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Research Article

Dynamical Aspects of an Equilateral Restricted Four-Body Problem

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The spatial equilateral restricted four-body problem (ERFBP) is a four body problem where a mass point of negligible mass is moving under the Newtonian gravitational attraction of three positive masses (called the primaries) which move on circular periodic orbits around their center of mass fixed at the origin of the coordinate system such that their configuration is always an equilateral triangle. Since fourth mass is small, it does not affect the motion of the three primaries. In our model we assume that the two masses of the primaries m_2 and m_3 are equal to μ and the mass m_1 is $1-2\mu$. The Hamiltonian function that governs the motion of the fourth mass is derived and it has three degrees of freedom depending periodically on time. Using a synodical system, we fixed the primaries in order to eliminate the time dependence. Similarly to the circular restricted three-body problem, we obtain a first integral of motion. With the help of the Hamiltonian structure, we characterize the region of the possible motions and the surface of fixed level in the spatial as well as in the planar case. Among other things, we verify that the number of equilibrium solutions depends upon the masses, also we show the existence of periodic solutions by different methods in the planar case.

1. Introduction

Dynamical systems with few bodies (three) have been extensively studied in the past, and various models have been proposed for research aiming to approximate the behavior of real celestial systems. There are many reasons for studying the four-body problem besides the historical ones, since it is known that approximately two-thirds of the stars in our Galaxy exist as part of multistellar systems. Around one-fifth of these is a part of triple systems, while a rough estimate suggests that a further one-fifth of these triples belongs to quadruple or higher systems, which can be modeled by the four-body problem. Among these models, the configuration used by Maranhão [1] and Maranhão and Llibre [2], where three point masses form at any time a collinear central configuration (Euler configuration, see [3]), is of particular interest not only for its simplicity but mainly because in the last 10 years, an increasing number of extrasolar systems have been detected, most of them consisting of a "sun" and a planet or of a "sun" and two planets.

We study the motion of a mass point of negligible mass under the Newtonian gravitational attraction of three mass points of masses m_1, m_2, and m_3 (called primaries) moving in circular periodic orbits around their center of mass fixed at the origin of the coordinate system. At any instant of time, the primaries form an equilateral equilibrium configuration of the three-body problem which is a particular solution of the three-body problem given by Lagrange (see [4] or [3]). Two of these primaries have equal masses and are located symmetrically with respect to the third primary.

We choose the unity of mass in such a way that $m_1 = 1 - 2\mu$ and $m_2 = m_3 = \mu$ are the masses of the primaries, where $\mu \in (0, 1/2)$. Units of length and time are chosen in such a way that the distance between the primaries is one.

For studying the position of the infinitesimal mass, m_4, in the plane of motion of the primaries, we use either the sideral system of coordinates, or the synodical system of coordinates (see [5] for details). In the synodical coordinates, the three point masses m_1, m_2, and m_3 are fixed at $(\sqrt{3}\mu, 0, 0)$, $(-(\sqrt{3}/2)(1-2\mu), 1/2, 0)$, and $(-(\sqrt{3}/2)(1-2\mu), -1/2, 0)$, respectively. In this paper, *the equilateral restricted four-body problem* (shortly, ERFBP) consists in describing the motion of the infinitesimal mass, m_4, under the gravitational attraction of the three primaries m_1, m_2, and m_3. Maranhão's PhD thesis [1] and the paper [2] by Maranhão and Llibre studied a restricted four body problem, where three primaries rotating in a fixed circular orbit define a collinear central configuration.

In the ERFBP, the equations of motion of m_4 in synodical coordinates (x, y, z) are

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} + 2\dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z, \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} \Omega = \Omega(x,y,z) &= \frac{1}{2}\left(x^2+y^2\right) + \frac{1-2\mu}{\rho_1} + \frac{\mu}{\rho_2} + \frac{\mu}{\rho_3}, \\ \rho_1 &= \sqrt{\left(x-\sqrt{3}\mu\right)^2 + y^2 + z^2}, \quad \rho_2 \\ &= \sqrt{\left(x+\frac{\sqrt{3}}{2}(1-2\mu)\right)^2 + \left(y-\frac{1}{2}\right)^2 + z^2}, \\ \rho_3 &= \sqrt{\left(x+\frac{\sqrt{3}}{2}(1-2\mu)\right)^2 + \left(y+\frac{1}{2}\right)^2 + z^2}. \end{aligned} \tag{1.2}$$

We remark that the ERFBP becomes *the central force problem* when $\mu = 0$, and $m_1 = 1$ is situated in the origin of the system, while $\mu = 1/2$ results in *the restricted three-body problem* with the bodies m_2 and m_3 of mass $1/2$.

Our paper is organized as follows: Section 2 is devoted to describing the most important dynamical phenomena that governs the evolution of asteroid movement and states the problem under consideration in the present study. In Section 3 reductions of the problem are discussed and a comprehensive treatment of streamline analogies is given. Section 4 is devoted to the principal qualitative aspect of the restricted problem—the surfaces and curves of zero velocity, several uses of which are discussed. The regions of allowed motion and the location and properties of the equilibrium points are established. We describe the Hill region. The description of the number of equilibrium points is given in Section 5, and in the symmetrical case (i.e., $\mu = 1/3$), we describe the kind of stability of each equilibrium. In Section 6, the planar case is considered. There, we prove the existence of periodic solutions as a continuation of periodic Keplerian orbits, and also when the parameter μ is small and when it is close to $1/2$. Finally, in Section 8 we present the conclusions of the present work.

Next, we will enunciate some four-body problem that has been considered in the literature. Cronin et al. in [6, 7] considered the models of four bodies where two massive bodies move in circular orbits about their center of mass or barycenter. In addition, this barycenter moves in a circular orbit about the center of mass of a system consisting of these two bodies and a third massive body. It is assumed that this third body lies in the same plane as the orbits of the first two bodies. The authors studied the motion of a fourth body of small mass which moves under the combined attractions of these three massive bodies. This model is called bicircular four-body problem. Considering this restricted four-body problem consisting of Earth, Moon, Sun, and a massless particle, this problem can be used as a model for the motion of a space vehicle in the Sun-Earth-Moon system. Several other authors have considered the study of this problem, for example, [8–11] and references therein. The quasi-bicircular problem is a restricted four body problem where three masses, Earth-Moon-Sun, are revolving in a quasi-bicircular motion (i.e. a coherent motion close to bicircular) also has been studied, see [12] and references therein. The restricted four-body problem with radiation pressure was considered in [13], while the photogravitational restricted four body problem was considered in [14].

2. Statement of the Problem

It is known that equilateral configurations of three-bodies with arbitrary masses m_1, m_2, and m_3 on the same plane, moving with the same angular velocity, form a relative equilibrium solution of the three-body problem (see e.g., [4] or [3]). More precisely, we consider three particles of masses m_1, m_2, and m_3 (called primaries) each describing, at any instant, a circle around their center of masses (which is fixed at the origin), with the same angular velocity ω and such that its configuration at any instant is an equilateral triangle (see Figure 1). Now, we consider an infinitesimal particle m_4 attracted by the primaries m_1, m_2, and m_3 according to Newton's gravitational law. Let $\mathbf{r}$ be the position vector of m_4.

The equations of motion can be written as

$$\mathbf{r}'' = \nabla U, \tag{2.1}$$

where $(\,)'$ denotes derivative with respect to t and

$$U = U(\mathbf{r}; t, m_1, m_2, m_3) = \frac{m_1}{\|\mathbf{r} - \mathbf{r}_1(t)\|} + \frac{m_2}{\|\mathbf{r} - \mathbf{r}_2(t)\|} + \frac{m_3}{\|\mathbf{r} - \mathbf{r}_3(t)\|}, \tag{2.2}$$

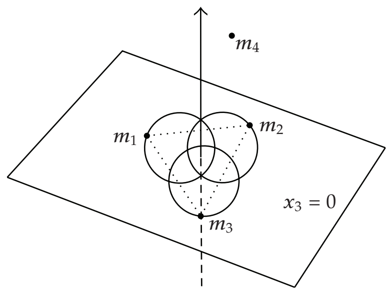


Figure 1: The equilateral restricted four body problem in inertial coordinates.

