality, in their thermodynamic framework, they assume specific forms for the Helmholtz potential and the rate of dissipation—reflecting on how the energy is stored in the body and the way in which the body dissipates it. In the next section, we discuss a general method on how to obtain the balance equations of mixture if the balance equations of each component are given.

3. Equations of motion for the mixture

To obtain the conservation of mass for the mixture, we add (2.8) and (2.9) to obtain

$$\frac{\partial(\rho_1 + \rho_2)}{\partial t} + \text{div}(\rho_1 \mathbf{v}_1 + \rho_2 \mathbf{v}_2) = 0. \tag{3.1}$$

Now, the mass-weighted velocity of the mixture is defined as (see Bowen [19])

$$\rho_m \mathbf{w_m} = \rho_1 \mathbf{v_1} + \rho_2 \mathbf{v_2}. \tag{3.2}$$

Thus, substituting this equation and recalling that $\rho_m = \rho_1 + \rho_2$, we obtain the conservation of mass for the mixture:

$$\frac{\partial \rho_m}{\partial t} + \text{div}(\rho_m \mathbf{w}_m) = 0. \tag{3.3}$$

Similarly, to obtain the conservation of linear momentum for the mixture, we can add (2.10) and (2.11):

$$\rho_1 \frac{D_1 \mathbf{v}_1}{Dt} + \rho_2 \frac{D_2 \mathbf{v}_2}{Dt} = \text{div}(\mathbf{T}_1 + \mathbf{T}_2) + \rho_1 \mathbf{b}_1 + \rho_2 \mathbf{b}_2. \tag{3.4}$$

Defining

$$\rho_m \mathbf{b}_m = \rho_1 \mathbf{b}_1 + \rho_2 \mathbf{b}_2. \tag{3.5}$$

Equation (3.4) is rewritten as, using (2.6),

$$\rho_1 \frac{D_1 \mathbf{v}_1}{Dt} + \rho_2 \frac{D_2 \mathbf{v}_2}{Dt} = \text{div}\,\mathbf{T}_m + \rho_m \mathbf{b}_m. \tag{3.6}$$

This is the balance of linear momentum for the mixture, and unlike (3.3) which looks similar to the conservation of mass for a single continuum, this equation appears different from that of the linear momentum equation for a single component continuum, due to the two time derivatives on the left-hand side of the equation. An alternative form for the balance of linear momentum for the mixture as a whole, as prescribed in the classical mixture theory (see, e.g., Müller [54], Bowen [19]), is given by

$$\rho_m \frac{D\mathbf{w_m}}{Dt} = \text{div}\,\mathbf{T}_{\text{mix}} + \rho_m \mathbf{b}_m, \tag{3.7}$$

where now

$$\mathbf{T}_{\text{mix}} = \mathbf{T}^{(1)} + \mathbf{T}^{(2)} - \sum_{\alpha=1}^{2} \rho_\alpha \mathbf{q}^{(\alpha)} \otimes \mathbf{q}^{(\alpha)}, \tag{3.8}$$

where $\mathbf{q}^{(\alpha)}$ is the diffusion velocity vector for the αth constituent at $(\mathbf{x}, t)$ defined by

$$\mathbf{q}^{(\alpha)} = \mathbf{V}^{(\alpha)} - \mathbf{w}, \tag{3.9}$$

where

$$\mathbf{V}^{(1)} = \mathbf{u}, \tag{3.10a}$$

$$\mathbf{V}^{(2)} = \mathbf{v}. \tag{3.10b}$$

Thus,

$$\mathbf{T}_{\text{mix}} = \mathbf{T}^{(1)} + \mathbf{T}^{(2)} - [\rho_1(\mathbf{u} - \mathbf{w}) \otimes (\mathbf{u} - \mathbf{w}) + \rho_2(\mathbf{v} - \mathbf{w}) \otimes (\mathbf{v} - \mathbf{w})]. \tag{3.11}$$

It is clearly seen that with this definition for mixture stress tensor, the balance of linear momentum for the mixture, (3.7), would appear differently, since the terms included in the bracket on the right-hand side of (3.11) now would appear in (3.7). As indicated by Massoudi [16], this form is not very amenable to practical engineering problems where one is interested in mixture properties, such as viscosity. As mentioned earlier, another way of defining the mixture stress tensor is due to Green and Naghdi [55, 56], where

$$\mathbf{T_m} = \mathbf{T}^{(1)} + \mathbf{T}^{(2)}, \tag{3.12}$$

where $\mathbf{T}^{(1)}$ and $\mathbf{T}^{(2)}$ are the constituent (partial) stress tensor relations for the two components as used in the above equations. Whether we use (3.11) or (3.12), the specific forms of $\mathbf{T}^{(1)}$ and $\mathbf{T}^{(2)}$ depend upon the way in which the constitutive relations are obtained and the types of restrictions imposed on the materials properties. The advantage of presenting the balance equations for the mixture, (3.3) and (3.4), as opposed to the balance equations for each component, is that in this formulation the interaction forces cancel each other and one no longer needs to consider constitutive relations for interaction forces. The disadvantage of this approach is that one can no longer obtain detailed information about velocity, pressure, and concentration distributions.

In the next section, we will discuss the approach of Massoudi and Rajagopal, as one of the possible approaches within mixture theory, in formulating constitutive equations for the stress tensors and the interaction forces.

4. Constitutive relations

Mathematically, the purpose of the constitutive relations is to supply connections between kinematical, mechanical, electromagnetic, and thermal fields that are compatible with the balance equations and that, in conjunction with them, provide a theory that is solvable for properly posed problems. Deriving constitutive relations for the stress tensors and the interaction forces is among the outstanding issues of research in multicomponent flows. In general, the constitutive expressions for $\mathbf{T}_1$ and $\mathbf{T}_2$ depend on the kinematical quantities associated with both the constituents. However, it can be assumed that $\mathbf{T}_1$ and $\mathbf{T}_2$ depend only on the kinematical quantities associated with the particles (component 1) and fluid (component 2), respectively (sometimes called the principle of component separation, see Adkins [57, 58]).

In this part of the paper, we provide a brief description of Massoudi and Rajagopal [M-R] approach. We then modify the M-R model to obtain a constitutive relation for blood. The M-R model considers a mixture of an incompressible fluid infused with solid particles, wherein the principles of mechanics of granular materials are used to describe the behavior of particles. This model has been used and discussed in Massoudi [2, 16, 41, 59–62], Johnson et al. [9, 10], Rajagopal et al. [63–65], Massoudi et al. [12], Massoudi and Johnson [66], Massoudi and Lakshmana Rao [67], and Ravindran et al. [40].