with $\mathbf{r}_1(t), \mathbf{r}_2(t)$, and $\mathbf{r}_3(t)$ representing the position of each primary, respectively. To remove the time dependence of the system (2.1), we consider the orthonormal moving frame in $\mathbb{R}^3$, given by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ where

$$\mathbf{e}_1 = \mathbf{e}_1(t) = e^{i\omega t}, \qquad \mathbf{e}_2 = \mathbf{e}_2(t) = i\mathbf{e}_1, \qquad \mathbf{e}_3 = \mathbf{e}_3(t) = (0, 0, 1) \tag{2.3}$$

with $i^2 = -1$. This orthonormal moving frame corresponds to the synodical system. Then, (2.1) can be written as

$$\begin{aligned} x_1'' - 2\omega x_2' - \omega^2 x_1 &= \frac{\partial U}{\partial x_1}, \\ x_2'' + 2\omega x_1' - \omega^2 x_2 &= \frac{\partial U}{\partial x_2}, \\ x_3'' &= \frac{\partial U}{\partial x_3}, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} U = U(x_1, x_2, x_3) &= \frac{m_1}{d_1} + \frac{m_2}{d_2} + \frac{m_3}{d_3}, \\ d_1 &= \sqrt{(x_1 - \alpha_1)^2 + (x_2 - \beta_1)^2 + x_3^2}, \\ d_2 &= \sqrt{(x_1 - \alpha_2)^2 + (x_2 - \beta_2)^2 + x_3^2}, \\ d_3 &= \sqrt{(x_1 - \alpha_3)^2 + (x_2 - \beta_3)^2 + x_3^2}, \end{aligned} \tag{2.5}$$

where $\mathbf{r}_j(t) = e^{i\omega t}\zeta_j$, with $\zeta_j = \alpha_j + \mathrm{i}\beta_j$ for $j = 1, 2, 3$. Applying the above notation, we can write $\mathbf{r} = (x_1 + ix_2)\mathbf{e}_1 + x_3\mathbf{e}_3$, $\mathbf{r}_1 = \zeta_1\mathbf{e}_1$, $\mathbf{r}_2 = \zeta_2\mathbf{e}_1$, $\mathbf{r}_3 = \zeta_3\mathbf{e}_1$, and so $\|\mathbf{r} - \mathbf{r}_j\| = \|(x_1 + ix_2) + x_3\mathbf{e}_3 - \zeta_j\|$ for $j = 1, 2, 3$.

We perform the reparametrization of time $d\tau = \omega dt$, then the system (2.4) is transformed into

$$\begin{aligned}
\ddot{x}_1 - 2\dot{x}_2 - x_1 &= \frac{1}{\omega^2}\frac{\partial W}{\partial x_1}, \\
\ddot{x}_2 + 2\dot{x}_1 - x_2 &= \frac{1}{\omega^2}\frac{\partial W}{\partial x_2}, \\
\ddot{x}_3 &= \frac{1}{\omega^2}\frac{\partial W}{\partial x_3},
\end{aligned} \tag{2.6}$$

where the dot denotes the derivative with respect to τ, and the potential W is given by

$$W = W(x_1, x_2, x_3) = \frac{m_1}{\rho_1} + \frac{m_2}{\rho_2} + \frac{m_3}{\rho_3} \tag{2.7}$$

with

$$\begin{aligned}
\rho_1 &= \sqrt{(x_1 - \alpha_1)^2 + (x_2 - \beta_1)^2 + x_3^2}, \\
\rho_2 &= \sqrt{(x_1 - \alpha_2)^2 + (x_2 - \beta_2)^2 + x_3^2}, \\
\rho_3 &= \sqrt{(x_1 - \alpha_3)^2 + (x_2 - \beta_3)^2 + x_3^2}.
\end{aligned} \tag{2.8}$$

If we define $\mu_1 = m_1/M$, $\mu_2 = m_2/M$, and $\mu_3 = m_3/M$, where $M = m_1 + m_2 + m_3$, the equations of motions (2.4) become

$$\begin{aligned}
\ddot{x}_1 - 2\dot{x}_2 - x_1 &= \frac{M}{\omega^2}\frac{\partial W}{\partial x_1}, \\
\ddot{x}_2 + 2\dot{x}_1 - x_2 &= \frac{M}{\omega^2}\frac{\partial W}{\partial x_2}, \\
\ddot{x}_3 &= \frac{M}{\omega^2}\frac{\partial W}{\partial x_3},
\end{aligned} \tag{2.9}$$

where

$$W = W(x_1, x_2, x_3) = \frac{M\mu_1}{\rho_1} + \frac{M\mu_2}{\rho_2} + \frac{M\mu_3}{\rho_3}. \tag{2.10}$$

For simplicity, we will consider an equilateral triangle of side 1 and so we obtain that $M/\omega^2 = 1$.

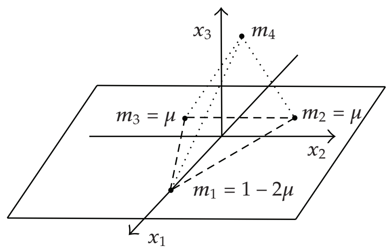


Figure 2: The equilateral restricted four body problem in a rotating frame.

3. Equations of Motion and Preliminary Results

From (2.9), we deduce that the equations of motion of the ERFBP in synodical coordinates are given by the system of differential equations

$$\begin{aligned}\ddot{x}_1 - 2\dot{x}_2 &= \Omega_{x_1},\\ \ddot{x}_2 + 2\dot{x}_1 &= \Omega_{x_2},\\ \ddot{x}_3 &= \Omega_{x_3},\end{aligned}\tag{3.1}$$

where

$$\begin{aligned}\Omega = \Omega(x_1, x_2, x_3) &= \frac{1}{2}\left(x_1^2 + x_2^2\right) + W(x_1, x_2, x_3),\\ W = W(x_1, x_2, x_3) &= \frac{1-2\mu}{\rho_1} + \frac{\mu}{\rho_2} + \frac{\mu}{\rho_3},\end{aligned}\tag{3.2}$$

with

$$\begin{aligned}\rho_1 &= \sqrt{\left(x_1 - \sqrt{3}\mu\right)^2 + x_2^2 + x_3^2},\\ \rho_2 &= \sqrt{\left(x_1 + \frac{\sqrt{3}}{2}(1-2\mu)\right)^2 + \left(x_2 - \frac{1}{2}\right)^2 + x_3^2},\\ \rho_3 &= \sqrt{\left(x_1 + \frac{\sqrt{3}}{2}(1-2\mu)\right)^2 + \left(x_2 + \frac{1}{2}\right)^2 + x_3^2}.\end{aligned}\tag{3.3}$$

Analogously to the circular three-body problem, we can verify that the system (3.1) possesses a first Jacobi type integral given by

$$C = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right) - \Omega(x_1, x_2, x_3).\tag{3.4}$$

Thus we have the following result.

Proposition 3.1. *The Jacobi-type function* (3.4) *is a first integral of the ERFBP for any value of* μ.

Proof. Differentiating (3.4) with respect to the time, we get

$$\frac{dC}{dt} = \dot{x}_1\ddot{x}_1 + \dot{x}_2\ddot{x}_2 + \dot{x}_3\ddot{x}_3 - x_1\dot{x}_1 - x_2\dot{x}_2 - \frac{\partial W}{\partial x_1}\dot{x}_1 - \frac{\partial W}{\partial x_2}\dot{x}_2 - \frac{\partial W}{\partial x_3}\dot{x}_3, \tag{3.5}$$

and using (2.9) we can reduce the obtained expression to

$$\begin{aligned} \frac{dC}{dt} &= \dot{x}_1\left(x_1 + \frac{\partial W}{\partial x_1} + 2\dot{x}_2\right) + \dot{x}_2\left(x_2 + \frac{\partial W}{\partial x_2} - 2\dot{x}_1\right) + \dot{x}_3\frac{\partial W}{\partial x_3} \\ &\quad - x_1\dot{x}_1 - x_2\dot{x}_2 - \frac{\partial W}{\partial x_1}\dot{x}_1 - \frac{\partial W}{\partial x_2}\dot{x}_2 - \frac{\partial W}{\partial x_3}\dot{x}_3 = 0. \end{aligned} \tag{3.6}$$

Hence C is a constant of motion. □

In order to write the Hamiltonian formulation of the ERFBP we introduce the new variables

$$\begin{aligned} x &= x_1, \qquad y = x_2, \qquad z = x_3, \\ X &= \dot{x} - y, \qquad Y = \dot{y} + x, \qquad Z = \dot{z}. \end{aligned} \tag{3.7}$$

Hence, it is verified that system (3.1) is equivalent to an autonomous Hamiltonian system with three degrees of freedom with Hamiltonian function given by

$$H = H(x, y, z, X, Y, Z) = \frac{1}{2}\left(X^2 + Y^2 + Z^2\right) + (yX - xY) - W. \tag{3.8}$$

Therefore, the Hamiltonian system associated is

$$\begin{aligned} \dot{x} &= y + X, \qquad \dot{X} = Y + W_x \\ \dot{y} &= -x + Y, \qquad \dot{Y} = -X + W_y \\ \dot{z} &= Z, \qquad \dot{Z} = W_z. \end{aligned} \tag{3.9}$$

Of course, the phase space where the equations of motion are well defined is

$$\begin{aligned} \mathcal{M} = \Bigg\{ (x, y, z, X, Y, Z) \in \Bigg(\mathbb{R}^3 \setminus \Bigg\{ \left(\sqrt{3}\mu, 0, 0\right), \left(-\frac{\sqrt{3}}{2}(1-2\mu), \frac{1}{2}, 0\right), \\ \left(-\frac{\sqrt{3}}{2}(1-2\mu), -\frac{1}{2}, 0\right) \Bigg\} \Bigg) \times \mathbb{R}^3 \Bigg\}, \end{aligned} \tag{3.10}$$

where the points that have been removed correspond to binary collisions between the massless particle and one of the primaries.

Additionally, the spatial ERFBP admits the planar case as a subproblem, that is, $z = Z = 0$ is invariant under the flow defined by (3.9).

On the other hand, we see that there are two limiting cases in the ERFBP, which we described below.

(a) If $\mu = 0$, we obtain a *central force problem*, with the body of mass $m_1 = 1$ at the origin of the coordinates.

(b) If $\mu = 1/2$, we obtain *the circular restricted three-body problem*, with masses $m_2 = m_3 = 1/2$.

Note that $\mu = 1/3$ corresponds to the symmetric case, that is, where the masses of the primaries are all equal to $1/3$.

It is easily seen that the equations of motion (3.9) are invariant by the symmetry

$$S: \ (x, y, z, X, Y, Z, \tau) \longrightarrow (x, -y, z, -X, Y, -Z, -\tau). \tag{3.11}$$

This means that if $\varphi(\tau) = (x(\tau), y(\tau), z(\tau), X(\tau), Y(\tau), Z(\tau))$ is a solution of the system (3.9), then $\varphi(t) = (x(-\tau), -y(-\tau), z(-\tau), -X(-\tau), Y(-\tau), -Z(-\tau))$ is also a solution. We note that this symmetry corresponds to a symmetry with respect to the xz-plane. In the planar case, the symmetry corresponds to symmetry with respect to the x-axis.

4. Permitted Regions of Motion

In this section, we will see that the function $\Omega(x, y, z)$ allows us to establish regions in the (x, y, z) space, where the motion of the infinitesimal particle could take place. We will use similar ideas to those developing in [15, 16].

By using (3.4), the surface of zero velocity is defined by the set

$$\mathcal{R}_C : (x, y, z) \in \mathbb{R}^3 \quad \text{such that } \Omega(x, y, z) = -C, \qquad \text{for any level } C. \tag{4.1}$$

This set corresponds to the so-called Hill region. We note that $C \geq 0$ implies $\mathcal{R}_C = \mathbb{R}^3 \setminus \{(\sqrt{3}\mu, 0, 0), (-\sqrt{3}/2(1-2\mu), 1/2, 0), (-\sqrt{3}/2(1-2\mu), -1/2, 0)\}$. That is, the region of all possible motions is given by the whole phase space and so the infinitesimal particle is free to move; in particular escape solutions are permitted.

In the spatial case, the surfaces that separate allowed and nonallowed motions are called zero-velocity surfaces, and for the planar case the set that separates the allowed and nonallowed motions is called zero-velocity curve. The shape and size of zero velocity sets $-C = \Omega(x, y, z)$ depend on C and μ. They correspond to the boundary of the Hill regions. The zero-velocity set $(\partial\mathcal{R}_C)$ is defined by the equation

$$\begin{aligned} \Omega = \frac{1}{2}\left(x^2 + y^2\right) &+ \frac{1-2\mu}{\sqrt{\left(x - \sqrt{3}\mu\right)^2 + y^2 + z^2}} + \frac{\mu}{\sqrt{\left(x + (\sqrt{3}/2)(1-2\mu)\right)^2 + (y - 1/2)^2 + z^2}} \\ &+ \frac{\mu}{\sqrt{\left(x + (\sqrt{3}/2)(1-2\mu)\right)^2 + (y + 1/2)^2 + z^2}} = -C, \end{aligned} \tag{4.2}$$

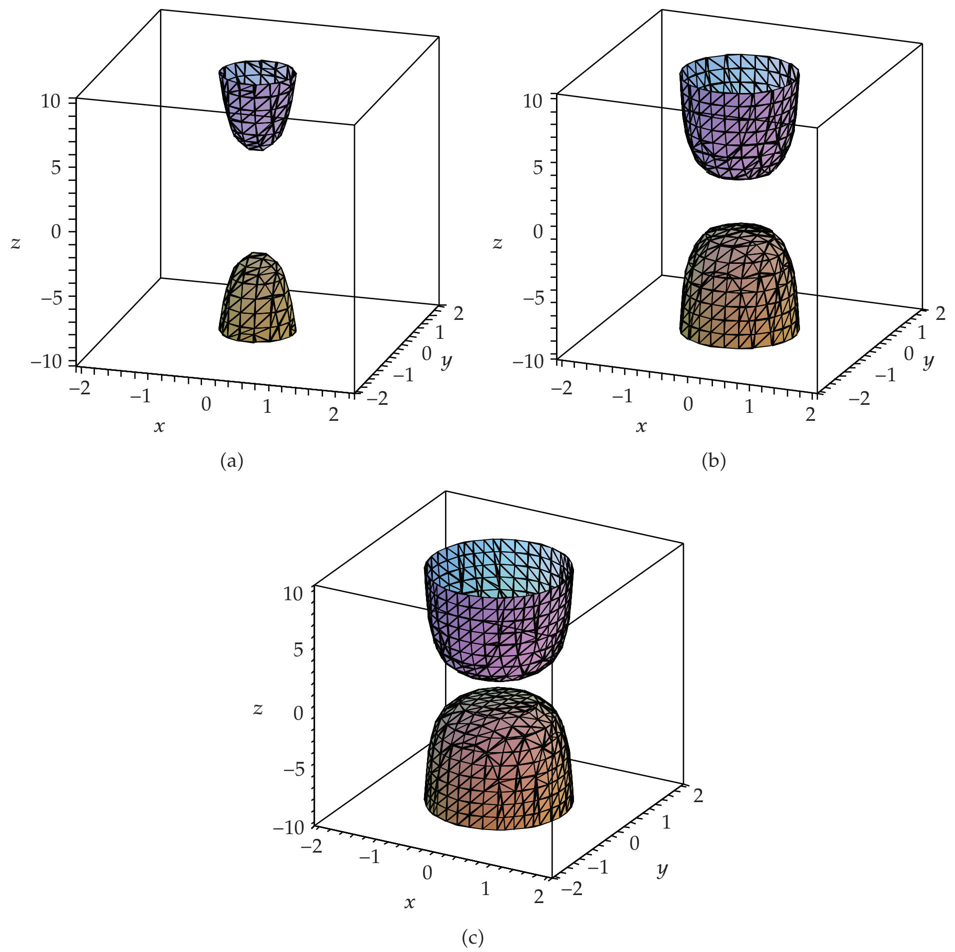


Figure 3: Evolution of zero-velocity surface in the three-dimensional ERFBP for $\mu = 1/3$. (a) $C = -1/4$. (b) $C = -1/2$. (c) $C = -3/4$.