4.1. Stress tensor for the fluid component

In most practical engineering problems studied by Massoudi and Rajagopal, the fluid can be assumed to behave as a linearly viscous fluid:

$$\mathbf{T}_1 = [-p(\rho_1) + \lambda_1(\rho_1)\operatorname{tr}\mathbf{D}_1]\mathbf{1} + 2\mu_1(\rho_1)\mathbf{D}_1, \tag{4.1}$$

where p is the fluid pressure, μ_1 is the viscosity, and $\mathbf{D}_1$ is the symmetric part of the velocity gradient of the fluid, and λ_1 is the second coefficient of viscosity. Obviously, this assumption can be modified if the fluid component is a nonlinear (non-Newtonian) fluid.

4.2. Stress tensor for the solid component

The stress tensor $\mathbf{T}_2$ in a flowing granular material may depend on the manner in which the granular material is distributed, that is, the volume fraction ϕ and possibly also its gradient, and the symmetric part of the velocity gradient tensor $\mathbf{D}_2$. Based on this observation, they assume that (see also Cowin [68])

$$\mathbf{T}_2 = \mathbf{f}(\phi, \nabla\phi, \mathbf{D}_2). \tag{4.2}$$

Using standard arguments in mechanics, restrictions can be found on the form of the above constitutive expression based on the assumption of frame-indifference, isotropy, and so forth. There could be further restrictions on the form of the constitutive expression because of internal constraints, such as, incompressibility and thermodynamics restrictions due to Clausius-Duhem inequality. A constitutive model that predicts the possibility of normal stress-differences and is properly frame invariant was given by Rajagopal and Massoudi [69]

$$\mathbf{T}_2 = [\beta_o + \beta_1 \nabla\phi \cdot \nabla\phi + \beta_2 \operatorname{tr}\mathbf{D}_2]\mathbf{1} + \beta_3\mathbf{D}_2 + \beta_4 \nabla\phi \otimes \nabla\phi + \beta_5\mathbf{D}_2^2, \tag{4.3}$$

where

$$\mathbf{D}_2 = \frac{1}{2}[\nabla\mathbf{v}_2 + (\nabla\mathbf{v}_2)^T], \tag{4.4}$$

where $\rho_2 = \rho_s\phi$, with ρ_s being constant, and the β's are material properties, which in general are functions of the appropriate principal invariants of the density gradient and the symmetric part of the velocity gradient. They assumed the following forms:

$$\beta_o = k\phi, \quad k < 0, \tag{4.5}$$

$$\beta_1 = \beta_1^*(1 + \phi + \phi^2),$$

$$\beta_2 = \beta_2^*(\phi + \phi^2),$$

$$\beta_3 = \beta_3^*(\phi + \phi^2), \tag{4.6}$$

$$\beta_4 = \beta_4^*(1 + \phi + \phi^2),$$

$$\beta_5 = \beta_5^*(1 + \phi + \phi^2).$$

The above representation can be viewed as Taylor series approximation for the material parameters (Rajagopal et al. [65]). Such a quadratic dependence, at least for the viscosity β_3, is on the basis of dynamic simulations of particle interactions (Walton and Braun [70, 71]). Further restrictions on the coefficients have been obtained by using the argument that the stress should vanish as $\phi \to 0$, therefore

$$\beta_{50} = \beta_{30} = \beta_{20}. \tag{4.7}$$

This is a principle of the limiting case, that is, if there are no particles, then ϕ and $\operatorname{grad} \phi$ are zero, and the stress should be zero. Since, the kinematical terms $\mathbf{D}_2$, $\mathbf{D}_2^2$, and $\operatorname{tr} \mathbf{D}_2$ multiplied by β_2, β_3, and β_5 do not necessarily go to zero when there are no particles, the above restrictions are necessary. Furthermore, Rajagopal and Massoudi [69] and Rajagopal et al. [63] have shown that

$$\beta_1 + \beta_4 > 0,$$
$$k < 0, \tag{4.8}$$

as compression should lead to densification of the material. They suggested the following rheological interpretation to the material parameters: $\beta_0(\phi)$ is similar to pressure in a compressible fluid and is to be given by an equation of state, $\beta_2(\phi)$ is like the second coefficient of viscosity in a compressible fluid, $\beta_1(\phi)$ and $\beta_4(\phi)$ are the material parameters connected with the distribution of the granular materials, $\beta_3(\phi)$ is the viscosity of the granular materials, and $\beta_5(\phi)$ is similar to the viscosity term in the Reiner-Rivlin model (Reiner [72, 73], Rivlin [74]), often referred to as the *cross-viscosity*. For the rheological parameters, β_2, β_3, and β_5, one can use the methods available in the mechanics of non-Newtonian fluids to find out more information about the signs. Obviously, since β_3 is related to the shear viscosity, it is assumed to be positive. The simulation studies of Walton and Braun [70, 71] suggest a structure of β_3 similar to the one assumed by Rajagopal and Massoudi. As shown in Rajagopal et al. [65] by using an orthogonal rheometer, and measuring the forces and moments exerted on the disks, one can characterize the materials moduli β's. Rajagopal et al. [75] and Baek et al. [76] discuss the details of experimental techniques using orthogonal and torsional rheometers to measure the material properties β_1 and β_4. In a certain sense, the model represented by (4.3) has the same structure as the Reiner-Rivlin model, whereby the effects of density (or volume fraction) gradients are also included. This model has been used in a variety of simple applications such as inclined flow and flow in a vertical pipe.

Looking at (4.3) with (4.5)–(4.7), it can be shown that this model is capable of predicting both normal stress differences in a simple shear flow problem (for details see Massoudi and Mehrabadi [77]). The volume fraction field $\phi(\mathbf{x}, t)$ plays a major role in this model. That is, even though we talk of distinct solid particles with a certain diameter, in this theory, the particles through the introduction of the volume fraction field have been homogenized. In other words, it is assumed that the material properties of the ensemble are continuous functions of position. That is, the material may be divided indefinitely without losing any of its defining properties (see also Collins [78]). A distributed volume $V_t = \int \phi \, dV$ and a distributed mass $M = \int \rho_s \phi \, dV$ can be defined, where the function ϕ is an independent kinematical variable called the volume distribution function and has the property

$$0 \leq \phi(x, t) < \phi_{\max} < 1. \tag{4.9}$$

The function ϕ is represented as a continuous function of position and time; in reality, ϕ in a granular system is either one or zero at any position and time, depending upon whether one is pointing to a granule or to the void space at that position. That is, the real volume distribution content has been averaged, in some sense, over the neighborhood of any given position. It should be mentioned that in practice ϕ is never equal to one; its maximum value, generally designated as the maximum packing fraction, depends on the shape, size, method of packing, and so forth. (For a review of constitutive modelling of (flowing) granular materials, see Massoudi [79]). This further complicates experimental identification of constitutive parameters, since it is not possible to study a "pure" sample of the granular phase.