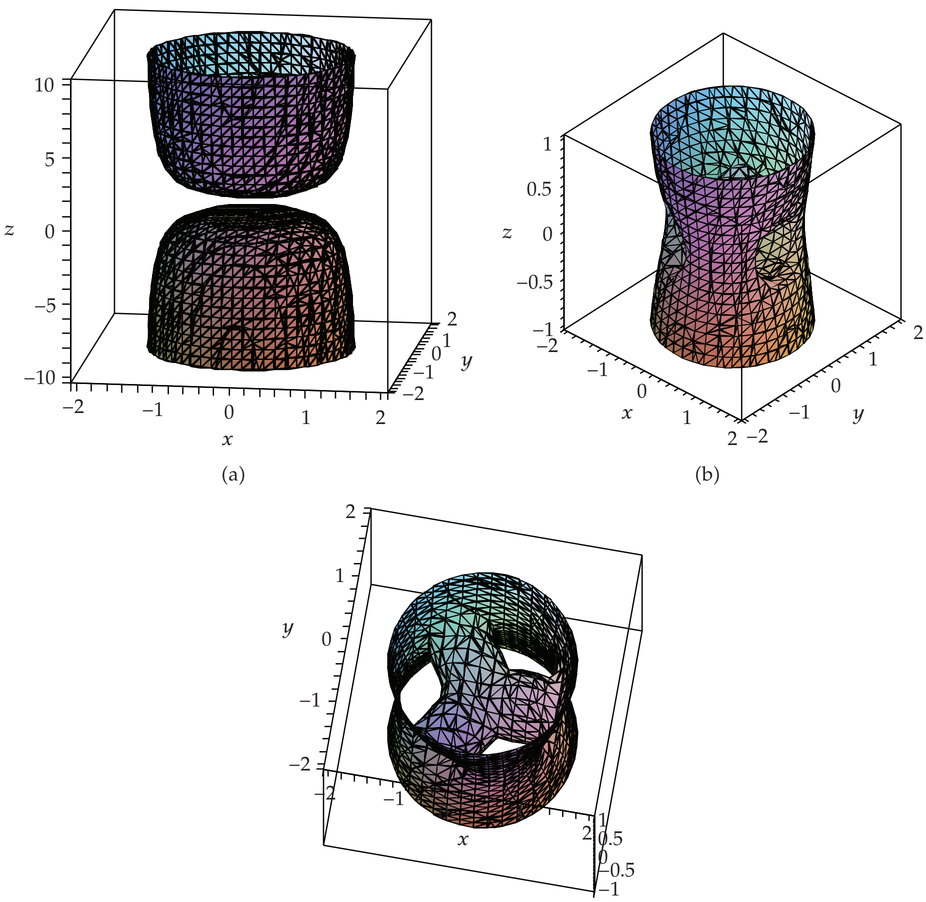
only for $C < 0$ and any value of μ. Next, we give a list of all possible situations that may appear when this condition is fulfilled.

(1) $z \to \pm\infty$ on the $\partial\mathcal{R}_C$ in which case $x^2 + y^2 \to -2C$, this means, that around the z-axis the variables (x, y) must be asymptotic to a circle of radius $\sqrt{-2C}$.

(2) $x \to \infty$ or $-\infty$ (resp., $y \to \infty$ or $-\infty$) on the $\partial\mathcal{R}_C$, when $C \to -\infty$.

(3) For $|C|$ very large this implies that (x, y) can be sufficiently close to one of the primaries, or the infinitesimal mass is close to infinity.

(4) Since $x^2 + y^2$ is a factor of Ω on $\partial\mathcal{R}_C$, then small values for $-C$ are not allowed.



(a)

(b)

(c)

Figure 4: Evolution of zero-velocity surface in the three dimensional ERFBP for $\mu = 1/3$. (a) $C = -1$. (b) and (c) $C = -1.6$ under different points of view.

By simplicity, we will only show zero-velocity surfaces for the case $\mu = 1/3$ and different values of the integral of motion C. Figures 3, 4, 5, and 6 show evolution of zero-velocity surfaces for several C values.

4.1. The Planar Case

As we mentioned in last section, the set $\{z = Z = 0\}$ is invariant under the flow, and so the motion of the infinitesimal body lies on the xy plane that contains the primaries. In Figure 7, we show the evolution of the function Ω in the planar case for different values of the parameter μ.

Next we show the evolution of the Hill's regions as well as the zero velocity curves, for $\mu = 1/3$ and many values of the Jacobian constant C; the permissible areas are shown on Figures 8, 9, 10, and 11 shading.

In Figure 12, we show the behavior of level curves in the planar case for some values of μ and for different energy levels.

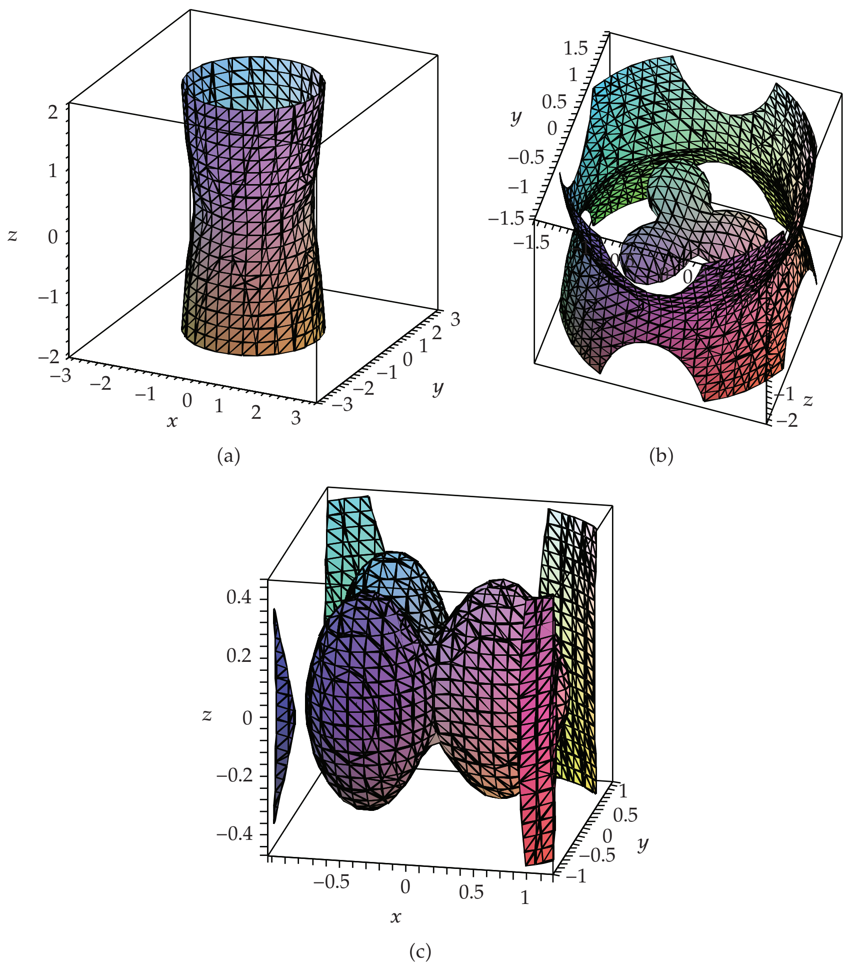


Figure 5: Evolution of zero-velocity surface in the three dimensional ERFBP for $\mu = 1/3$. All cases correspond to $C = -1.7$ under different points of view.

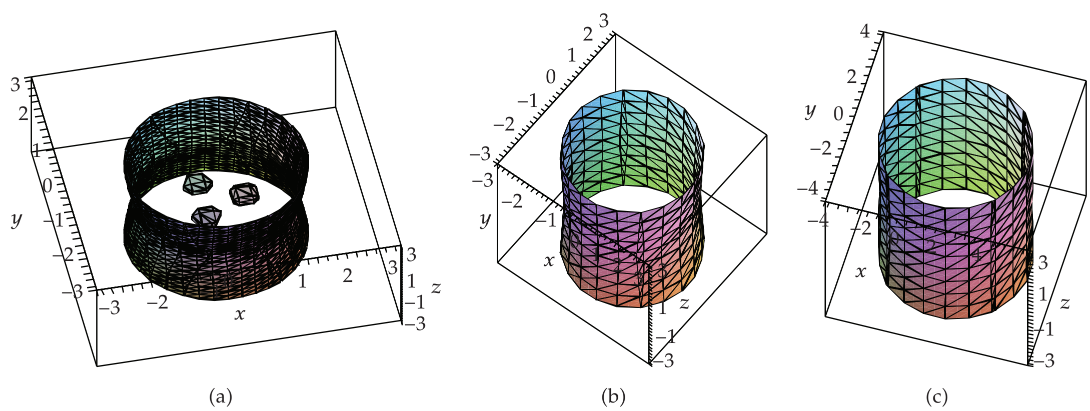


Figure 6: Evolution of zero-velocity surface in the three dimensional ERFBP for $\mu = 1/3$. (a) $C = -2$. (b) $C = -3$. (c) $C = -5$.

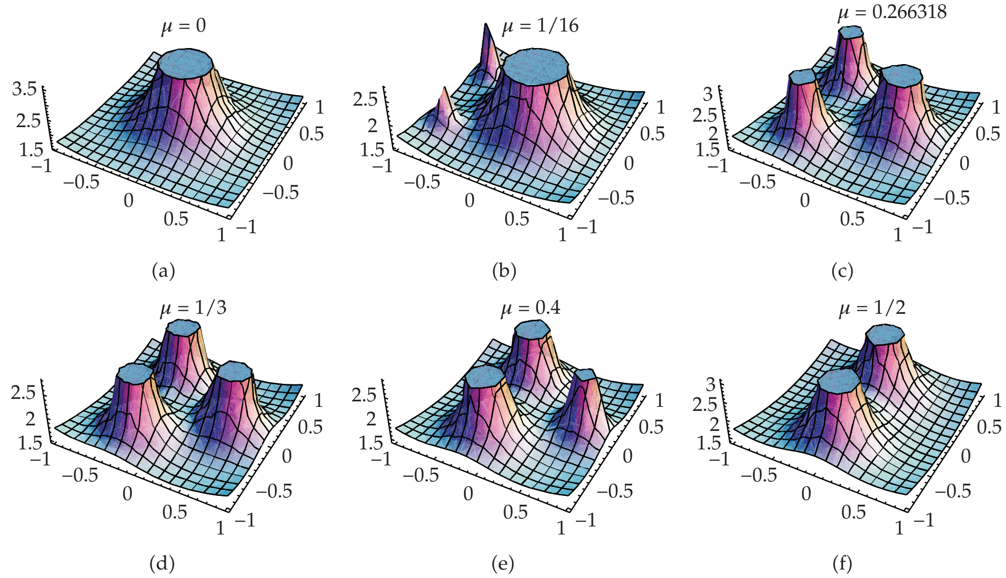


Figure 7: Evolution of the graph of $\Omega(x, y)$ on the xy plane for different values of μ.

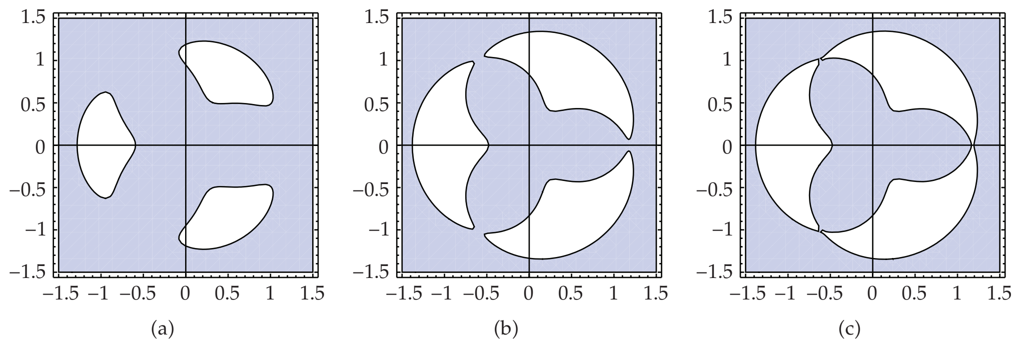


Figure 8: Evolution of zero-velocity curves and Hill's region in the planar ERFBP for $\mu = 1/3$, where shading represents permissible areas. (a) $C = -1.6$, (b) $C = -1.6775$, (c) $C = -1.6795$.

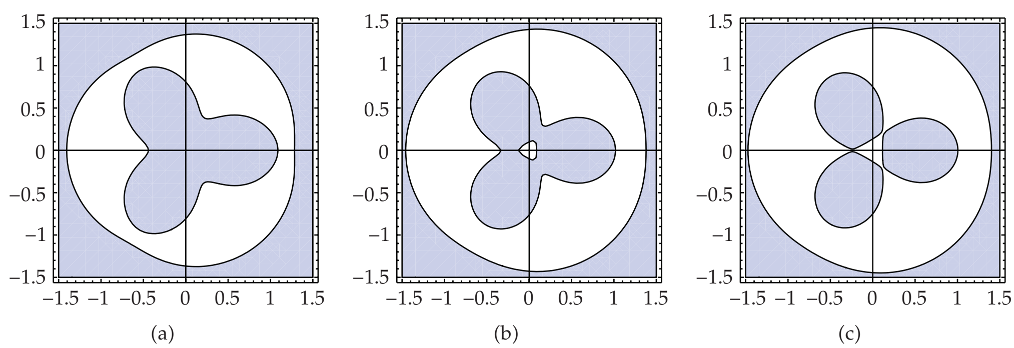


Figure 9: Evolution of zero-velocity curves and Hill's region in the planar ERFBP for $\mu = 1/3$, where shading are permissible areas. (a) $C = -1.7$, (b) $C = -1.75$, (c) $C = -1.765$.

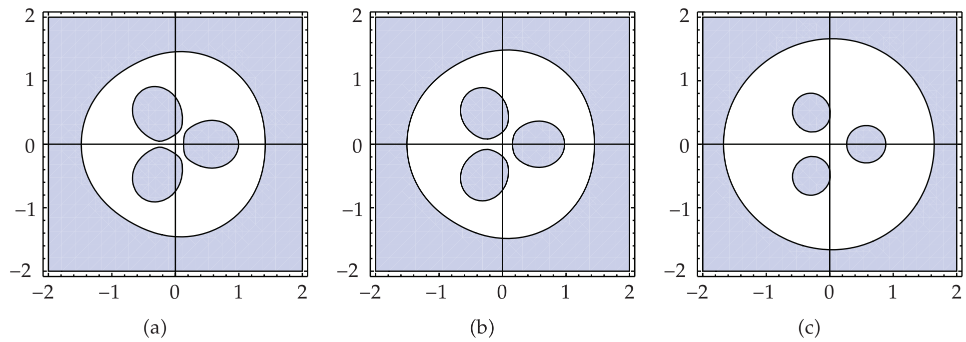


Figure 10: Evolution of zero-velocity curves and Hill's region in the planar ERFBP for $\mu = 1/3$, where shading are permissible areas. (a) $C = -1.775$. (b) $C = -1.8$. (c) $C = -2$.

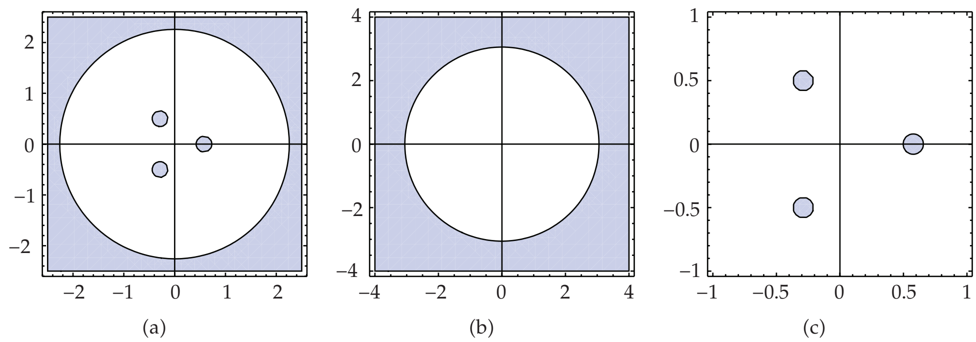


Figure 11: Evolution of zero-velocity curves and Hill's region in the planar ERFBP for $\mu = 1/3$, where shading are permissible areas. (a) $C = -3$, (b) and (c) both figures correspond to $C = -5$ with different window size.

5. Equilibrium Solutions

It is verified that the equilibrium solutions of the system (3.9) or equivalently (3.1) are given by the critical points of the function $\Omega = \Omega(x, y, z)$ or simply they are the solutions of the following system of equations:

$$
\begin{aligned}
&(1-2\mu)\frac{x-\sqrt{3}\mu}{\rho_1^3} + \mu\left(x + \frac{\sqrt{3}}{2}(1-2\mu)\right)\left(\frac{1}{\rho_2^3} + \frac{1}{\rho_3^3}\right) = x,\\
&(1-2\mu)\frac{y}{\rho_1^3} + \mu\left(\frac{y-1/2}{\rho_2^3} + \frac{y+1/2}{\rho_3^3}\right) = y,\\
&-\left(\frac{1-2\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3} + \frac{\mu}{\rho_3^3}\right)z = 0.
\end{aligned}
\tag{5.1}
$$

From the last equation we see that the coordinate z must be zero, so the critical points are restricted to the plane xy, and are given by the solutions of the first two equations.