4.3. Constitutive relation for the interaction forces

The interaction force, in general, depends on the fluid pressure gradient, the density gradient (the buoyancy forces), the relative velocity (the drag force on the particles), the relative acceleration (the virtual mass of the particles), the magnitude of the rate of the deformation tensor of the fluid (the lift force on the particles), the spinning motion, as well as the translation of particles (the Faxen's force), the particles tendency to move toward the region of higher velocity (the Magnus force), the history of the particle motion (the Bassett force), the temperature gradient, and so forth. For laminar flow of a mixture of an incompressible fluid infused with particles, the mechanical interaction force can be assumed to be of the form (Johnson et al. [80])

$$\mathbf{f}_I = A_1 \operatorname{grad} \phi + A_2 F(\phi)(\mathbf{v_2} - \mathbf{v_1}) + A_3 \phi (2 \operatorname{tr} \mathbf{D}_1^2)^{-1/4} \mathbf{D}_1 (\mathbf{v_2} - \mathbf{v_1}) + A_4 \phi (\mathbf{W}_2 - \mathbf{W}_1)(\mathbf{v_2} - \mathbf{v_1}) + A_5 \mathbf{a}_{vm},$$
(4.10)

where $\mathbf{W}_2$ and $\mathbf{W}_1$ are the spin tensors for the two components. The terms on the right-hand side of this equation reflect the presence of (nonuniform) concentration distribution (diffusion), drag, slip-shear lift, spin lift, and virtual mass. If the flow is assumed to be steady, the virtual mass effects can be neglected. Most of these coefficients have not been measured or extensively studied for general two-component flows. Johnson et al. [80], based on previous experimental observations and certain analytical approximations, suggested the following coefficients:

$$A_2 = \frac{9}{2} \frac{\mu_f}{a^2}; \qquad A_3 = \frac{3(6.46)}{4\pi} \frac{\rho_f^{1/2} \mu_f^{1/2}}{a}, \qquad A_4 = \frac{3}{4} \rho_f, \qquad A_5 = 2 \frac{\pi}{3} a^3 \phi \frac{1 + 2\phi}{1 - \phi}.$$
(4.11)

Most of these coefficients are given either as a generalization of single particle result or as a result of some other limiting conditions. If the particles are nonspherical, such as fibers, or a blood cell then the directionality may become an important element, and in that case we need to use (or develop) the constitutive relations which take the microstructure and directionality, that is, anisotropic nature of the material, into account (cf., Ericksen [37, 81], Leslie [82]). There are obvious "gray" areas that await further studies for clarification. For example, there have been no investigations as to what form A_1 may have. The remaining coefficients have not been extensively studied for general two-component flows; thus, the forms given above are ad hoc applications of results that are strictly valid under more restricted conditions. Despite the assumptions involved, these expressions effectively describe how the interaction forces vary with the system parameters. The coefficients can be considered functions of ϕ only

(because ϕ varies with position in the flow) for the purpose of performing numerical studies. The way in which Massoudi and Rajagopal use (4.10) along with (4.11) is that since the governing equations are made dimensionless, the material parameters such as viscosity and density along with the numerical coefficients appearing in these correlations are absorbed in the dimensionless numbers. Therefore, the actual numerical values of these experimentally obtained coefficients do not enter into their calculations directly; instead they perform a parametric study for a range of these dimensionless numbers. Massoudi [2, 62] has recently reviewed the subject of interaction mechanisms in multicomponent flows. In the next section, we discuss a possible extension of the fluid-solid mixture theory formulation of Massoudi and Rajagopal to blood.

5. Constitutive relations for plasma and red blood cells

The physical and biological processes governing the flow of blood are intimately responsible for safety and efficacy of all blood-wetted medical devices. The quest to design improved cardiovascular devices is however stifled by the inadequacies of current understanding of blood trauma and thrombosis. Contemporary design relies upon formula for blood describing (1) rheology, (2) cell trauma (hemolysis), and (3) thrombosis, that are based primarily on empiricism. These relations are application-specific at best, and are more *descriptive* than *predictive.* Furthermore, we now understand that these three phenomena are more closely coupled than that previously appreciated. Unfortunately, the deficiencies of accurate predictive mathematical models have prevented engineering rigor to supplant the common practices which rely on intuition and experimental trial-and-error. Chief on the list of biological phenomena that are difficult to predict a priori is thrombosis and hemolysis, collectively referred to as *blood trauma.* It is becoming evident that red cells may have a significant influence on hemostasis and thrombosis; the nature of the effect is related to the flow conditions.

One of the most important rheological consequences of the multicomponent nature of blood in small vessels is the migration of red cells towards the core of the flow and commensurate enhancement of platelet concentration near the boundaries. Several investigators attempting to simulate the deposition of platelets on biological and artificial surfaces have found that single-continuum models significantly under-predict the concentration of platelets near the boundary (see Sorenson et al. [83, 84], Jordan et al. [85], Anand et al. [86], and Goodman et al. [87]). This error was attributed to the absence of RBC-induced platelet margination which enhances the lateral diffusion of platelets towards the wall. The conclusions of Sorensen et al. and Jordan et al. strongly suggest the existence of an anisotropic (directionally-dependent) diffusivity of platelets.

The mechanism for the above phenomenon, whereby platelets in flowing blood are propelled towards the surface, can be understood in terms of their interaction with red blood cells. Extensive experimental studies dating to the late 1920s (see Fahraeus and Lindqvist [88, 89]) have demonstrated the phenomenon of lateral migration of RBCs, resulting in an excess layer of plasma (and platelets) near the wall. The pioneering work performed by Chien et al. [90–92], Sutera et al. [93], Blackshear et al. [94, 95], Goldsmith [96–98], and Hochmuth et al. [99] should be acknowledged. These investigators provided the earliest insights to understand the microstructural basis for blood rheology. Most notable is the microscopic "freeze-capture" experiments by Goldsmith et al. to visualize the margination of red cells in small (65 to 200 mm dia.) cylindrical glass tubes. The associated phenomenon of near-wall excess of platelets was subsequently studied by Eckstein et al. [100, 101] and Aarts et al. [102],

who empirically revealed the importance of several independent parameters, including shear rate, hematocrit, the size and concentration of platelets (or platelet-sized particles), and vessel geometry.