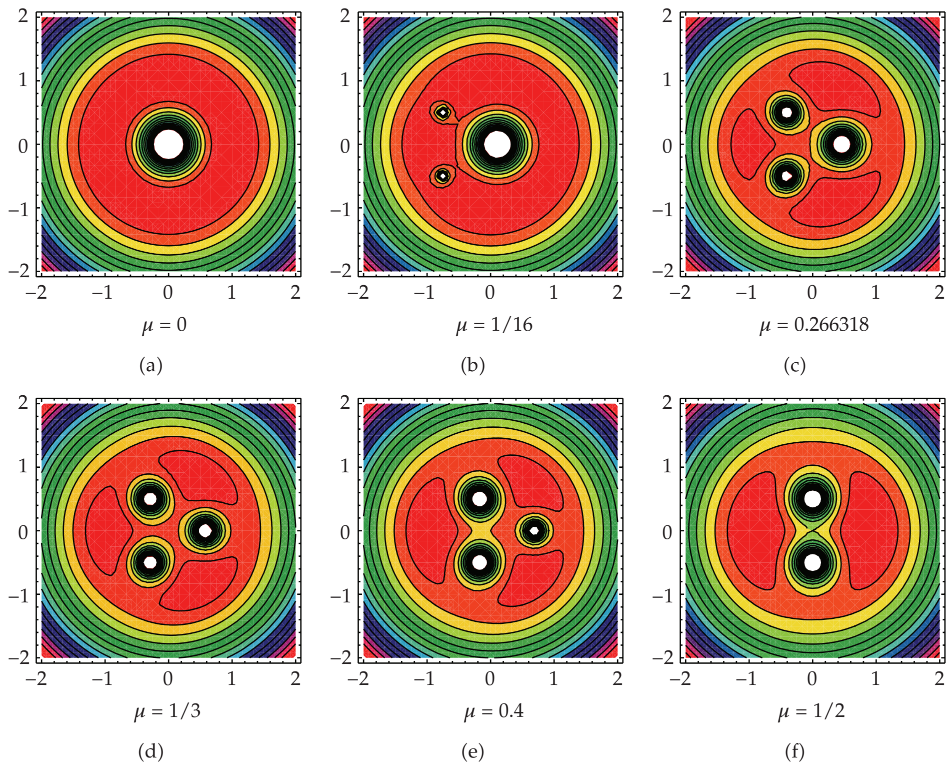


Figure 12: Energy level curves for some values of the parameter μ in the planar case.

It is known (see [17]) that the number of equilibrium solutions of the system (5.1) is 8, 9 or 10 depending on the values of the masses, m_1, m_2 and m_3 which must be positive. Six of them are out of the symmetry axis (i.e., out of the x-axis), therefore on the axis of symmetry we must have 2, 3 or 4. From the analysis done it follows that the number of the equilibrium solutions depends on the parameter μ. This implies that finding the critical points is a non-trivial problem, and this is one of the main differences with the problem studied by Maranhão in his doctoral thesis [1], because there, the number of critical points did not depend on the parameter μ.

The critical points on the axis $y = 0$ are the zeros of the function

$$F_\mu(x) = (1-2\mu)\frac{\left|x-\sqrt{3}\mu\right|}{\left(x-\sqrt{3}\mu\right)^3} + 2\mu\frac{x+\left(\sqrt{3}/2\right)(1-2\mu)}{\rho_2^3} - x, \tag{5.2}$$

where

$$\rho_1 = \left|x-\sqrt{3}\mu\right|, \qquad \rho_2 = \rho_3 = \sqrt{\left(x+\frac{\sqrt{3}}{2}(1-2\mu)\right)^2 + \frac{1}{4}}. \tag{5.3}$$

Table 1: Number of critical points on the x-axis.

$\mu = 0$	$0 < \mu < \mu^*$	$\mu = \mu^*$	$\mu^* < \mu < 1/2$	$\mu = 1/2$
2	2	3	4	3

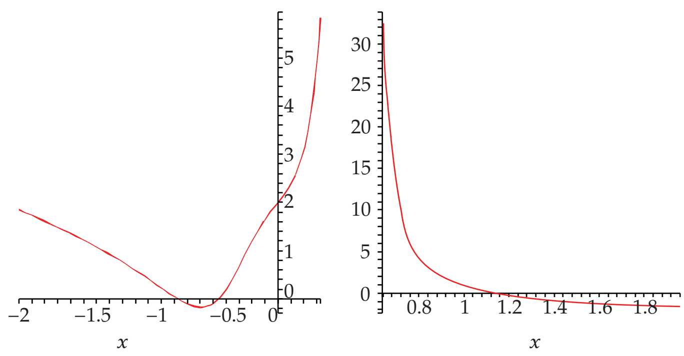


Figure 13: Graph of $F_{1/3}$. (a) $x < \sqrt{3}/3$. (b) $x > \sqrt{3}/3$.

An explicit computation shows that in the limit case problems the number of equilibrium points corresponding to the system (5.1) is as follows.

(a) The function (5.2) with $\mu = 0$ results in

$$F_0(x) = \frac{x}{|x|^3} - x \tag{5.4}$$

whose zeros are $x = -1$ and $x = 1$, and so there are two equilibrium points.

(b) Taking $\mu = 1/2$ in (5.2) becomes

$$F_{1/2}(x) = \frac{x}{\left(x^2 + (1/4)\right)^{3/2}} - x \tag{5.5}$$

with zeros given by $x = -(\sqrt{3}/2)$, $x = 0$ and $x = (\sqrt{3}/2)$. We conclude that there are three equilibrium points.

From numerical simulations we get that the number of critical points along the x–axis is given in Table 1. Observe that $\mu^* := 0.266318$ is the bifurcation value.

In the symmetric case when all the masses are equals (i.e., $\mu = 1/3$) we have that the graph of $F_{1/3}$ is similar to the one shown in Figure 13. As a consequence, there are exactly 4 equilibrium solutions on the x-axis, and therefore there are exactly 10 equilibrium solutions. Of course, $(0, 0, 0)$ is an equilibrium solution.

In general for any equilibrium solution of the form $(x_0, y_0, 0)$, the linearized system (3.9) in the planar case give us that the characteristic polynomial is

$$\begin{aligned}C_A(\lambda) =& \left(\lambda^2 - W_{zz}(x_0, y_0, 0)\right)\lambda^4 + (2 - W_{xx}(x_0, y_0, 0) - W_{yy}(x_0, y_0, 0))\lambda^2 \\ &+ \left(1 + W_{xx}(x_0, y_0, 0) + W_{yy}(x_0, y_0, 0) + W_{xx}(x_0, y_0, 0)W_{yy}(x_0, y_0, 0)\right. \\ &\left. - W_{xy}^2(x_0, y_0, 0)\right),\end{aligned} \tag{5.6}$$

whose roots are

$$\lambda = \pm\sqrt{W_{zz}(x_0, y_0, 0)}, \qquad \lambda = \pm\frac{1}{2}\sqrt{\rho^{\pm}}, \tag{5.7}$$

where $\rho^{\pm}$is given by

$$\rho^{\pm} = -4 + 2(a + c) \pm 2\sqrt{(a - c)^2 + 4b^2 - 8(a + c)} \tag{5.8}$$

with $a = W_{xx}(x_0, y_0, 0)$, $b = W_{xy}(x_0, y_0, 0)$ and $c = W_{yy}(x_0, y_0, 0)$. A very simple result is the following.

Lemma 5.1. *The roots of $p(\rho) = \rho^2 + A\rho + B$ are real and negative if and only if $A > 0$, $B > 0$ and $\Delta = A^2 - 4B \geq 0$.*

Associating to our characteristic polynomial (5.6) we have

$$\begin{aligned}&A = 2 - W_{xx}(x_0, y_0, 0) - W_{yy}(x_0, y_0, 0), \\ &B = 1 + W_{xx}(x_0, y_0, 0) + W_{yy}(x_0, y_0, 0) + W_{xx}(x_0, y_0, 0)W_{yy}(x_0, y_0, 0) - W_{xy}^2(x_0, y_0, 0).\end{aligned} \tag{5.9}$$

Now, we remark that

$$W_{zz}(x_0, y_0, 0) = -\left[\frac{1 - 2\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3} + \frac{\mu}{\rho_3^3}\right] < 0. \tag{5.10}$$

Consequently we have the following result:

Corollary 5.2. *In the spatial ERFBP for any equilibrium solution $(x_0, y_0, 0)$ we have at least two pure imaginary eigenvalues associated to the linear part, which are given by $\lambda = \pm\sqrt{-W_{zz}(x_0, y_0, 0)}i$.*

From this corollary we deduce that to study the nonlinear stability in the Lyapunov sense of each equilibrium solution of the spatial ERFBP is not a simple problem, because we need to take into account the existence or not of resonance in each situation. Leandro in [17] studied the spectral stability in some situations (according to the localization of the equilibrium solution along the symmetry-axis). In a future work we intend to study the Lyapunov stability of each equilibrium.

5.1. Analysis of the Symmetrical Case, $\mu = 1/3$

As we have said previously in the symmetrical case (i.e., $\mu = 1/3$) there are 10 equilibrium solutions and one of them is $(0, 0, 0)$. Here we have $\rho_1 = \rho_2 = \rho_3 = 1/\sqrt{3}$, $a = b = 3\sqrt{3}/2$ and $c = 0$. Consequently, the characteristic roots are

$$\begin{aligned} \lambda_1 &= -\sqrt{3\sqrt{3}}i, & \lambda_2 &= \sqrt{3\sqrt{3}}i, \\ \lambda_3 &= -\frac{1}{2}\sqrt{6\sqrt{3} - 4 + 4\sqrt{-6\sqrt{3}}}, & \lambda_4 &= \frac{1}{2}\sqrt{6\sqrt{3} - 4 + 4\sqrt{-6\sqrt{3}}}, \\ \lambda_5 &= -\frac{1}{2}\sqrt{6\sqrt{3} - 4 - 4\sqrt{-6\sqrt{3}}}, & \lambda_6 &= \frac{1}{2}\sqrt{6\sqrt{3} - 4 - 4\sqrt{-6\sqrt{3}}}. \end{aligned} \tag{5.11}$$

Therefore, we have the following result.

Corollary 5.3. *In the symmetrical spatial ERFBP the equilibrium solution* $(0, 0, 0)$ *is unstable in the Lyapunov sense.*

In general, it is possible to prove that the equilibrium solutions on the x–axis are $x_1 = -0.9351859666722429$, $x_2 = -0.23895830919534947$ and $x_3 = 1.1799984048894328$, and by symmetry it follows:

Corollary 5.4. *In the symmetrical spatial ERFBP all the equilibrium solutions are unstable in the Lyapunov sense.*

According to [17] we have the following corollary.

Corollary 5.5. *In the symmetrical planar ERFBP all the equilibrium solutions are unstable in the Lyapunov sense.*

6. Continuation of Periodic Solutions in the Planar Case

In this section we prove the existence of periodic solutions in the ERFBP for μ sufficiently small in the planar case and by the use of the Lyapunov Center Theorem when μ is close to 1/2. In order to find periodic orbits of our problem we will use the continuation method developed by Poincaré which is one of the most frequently used methods to prove the existence of periodic orbits in the planar circular restricted three-body problem (see [15]). This method has been also used by other authors in different problems. In Meyer and Hall [5], we find a good discussion of the Poincaré continuation method to different n-body problem (see also [18]).

In our approach we will continue circular and elliptic solutions of the Kepler problem with the body fixed in the origin of the system with mass 1. We know that all the orbits of the Kepler problem with angular momentum zero are collision orbits with the origin. We assume that the angular momentum is not zero and we study the orbits that have positive distance of $(-\sqrt{3}/2, 1/2)$ and $(-\sqrt{3}/2, -1/2)$. In the following lemma we resume the kind of orbits that we will consider.

Lemma 6.1. *Fixed $a > 0$ there exists a finite number of elliptic orbits with semi-major axis a, such that its trajectories are periodic in the rotating system and pass through the singularity of the other primaries $(-\sqrt{3}/2, 1/2)$ or $(-\sqrt{3}/2, -1/2)$.*

The proof of this lemma can be found in [19].

6.1. Continuation of Circular Orbits

In this section we show that circular solutions of the unperturbed Kepler problem can be continued to periodic solutions of the ERTBP for small values of μ. We introduce the polar coordinates given as $x = r\cos\theta, y = r\sin\theta$, thus $\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$ and $\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$. So, $\dot{X} = \dot{r}\cos\theta - r(\dot{\theta}+1)\sin\theta$ and $\dot{Y} = \dot{r}\sin\theta + r(\dot{\theta}+1)\cos\theta$, consequently $X^2 + Y^2 = \dot{r}^2 + r^2(\dot{\theta}+1)^2$ and $yX - xY = -r^2(\dot{\theta}+1)$. Thus, the Hamiltonian (3.8) now is

$$H = \frac{\dot{r}^2 + r^2(\dot{\theta}+1)^2}{2} - r^2(\dot{\theta}+1) - V(r,\theta), \tag{6.1}$$

where

$$V(r,\theta) = \frac{1-2\mu}{\rho_1} + \mu\left(\frac{1}{\rho_2} + \frac{1}{\rho_3}\right), \tag{6.2}$$

where

$$\begin{gathered}
\rho_1 = \sqrt{\left(r\cos\theta - \sqrt{3}\mu\right)^2 + r^2\sin^2\theta}, \\
\rho_2 = \sqrt{\left(r\cos\theta + \sqrt{3}/2(1-2\mu)\right)^2 + (r\sin\theta - 1/2)^2}, \\
\rho_3 = \sqrt{\left(r\cos\theta + \sqrt{3}/2(1-2\mu)\right)^2 + (r\sin\theta + 1/2)^2}.
\end{gathered} \tag{6.3}$$

The new coordinates are not symplectic. In order to obtain a set of symplectic coordinates (r, θ, R, Θ) we define $R = \dot{r}$ (radial velocity in the sideral system) and $\Theta = r^2(\dot{\theta}+1)$ (angular momentum in the sideral system), then H is

$$H = \frac{R^2 + \Theta/r^2}{2} - \Theta - V(r,\theta). \tag{6.4}$$

When $\mu = 0$ we have that

$$H = \frac{R^2 + \Theta^2/r^2}{2} - \Theta - \frac{1}{r}, \tag{6.5}$$

is the Hamiltonian of the Kepler problem in polar coordinates. So, if μ is a small parameter, the Hamiltonian (3.8) assumes the form

$$H = \frac{R^2 + \Theta^2/r^2}{2} - \Theta - \frac{1}{r} + \mathcal{O}(\mu). \tag{6.6}$$

For $\mu = 0$, the Hamiltonian system associated is

$$\begin{aligned} \dot{r} &= R, \qquad \dot{R} = \frac{\Theta^2}{r^3} - \frac{1}{r^2}, \\ \dot{\theta} &= \frac{\Theta}{r^2} - 1, \qquad \dot{\Theta} = 0. \end{aligned} \tag{6.7}$$

Let $\Theta = c$ be a fixed constant. For $c \neq 1$, the circular orbit $R = 0$, $r = c^2$ is a periodic solution with period $|2\pi c^3/(1 - c^3)|$. Linearizing the r and R equations about this solution gives

$$\dot{r} = R, \qquad \dot{R} = -c^{-6} r, \tag{6.8}$$

which has solutions of the form $\exp(\pm i t/c^3)$, and so the nontrivial multipliers of the circular orbits are $\exp(\pm i 2\pi/(1 - c^3))$ which are not $+1$, provided $1/(1 - c^3)$ is not an integer. Thus we have proved the following theorem (see details in [5]).

Theorem 6.2. *If $c \neq 1$ and $1/(1 - c^3)$ is not an integer, then the circular orbits of the Kepler problem in rotating coordinates with angular momentum c can be continued into the equilateral restricted four body problem for small values of μ.*

6.2. Continuation of Elliptic Orbits

In Section 3, we saw that the ERFBP has the S-symmetry which when exploited properly proves that some elliptic orbits can be continued from the Kepler problem. The main idea is given in the following lemma, which is a consequence of the uniqueness of the solution of the differential equations and the symmetry of the problem.

Lemma 6.3. *A solution of the equilateral restricted problem which crosses the line of syzygy (the x-axis) orthogonally at a time $t = 0$ and later at a time $t = T/2 > 0$ is T-periodic and symmetric with respect to the line syzygy.*

That is, if $x(t)$ and $y(t)$ is a solution of the equilateral restricted four body problem such that $y(0) = \dot{x}(0) = y(T/2) = \dot{x}(T/2) = 0$, where $T > 0$, then this solution is T-periodic and symmetric with respect to the x-axis.

In Delaunay variables (l, g, L, G), an orthogonal crossing of the line of sizygy at a time t_0 is

$$l(t_0) = n\pi, \quad g(t_0) = m\pi, \quad n, m \text{ integers}. \tag{6.9}$$

These equations will be solved using the Implicit Function theorem to yield the following theorem (see details in [5]).

Theorem 6.4. *Let m, k be relatively prime integers and $T = 2\pi m$. Then the elliptic T-periodic solution of the Kepler problem in rotating coordinates which satisfies*

$$l(0) = \pi, \qquad g(0) = \pi, \qquad L^3(0) = \frac{m}{k} \tag{6.10}$$

and does not go through $(-\sqrt{3}/2, 1/2)$ and $(-\sqrt{3}/2, -1/2)$ can be continued into the equilateral restricted four body problem for small μ. This periodic solution is symmetric with respect to the line of syzygy.