In general, blood can be viewed as a suspension and modeled using the techniques of non-Newtonian fluid mechanics. We, however, assume that blood is a two-component mixture, composed of the red blood cells (RBCs) suspended in a (platelet rich) plasma. In the following description, the plasma in the mixture will be represented by S_1 and the RBCs by S_2. We also assume that ϕ represents the concentration of the RBCs, also commonly referred to as *hematocrit*. Now, ρ_1 and ρ_2 are the bulk densities of the mixture components given by $\rho_1 = \gamma \rho_P$, $\rho_2 = \phi \rho_{RBC}$, where ρ_P is the density of the plasma, ρ_{RBC} is the density of the RBCs, ϕ is the volume fraction of the RBCs, and γ is the volume fraction of the plasma. Once the individual stress tensors are derived (or proposed), a mixture stress tensor can be defined as $\mathbf{T}_m = \mathbf{T}_1 + \mathbf{T}_2$, where $\mathbf{T}_1 = (1 - \phi)\,\mathbf{T}_P$ and $\mathbf{T}_2 = \mathbf{T}_R$, so that the mixture stress tensor reduces to that of plasma as $\phi \to 0$ and to that of a RBCs as $\phi \to 1$. Note that $\mathbf{T}_2$ may also be written as $\mathbf{T}_2 = \phi \widehat{\mathbf{T}}_R$, where $\widehat{\mathbf{T}}_R$ may be thought of as representing the stress tensor in a reference configuration.

We assume that the mixture is saturated, that is, $\gamma = 1 - \phi$ and that the plasma can be represented by the fluid component of M-R model, where

$$p(\rho_1) = p(1 - \phi),$$
$$\lambda_1(\rho_1) = \lambda(1 - \phi), \tag{5.1}$$
$$\mu_1(\rho_1) = \mu(1 - \phi).$$

And thus (4.1) becomes

$$\mathbf{T}_1 = [-p(1 - \phi) + \lambda(1 - \phi)\operatorname{tr}\mathbf{D}_1]\mathbf{1} + 2\mu(1 - \phi)\mathbf{D}_1, \tag{5.2}$$

where p is the pressure, μ is the viscosity, and $\mathbf{D}_1$ is the symmetric part of the velocity gradient of the plasma, and λ is the second coefficient of viscosity.

Modelling the RBCs is practically more complicated since they are anisotropic and deformable. The main reason for this difficulty is the orientation or the alignment of these nonspherical cells. Materials possessing microstructures, for example, with the internal couples or couple stresses, were first studied in the early twentieth century by D. Cosserat and F. Cosserat (Truesdell and Toupin [103]). To study this effect, there are at least two distinct yet related methods based on continuum mechanics. The first method is to use an orientation distribution function, whereby one derives orientation tensors to characterize the behavior of the particles, in this case, RBCs. The general idea of using orientation tensors to account, in an averaged sense, for the distribution of fibers in a fluid, in a general situation, was suggested by Hand [104, 105]. The details of these techniques are given in Advani and Tucker [106], Advani [107], and Petrie [108]. The second method is to use the continuum mechanics theories whereby the microstructure is in some sense incorporated into the theory, for example, as is done in the micropolar or director theories (Truesdell and Noll [109]). A very powerful use of this method is the theory of liquid crystals developed by Ericksen and later generalized by Hand, Leslie, and others. In this approach, a unit vector $\mathbf{n}$ is introduced as one of the independent constitutive variables, and as a result the stress tensor would depend on $\mathbf{n}$ and its derivatives, as well as other important constitutive parameters such as velocity, velocity gradient, temperature, and so forth, in an appropriate frame-invariant form.

To model the stress tensor for the RBCs, we modify the constitutive relation derived by Massoudi [110] for an anisotropic granular media, where the granules were assumed to have a principal direction, denoted with a unit normal vector $\mathbf{n}$. A general representation for the stress tensor was given. Similar constitutive relations have been obtained, for example, by Rajagopal and Wineman [64] and Rajagopal and Ružička [111] within the context of continuum mechanics of electrorheological materials:

$$
\begin{aligned}
\mathbf{T_2} = {} & a_1\mathbf{1} + a_2\mathbf{m}\otimes\mathbf{m} + a_3\mathbf{n}\otimes\mathbf{n} + a_4(\mathbf{m}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{m}) + a_5\mathbf{D_2} + a_6\mathbf{D_2^2} \\
& + a_7(\mathbf{m}\otimes\mathbf{D_2 m} + \mathbf{D_2 m}\otimes\mathbf{m}) + a_8(\mathbf{m}\otimes\mathbf{D_2^2 m} + \mathbf{D_2^2 m}\otimes\mathbf{m}) \\
& + a_9(\mathbf{n}\otimes\mathbf{D_2 n} + \mathbf{D_2 n}\otimes\mathbf{n}) + a_{10}(\mathbf{n}\otimes\mathbf{D_2^2 n} + \mathbf{D_2^2 n}\otimes\mathbf{n}) \\
& + a_{11}[(\mathbf{m}\otimes\mathbf{D_2 n} + \mathbf{D_2 n}\otimes\mathbf{m}) - (\mathbf{n}\otimes\mathbf{D_2 m} + \mathbf{D_2 m}\otimes\mathbf{n})],
\end{aligned}
\tag{5.3}
$$

or

$$
\begin{aligned}
T_{ij} = {} & a_1\delta_{ij} + a_2 m_i m_j + a_3 n_i n_j + a_4(m_i n_j + n_i m_j) + a_5 D_{ij} + a_6 D_{ij}^2 \\
& + a_7(m_i D_{jk} m_k + D_{ik} m_k m_j) + a_8(m_i D_{jk}^2 m_k + D_{ik}^2 m_k m_j) \\
& + a_9(n_i D_{jk} n_k + D_{ik} n_k n_j) + a_{10}(n_i D_{jk}^2 n_k + D_{ik}^2 n_k n_j) \\
& + a_{11}[(m_i D_{jk} n_k + D_{ik} n_k m_j) - (n_i D_{jk} m_k + D_{ik} m_k n_j)],
\end{aligned}
\tag{5.4}
$$

where a_1–a_{11} are scalar functions of the set of invariants (see Spencer [112], Wang [113–115], Zheng [116]):