Proof. The Hamiltonian of the ERFBP in Delaunay coordinates for μ sufficiently small is

$$H = -\frac{1}{2L^2} - G + \mathcal{O}(\mu), \tag{6.11}$$

and the equations of motion are

$$\begin{aligned} \dot{l} &= \frac{1}{L^3} + \mathcal{O}(\mu), \qquad & \dot{L} &= 0 + \mathcal{O}(\mu), \\ \dot{g} &= -1 + \mathcal{O}(\mu), \qquad & \dot{G} &= 0 + \mathcal{O}(\mu). \end{aligned} \tag{6.12}$$

Let $L_0^3 = m/k$, and let $l(t, \Lambda, \mu)$, $g(t, \Lambda, \mu)$, $L(t, \Lambda, \mu)$ and $G(t, \Lambda, \mu)$ be the solution which goes through $l = \pi$, $g = \pi$, $L = \Lambda$, G arbitrary at $t = 0$; so, it is a solution with an orthogonal crossing of the line of syzygy at $t = 0$.

From (6.12) $l(t, \Lambda, 0) = t/\Lambda^3 + \pi$, $g(t, \Lambda, 0) = -t + \pi$. Thus, $l(T/2, L_0, 0) = (1 + k)\pi$ and $g(T/2, L_0, 0) = (1 - m)\pi$, and so when $\mu = 0$, this solution has another orthogonal crossing at time $T/2 = m\pi$. Also,

$$\det \begin{pmatrix} \dfrac{\partial l}{\partial t} & \dfrac{\partial l}{\partial \Lambda} \\ \\ \dfrac{\partial g}{\partial t} & \dfrac{\partial g}{\partial \Lambda} \end{pmatrix}_{t=T/2,\ L=L_0,\ \mu=0} = \det \begin{pmatrix} \dfrac{k}{m} & -3\pi \left(\dfrac{k^4}{m}\right)^{1/3} \\ \\ -1 & 0 \end{pmatrix} \neq 0. \tag{6.13}$$

Thus, the theorem follows by the Implicit Function theorem. □

6.3. Application of the Lyapunov Center Theorem

For $\mu = 1/2$, we have three equilibrium solutions on the x-axis which are $P_1 = (-\sqrt{3}/2, 0)$, $P_2 = (0, 0)$ and $P_3 = (\sqrt{3}/2, 0)$. At the point P_2, the associated eigenvalues are $\pm\sqrt{75 + 8\sqrt{2}}$ and $\pm\sqrt{75 - 8\sqrt{2}}$. Therefore, this equilibrium is unstable and by Lyapunov's Center Theorem (see [5]), we obtain the following theorem.

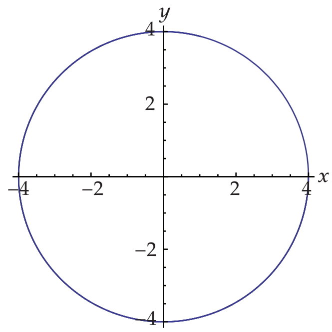


Figure 14: Circular orbit for $c = 2$ with $\mu = 10^{-2}$.

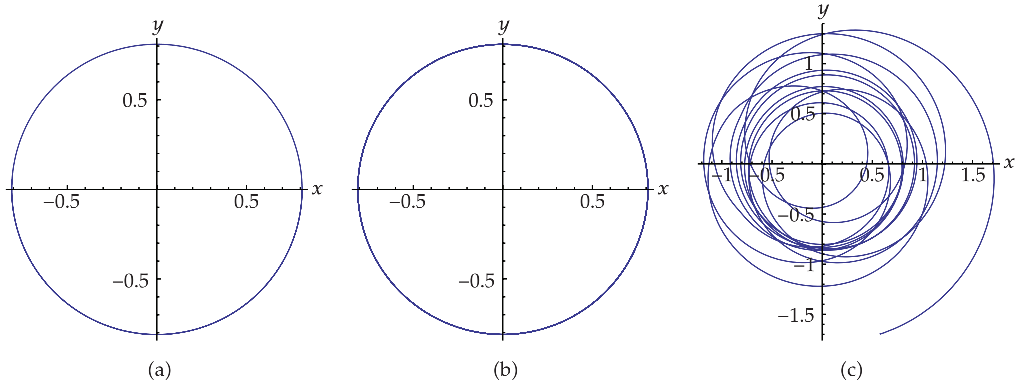


Figure 15: The circular orbit associated to $c = 9/10$ for $\mu = 0$ continued to $\mu = 10^{-4}$ which is not circular for $\mu = 10^{-2}$.

Theorem 6.5. *There exists a one-parameter family of periodic orbits of the ERFBP emanating from the Euler equilibrium (for $\mu = 1/2$). Moreover, when approaching the equilibrium point along the family, the periods tend to $2\pi/\sqrt{-3+8\sqrt{2}}$.*

7. Numerical Results

In the Section 8, we established theorems on the continuation of periodic solutions from the Kepler's problem in rotating coordinates to the ERFBP. In this section, we present some numerical experiments that illustrates the thesis of Theorem 6.2.

To find those circular orbits we first selected an angular momentum c such that $c \neq 1$ and $1/(1-c^3) \notin \mathbb{Z}$. By varying c we generated a set of initial conditions for Kepler problem in rotating coordinates given by the system (3.9) taking $\mu = 0$. We have chosen $y_0 = 0$ and $X_0 = 0$ for all orbits, ensuring that we were following a family of symmetric orbits; we have taken into account the fact that circular orbits satisfy $r = c^2$.

We have noticed that for values of $c = 2, 3, 4, 5, 6, 7, 8, 9, 10$ with $\mu = 10^{-2}$ the orbit is close to the circular orbit, see Figure 14.

However, the circular orbits associated to $c \approx 0, 1$ is close to the circular orbit if $\mu \leq 10^{-4}$, for instance $c = 9/10$ can be continued for μ small and of the order 10^{-4} but not for higher values. The orbits obtained as a consequence of numerical simulations are shown in Figure 15.

8. Conclusions and Final Remarks

The spatial equilateral restricted four-body problem (ERFBP) is considered. This model of four-body problem, we have that three masses, moving in circular motion such that their configuration is always an equilateral triangle, the fourth mass being small and not influencing the motion of the three primaries. In our model we assume that two masses of the primaries m_2 and m_3 are equal to μ and the mass m_1 is $1 - 2\mu$. In a synodical systems of coordinates the dynamics obeys to the system of differential equations

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} + 2\dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z, \end{aligned} \tag{8.1}$$

where

$$\begin{aligned} \Omega = \Omega(x, y, z) &= \frac{1}{2}\left(x^2 + y^2\right) + \frac{1 - 2\mu}{\rho_1} + \frac{\mu}{\rho_2} + \frac{\mu}{\rho_3}, \\ \rho_1 &= \sqrt{\left(x - \sqrt{3}\mu\right)^2 + y^2 + z^2}, \\ \rho_2 &= \sqrt{\left(x + \frac{\sqrt{3}}{2}(1 - 2\mu)\right)^2 + \left(y - \frac{1}{2}\right)^2 + z^2} \\ \rho_3 &= \sqrt{\left(x + \frac{\sqrt{3}}{2}(1 - 2\mu)\right)^2 + \left(y + \frac{1}{2}\right)^2 + z^2}. \end{aligned} \tag{8.2}$$

In Section 4 it is devoted to give the principal qualitative aspect of the restricted problem—the surfaces and curves of zero velocity, several uses of which are discussed. The regions of permitted motion and the location and properties of the equilibrium points are established. We describe the Hill region. The description of the number of equilibrium points is given in Section 5, and in the symmetrical case (i.e., $\mu = 1/3$) we are describing the kind of stability of each equilibrium. In Section 6 the planar case is considered. Here, we prove the existence of periodic solutions as continuation of periodic Keplerian orbits, when the parameter μ is small and when it is close to $1/2$. Finally, in Section 7 we present some numerical experiments that illustrates the thesis of theorem concerning with the continuation of circular orbits of the Kepler problem to the ERFBP with μ small enough.

In a work in progress we intend to continue the study of the ERFBP in different aspects of its dynamics. For example, the behavior of the flow near the singularities (collisions). The study of the escapes solutions (i.e., the unbounded solutions). Existence of chaos under the construction of a shift map. We desired to get periodic solutions under the use of numerical methods.

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