$$
\begin{aligned}
& I_1 = \operatorname{tr}\mathbf{D_2}, \qquad I_2 = \operatorname{tr}\mathbf{D_2}^2, \qquad I_3 = \operatorname{tr}\mathbf{D_2}^3, \\
& I_4 = \operatorname{tr}[\mathbf{m}\otimes\mathbf{m}], \qquad I_5 = \operatorname{tr}[\mathbf{n}\otimes\mathbf{n}], \qquad I_6 = \operatorname{tr}[\mathbf{m}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{m}], \\
& I_7 = \operatorname{tr}[\mathbf{m}\otimes\mathbf{D_2\,m}], \qquad I_8 = \operatorname{tr}[\mathbf{m}\otimes\mathbf{D_{22}}^2\mathbf{m}], \qquad I_9 = \operatorname{tr}[\mathbf{n}\otimes\mathbf{D_2\,n}], \\
& I_{10} = \operatorname{tr}[n\otimes\mathbf{D_2}^2\mathbf{n}], \qquad I_{11} = \operatorname{tr}[\mathbf{m}\otimes\mathbf{D_2\,n}], \qquad I_{12} = \operatorname{tr}[\mathbf{m}\otimes\mathbf{D_2}^2\mathbf{n}],
\end{aligned}
\tag{5.5}
$$

where $\mathbf{D_2} = (1/2)[\nabla\mathbf{v_2} + (\nabla\mathbf{v_2})^T]$, and

$$
\mathbf{M} = \mathbf{m}\otimes\mathbf{m} = \operatorname{grad}\rho\otimes\operatorname{grad}\rho,
\tag{5.6a}
$$

$$
M_{ij} = \rho_{,i}\rho_{,j},
\tag{5.6b}
$$

$$
\mathbf{N} = \mathbf{n}\otimes\mathbf{n},
\tag{5.7a}
$$

$$
N_{ij} = n_i n_j.
\tag{5.7b}
$$

This general equation not only depends on $\mathbf{D_2}$ (the symmetric part of the velocity gradient) and its higher order powers, but also on the density gradient and the RBCs orientation vector $\mathbf{n}$ (a measure of their anisotropy). There are at least 11 material coefficients in (5.3) which in some ways have to be specified before a meaningful study can be carried out. Again, for certain cases, without performing any stability (see Rajagopal et al. [63]) or thermodynamic analysis, we can gain some information about the sign of these parameters. Some of the

rheological properties can be measured, for example, using orthogonal rheometers (see Rajagopal et al. [75]). If, for simplicity, we assume

$$a_3 = a_4 = a_7 = a_8 = a_9 = a_{10} = a_{11} = 0 \tag{5.8}$$

corresponding to, for example, a dispersed component such as spherical particles, where there is a degree of symmetry and the anisotropy of the material does not play a role. However, density (or volume fraction) gradient is still important. For such a medium, (5.3) becomes

$$\mathbf{T} = b_1\mathbf{1} + b_2\mathbf{m} \otimes \mathbf{m} + b_3\mathbf{D_2} + b_4\mathbf{D_2}^2, \tag{5.9}$$

where b_1–b_4 are now scalar functions of the appropriate invariants. Let us furthermore assume

$$\begin{aligned} b_1 &= b_1(\rho, \operatorname{tr} \mathbf{D_2}, \operatorname{tr}(\mathbf{m} \otimes \mathbf{m})), \\ b_2 &= b_2(\rho), \qquad b_3 = b_3(\rho), \qquad b_4 = b_4(\rho). \end{aligned} \tag{5.10}$$

Now, if b_1 is given by

$$b_1 = \beta_0(\rho) + \beta_1 \operatorname{grad} \rho \cdot \operatorname{grad} \rho + \beta_2(\rho)\operatorname{tr} \mathbf{D_2}, \tag{5.11}$$

then (5.9) can be written as

$$\mathbf{T} = [\beta_0(\rho) + \beta_1(\rho)\operatorname{grad} \rho \cdot \operatorname{grad} \rho + \beta_2(\rho)\operatorname{tr} \mathbf{D_2}]\mathbf{1} + b_2 \operatorname{grad} \rho \otimes \operatorname{grad} \rho + b_3\mathbf{D_2} + b_4\mathbf{D_2}^2. \tag{5.12}$$

This equation was derived by Rajagopal and Massoudi [69]. A special case of this model, with $b_4 = 0$, has been used extensively by Massoudi and Rajagopal in a variety of applications (Massoudi et al. [12]). If we furthermore, for the sake of simplicity, assume that the effects of the volume fraction gradient are negligible, implying that

$$\beta_1 = b_2 = 0, \tag{5.13}$$

then the stress tensor for the RBCs now has the structure

$$\mathbf{T_2} = [\beta_o(\phi) + \beta_2(\phi)\operatorname{tr} \mathbf{D_2}]\mathbf{1} + \beta_3(\phi)\mathbf{D_2} + \beta_5(\phi)\mathbf{D_2^2}, \tag{5.14}$$

where (based on suggestions made by Massoudi and Rajagopal) the following additional assumptions are made:

$$\begin{aligned} \beta_0 &= -p\phi, \\ \beta_2(\phi) &= \beta_{20}(\phi + \phi^2), \\ \beta_3(\phi) &= \beta_{30}(\phi + \phi^2), \\ \beta_5(\phi) &= \beta_{50}(\phi + \phi^2). \end{aligned} \tag{5.15}$$

The first equation, namely, $\beta_0 = -p\phi$, is different from that suggested by Massoudi and Rajagopal. This equation, in conjunction with (5.1)a, that is, $p(\rho_1) = p(1 - \phi)$, implies that the total pressure is weighted (distributed) among the two components according to the

volume fraction. This is an accepted assumption in many two-component theories, and unless there are clear experimental observations pointing otherwise, this seems to be a reasonable assumption.

To account for the shear-thinning effects observed in blood flow, it is reasonable to assume that the viscosity coefficient β_3 should also depend on Π; that is

$$\beta_3(\phi, \operatorname{tr} \mathbf{D_2}) = \beta_{30}(\phi + \phi^2)\Pi^{m/2}, \tag{5.16}$$

where

$$\Pi = \frac{1}{2}\operatorname{tr}[2\mathbf{D_2}]^2, \tag{5.17}$$

when $m < 0$, the material is shear-thinning, and if $m > 0$, it is shear-thickening (see also Pontrelli [117]). When $m = 0$, we recover the M-R model. Alternatively, the shear-thinning effects of blood can also be included in the viscosity term in RBC by assuming (see Yeleswarapu [118])

$$\beta_{30} = \mu_\infty + (\mu_0 - \mu_\infty)\frac{1 + \ln(1 + \kappa\dot{\gamma})}{1 + \kappa\dot{\gamma}}, \tag{5.18}$$

where μ_∞ is a term related to asymptotic viscosity under infinity shear rate, μ_0 is a term related to viscosity at zero shear rate, $\dot{\gamma}$ is the shear rate, and κ is a materials parameter describing the shear thinning profile. Clearly, other types of viscoelastic fluid models such as the modified Maxwell or the Oldroyd type fluid models can also be used here. Substituting (5.15) and (5.16) into (5.14), we obtain, the stress tensor equation for the RBCs:

$$\mathbf{T_2} = [-p\phi + \beta_{20}(\phi + \phi^2)\operatorname{tr}\mathbf{D_2}]\mathbf{1} + \beta_{30}(\phi + \phi^2)\Pi^{m/2}\mathbf{D_2} + \beta_{50}(\phi + \phi^2)\mathbf{D_2^2}. \tag{5.19}$$

If for simplicity, we define

$$\begin{aligned}
\lambda_2 &= \beta_{20}(\phi + \phi^2), \\
\mu_2 &= \beta_{30}(\phi + \phi^2)\Pi^{m/2}, \\
\delta_2 &= \beta_{50}(\phi + \phi^2).
\end{aligned} \tag{5.20}$$

Then,

$$\mathbf{T_2} = [-p\phi + \lambda_2\operatorname{tr}\mathbf{D_2}]\mathbf{1} + \mu_2\mathbf{D_2} + \delta_2\mathbf{D_2^2}, \tag{5.21}$$

which interestingly has the same structure as a Reiner-Rivlin fluid where the material coefficients are given by (5.20).

Finally, it needs to be mentioned that the RBCs themselves actually form a mixture made of a central fluid-like region bounded by a viscoelastic solid. Many researchers have measured the viscoelastic properties of RBCs. However, for the sake of simplicity, we have decided to treat the RBCs not as a mixture, but as an isotropic granular-like material.

6. A stress tensor for blood

Blood is a complex mixture composed of plasma, red blood cells (RBCs), white blood cells (WBCs), platelets, and other proteins. A complete constitutive relation for the stress tensor of the (whole) blood, not only must capture and describe the rheological characteristics of its different components, but also must include the biochemistry and the chemical reactions occurring. To date, no such comprehensive and universal constitutive relation exists. As mentioned by Anand et al. [86]: "However, the numerous biochemical reactions that take place leading to the formation and lysis of clots, and the exact influence of hemodynamic factors in these reactions are incompletely understood." In fact, the majority, if not all, of the papers published on blood characteristics either deal with the biochemistry of clot formation and other biochemical issues, ignoring completely the hemodynamic (see, e.g., Kuharsky and Fogelson [119]), or deal exclusively with hemodynamic or homeostasis and pay no attention to the biochemical reactions (see Sorensen et al. [83, 84]). Anand and Rajagopal [120] developed a model for blood that is capable of incorporating platelet activation. More recently, Rajagopal and colleagues (Anand et al. [86, 121, 122]) have provided a framework whereby some of the biochemical aspects of blood along with certain rheological (viscoelastic) properties of blood are included in their formulation.

It has been reported that at low shear rates, blood seems to have a high apparent viscosity (due to RBC aggregation) while at high shear rates the opposite behavior is observed (due to RBC disaggregation) (see Anand and Rajagopal [120, 123], Anand et al. [86]). One of the successful models which has been able to capture the shear-thinning behavior of blood over a wide range of shear rates is the one proposed by Yeleswarapu [118] and Yeleswarapu et al. [124]; in this model, the authors proposed a generalization of a three-constant Oldroyd-B fluid. Most of the modeling efforts to use the non-Newtonian fluid mechanics approach to study blood fall into two categories: (i) those which predict shear-thinning (power law models or Oldroyd type models), and (ii) models which exhibit yield stress (Casson model or Herschel-Bulkley type models). However, it is well known that many suspensions composed of particle (deformable or nondeformable, spherical, or rod-like) also show normal stress effects giving rise to phenomena such as die-swell or rod-climbing (see, e.g., Macosko [125], Larson [126]). Interestingly there is not much experimental work reported on this subject, that is, whether blood also exhibits normal stress effects. Massoudi and Phuoc [127] modeled blood as a modified second grade fluid where the shear viscosity and the normal stress coefficients depend on the shear rate. It was assumed that blood near the wall behaves as Newtonian fluid, and at the core it behaves as non-Newtonian fluid.

An alternative way is to model blood as we have done here: a two-component mixture composed of plasma and RBCs. If we are interested in describing the global behavior of blood, then a stress tensor for blood can be assumed to be given by

$$\mathbf{T}_{\text{blood}} = \mathbf{T}^{(1)}_{\text{plasma}} + \mathbf{T}^{(2)}_{\text{RBC}}, \tag{6.1}$$

where by using (5.2) and (5.19), we have

$$\mathbf{T}_{\text{blood}} = [-p(1-\phi) + \lambda(1-\phi)\text{tr}\,\mathbf{D_1}]\mathbf{1} + 2\mu(1-\phi)\mathbf{D_1} + [-p\phi + \beta_{20}(\phi + \phi^2)\text{tr}\,\mathbf{D_2}]\mathbf{1}$$
$$+ \beta_{30}(\phi + \phi^2)\Pi^{m/2}\mathbf{D_2} + \beta_{50}(\phi + \phi^2)\mathbf{D_2^2}. \tag{6.2}$$

This can be rewritten as

$$\mathbf{T}_{\text{blood}} = [-p + \lambda(1-\phi)\text{tr}\,\mathbf{D_1} + \beta_{20}(\phi + \phi^2)\text{tr}\,\mathbf{D_2}]\mathbf{1} + 2\mu(1-\phi)\mathbf{D_1}$$
$$+ \beta_{30}(\phi + \phi^2)\Pi^{m/2}\mathbf{D_2} + \beta_{50}(\phi + \phi^2)\mathbf{D_2^2}. \tag{6.3}$$

Now, if we assume that the two velocities are equal, that is,

$$\mathbf{v_1} = \mathbf{v_2} = \mathbf{v}. \tag{6.4}$$

This refers to an idealized case where the RBCs are basically being carried away with the velocity of the plasma. It follows that

$$\mathbf{D_1} = \mathbf{D_2} = \mathbf{D}, \tag{6.5}$$

and (6.3) can be rewritten as

$$
\begin{aligned}
\mathbf{T}_{\text{blood}} = \{-p + [\lambda(1 - \phi) + \beta_{20}(\phi + \phi^2)]\,\text{tr}\,\mathbf{D}\}\mathbf{1} \\
+ [2\mu(1 - \phi) + \beta_{30}(\phi + \phi^2)\Pi^{m/2}]\mathbf{D} + \beta_{50}(\phi + \phi^2)\mathbf{D^2}.
\end{aligned}
\tag{6.6}
$$

If we define

$$\lambda_{\text{blood}} = [\lambda(1 - \phi) + \beta_{20}(\phi + \phi^2)], \tag{6.7}$$

$$\mu_{\text{blood}} = [2\mu(1 - \phi) + \beta_{30}(\phi + \phi^2)\Pi^{m/2}], \tag{6.8}$$

then a simple constitutive relation for blood is obtained

$$\mathbf{T}_{\text{blood}} = [-p + \lambda_{\text{blood}}\text{tr}\,\mathbf{D}]\mathbf{1} + \mu_{\text{blood}}\mathbf{D} + \beta_{50}(\phi + \phi^2)\mathbf{D^2}. \tag{6.9}$$

Now, using a simple viscometer, the apparent blood viscosity can be measured, where μ_{blood} is given by (6.8) indicating that there is a contribution from the plasma component, and a contribution from the RBCs, appropriately weighted (see Massoudi [16] for a discussion of this issue. Massoudi [16] proposed the following hypothesis: "The material properties of one constituent which appear in the constitutive relation(s) of the other constituent(s), sometimes referred to as the cross terms, should be multiplied by a function $f(\phi)$ such that as $\phi \to 0$ or as $\phi \to 1$, $f(\phi) \to 0$ (the simplest case being, $\phi(1 - \phi)$), which implies that the cross terms approach zero, while all other material properties should be weighted according to the respective volume fraction of that constituent."). Furthermore, there is shear-rate dependence from the RBCs and a nonlinear viscosity term, also a function of the RBCs concentration given by the last term in the above equation.

Obviously, the form of (6.9) depends both on the form of the constitutive relations for plasma, (5.2), and for the RBCs, (5.19). Had we assumed an anisotropic representation for the RBCs, then the stress tensor for blood would have reflected a different fluid-type characteristic. For example, if in (5.3) we set

$$a_2 = a_4 = a_7 = a_8 = a_{11} = 0, \tag{6.10}$$

which means a flowing anisotropic material where density gradient does not have an impact on the stress, then we have

$$\mathbf{T_2} = \alpha_1\mathbf{1} + \alpha_3\mathbf{n} \otimes \mathbf{n} + \alpha_5\mathbf{D_2} + \alpha_6\mathbf{D_2^2} + \alpha_9(\mathbf{n} \otimes \mathbf{D_2 n} + \mathbf{D_2 n} \otimes \mathbf{n}) + \alpha_{10}(\mathbf{n} \otimes \mathbf{D_2^2 n} + \mathbf{D_2^2 n} \otimes \mathbf{n}), \tag{6.11}$$

or

$$T_{ij} = \alpha_1 \delta_{ij} + \alpha_3 n_i n_j + \alpha_5 D_{ij} + \alpha_6 D_{ij}^2 + \alpha_9 (n_i D_{jk} n_k + D_{ik} n_k n_j) + \alpha_{10}(n_i D_{jk}^2 n_k + D_{ik}^2 n_k n_j),$$

(6.12)

which is the same as Leslie-Ericksen (see Leslie [82, Equation (1)]), provided that

$$\alpha_1 = -p,$$

(6.13a)

and the α's are functions of

$$n_i n_i, D_{ij} n_i n_j, D_{ik} D_{kj} n_i n_j, D_{ij} D_{ij}, D_{ik} D_{kj} D_{ji}.$$

(6.13b)

Of course, this makes the problem more complicated in the sense that now, similar to anisotropic liquids, one needs additional governing (balance) equations and one has to deal with the unknown additional boundary conditions created by the associated nonlinearities (see Massoudi [110] for details). The formulation in this section reveals that the rheological behavior of blood, represented by (6.6), for example, is to a large extent determined and controlled by the rheological behavior of the RBCs, as given, for example, by (5.19). We have neglected the deformability of the cells. In the next section, we will look at a simple boundary value problem.

7. Simple shear flow between two flat plates

For a simple shear flow, that is, flow between two horizontal plates a distance "h" apart, with the lower plate fixed and the upper plate moving with a constant speed, the velocity field $\mathbf{v}$ and the volume function ϕ are assumed to be of the form (Furthermore, we have assumed that the walls are rigid and nondeformable; however, in reality the vessel walls are flexible and viscoelastic.)

$$\mathbf{v} = u(y)\mathbf{i},$$

(7.1a)

$$\phi = \phi(y).$$

(7.1b)

It then follows that

$$\mathbf{D} = \frac{1}{2} \begin{pmatrix} 0 & u' & 0 \\ u' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{D}^2 = \frac{1}{4} \begin{pmatrix} (u')^2 & 0 & 0 \\ 0 & (u')^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(7.2)

Also, notice that

$$\operatorname{tr} \mathbf{D} = 0,$$

(7.3a)

$$\operatorname{tr} \mathbf{D}^2 = \frac{1}{2} \left(\frac{du}{dy} \right)^2,$$

(7.3b)

$$\Pi^{m/2} = \left| \frac{du}{dy} \right|^m.$$

(7.3c)

Now, using (7.1a), (7.1b), (7.2), (7.3a), (7.3b), and (7.3c) in (6.9), we find that

$$T_{xy} = \frac{1}{2}\left[\mu_{\text{blood}}\frac{du}{dy}\right], \tag{7.4a}$$

$$T_{xx} - T_{yy} = 0, \tag{7.4b}$$

$$T_{yy} - T_{zz} = [\beta_{50}(\phi + \phi^2)]\left(\frac{du}{dy}\right)^2. \tag{7.4c}$$

Therefore, we can see that according to (6.9), blood exhibits only one of the normal stress differences. If the term $\beta_4(\phi)\nabla\phi \otimes \nabla\phi$ were kept in the constitutive expression in (5.12), the model would be capable of exhibiting both of the normal stress differences.

Recalling that the balance of linear momentum for blood as a mixture is given by (3.6), and for the case of $\mathbf{v_1} = \mathbf{v_2} = \mathbf{v}$, implying $\mathbf{D_1} = \mathbf{D_2} = \mathbf{D}$, we have

$$(\rho_1 + \rho_2)\frac{D\mathbf{v}}{Dt} = \text{div}\,\mathbf{T}_m + (\rho_1 + \rho_2)\mathbf{b}, \tag{7.5}$$

where $\rho_1 = (1 - \phi)\rho_{\text{plasma}}$ and $\rho_2 = \phi\rho_{\text{RBC}}$, and

$$\mathbf{T}_m = \mathbf{T}_{\text{blood}} = [-p + \lambda_{\text{blood}}\text{tr}\,\mathbf{D}]\mathbf{1} + \mu_{\text{blood}}\mathbf{D} + \beta_{50}(\phi + \phi^2)\mathbf{D}^2. \tag{7.6}$$

For the flowfield assumed by (7.1a), (7.1b), (7.5), with using (7.6), reduces to the three components in the x, y, and z direction, respectively,

$$\frac{1}{2}\frac{d}{dy}\left[\mu_{\text{blood}}\frac{du}{dy}\right] + (\rho_1 + \rho_2)b_x = 0, \tag{7.7a}$$

$$\frac{d}{dy}\left[-p + \frac{\beta_{50}(\phi + \phi^2)}{4}\left(\frac{du}{dy}\right)^2\right] + (\rho_1 + \rho_2)b_y = 0, \tag{7.7b}$$

$$(\rho_1 + \rho_2)b_z = 0, \tag{7.7c}$$

where b_x, b_y, and b_z are the components of the external body force, and μ_{blood} is given by (6.8). Thus, we see that a motion of the form of (7.1a) and (7.1b) is only possible if the z-component of the body force field is zero.

Equations (7.7a) and (7.7b) form a system of two coupled second-order nonlinear ordinary differential equations, and in general have to be solved numerically. Appropriate boundary conditions are needed in order to have a well-posed problem. For the simple shearing motion assumed by (7.1a) and (7.1b), we assume

$$u(0) = 0, \tag{7.8a}$$

$$u(h) = V, \tag{7.8b}$$

where "h" is the distance between the two plates. It is possible that the no-slip condition may not be appropriate for all cases, as the RBCs might roll or slip at the boundaries. And for this special case, where the gradients of volume fraction are ignored, we only need one boundary condition for ϕ:

$$\phi(0) = \phi_0, \tag{7.9}$$

where ϕ_0 is a constant (for different types of boundary conditions, see Langtangen and Munthe [128], and Massoudi [41]).

To obtain an analytical solution (closed form solution) to the above (7.7a), (7.7b) under some idealized conditions, we furthermore assume

$$b_x = b_y = b_z = 0. \tag{7.10}$$

Then, (7.7a) can be integrated once to give

$$\left[\mu_{\text{blood}} \frac{du}{dy} \right] = C_1, \tag{7.11}$$

where C_1 is a constant. Recalling that (7.3c), and (6.8), we have

$$\mu_{\text{blood}} = [2\mu(1 - \phi) + \beta_{30}(\phi + \phi^2)\Pi^{m/2}] = 2\mu(1 - \phi) + \beta_{30}(\phi + \phi^2)\left| \frac{du}{dy} \right|^m. \tag{7.12}$$

Substituting this in (7.11), we have

$$\left[2\mu(1 - \phi) + \beta_{30}(\phi + \phi^2)\left| \frac{du}{dy} \right|^m \right] \frac{du}{dy} = C_1. \tag{7.13}$$

Equation (7.13) can be integrated once to give us

$$-p + \frac{\beta_{50}(\phi + \phi^2)}{4}\left(\frac{du}{dy} \right)^2 = C_2, \tag{7.14}$$

where C_2 is a constant. Now, it can be shown that the above equations admit a solution of the form

$$u = \alpha y,$$
$$\phi = k, \tag{7.15}$$

where α and k are constants. However, the system of (7.7a) and (7.7b) is nonlinear and might admit additional solutions.

8. Comments

In this paper, we have discussed, based on the classical mixture theory and the approach taken by Massoudi and Rajagopal, a framework for modeling the rheological behavior of blood. The proposed constitutive relation depends on the form of the stress tensors for the plasma and the red-blood cells. In general, the RBCs are assumed to behave as an anisotropic density-gradient-dependent viscous fluid. As such the equations are highly nonlinear. After making many simplifying assumptions, a relatively simple constitutive relation for the stress tensor for blood is obtained (see (7.6)); its form is very similar to a Reiner-Rivlin fluid, where the shear viscosity coefficient is not only a function of the shear rate, but also of the concentration. It is noted that we have only discussed the development of a model and that specific boundary value problems need to be solved to test the efficacy of the model. A simple shear flow is studied, and an exact solution is obtained for a very special case;

for more general cases, it is necessary to solve the nonlinear coupled equations numerically. Furthermore, it should be mentioned that for higher order or higher gradient theories, assigning boundary conditions for certain terms, which appear in the governing equations, is a difficult task. Quite often these boundary conditions are not derived from first principles; instead they are given as ad hoc assumptions, or they are simply specified as mathematical conveniences. Sometimes experiments have been used successfully to specify these necessary additional boundary conditions.

Furthermore, it goes without saying that the model developed here is only appropriate for a healthy human, and it does not capture any blood disorder. It has also been shown that with regular exercise and physical training certain characteristics of blood can change, and blood undergoes what is known as "fluidification" (see Ernst and Matrai [129], Wannamethee et al. [130]).

To include the formation and growth of clots, and lysis of blood cells in blood, in general, the reaction-convection-diffusion equations are to be solved in conjunction with the balance laws for mass, linear and angular momentum, and energy (for each component). Although we have ignored the biochemical effects of blood in this paper, in principle, the theory is amenable to extension (see Anand et al. [86]).

Nomenclature

$\mathbf{a}$: Acceleration vector
$\mathbf{b}$: Body force vector
$\mathbf{D}$: Symmetric part of the velocity gradient
$\mathbf{f}_I$: Interaction force vector
$\mathbf{I}$: Identity tensor
$\mathbf{L}$: Gradient of velocity vector
p: Fluid pressure
$\mathbf{T}$: Stress tensor
$\mathbf{v}$: Velocity vector
$\mathbf{W}$: Spin tensor
$\mathbf{x}$: Position vector

Greek letters

λ_f: Second coefficient of (fluid) viscosity
μ: First coefficient of (fluid) viscosity
ν: Volume fraction of the solid
ρ: Density
ρ_o: Reference density
ϕ: Volume fraction of fluid

Subscripts

$1, f$: Referring to the fluid component
$2, s$: Referring to the solid component
m: Referring to the mixture

Superscripts

T: Transpose
$*$: Dimensionless quantity

Other symbols

div: Divergence operator
∇:   Gradient operator
tr: Trace of a tensor
$\otimes$:   Outer product
$\cdot$:   Dot product

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