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Research Article

Extended Stokes' Problems for Relatively Moving Porous Half-Planes

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A shear flow motivated by relatively moving half-planes is theoretically studied in this paper. Either the mass influx or the mass efflux is allowed on the boundary. This flow is called the extended Stokes' problems. Traditionally, exact solutions to the Stokes' problems can be readily obtained by directly applying the integral transforms to the momentum equation and the associated boundary and initial conditions. However, it fails to solve the extended Stokes' problems by using the integral-transform method only. The reason for this difficulty is that the inverse transform cannot be reduced to a simpler form. To this end, several crucial mathematical techniques have to be involved together with the integral transforms to acquire the exact solutions. Moreover, new dimensionless parameters are defined to describe the flow phenomena more clearly. On the basis of the exact solutions derived in this paper, it is found that the mass influx on the boundary hastens the development of the flow, and the mass efflux retards the energy transferred from the plate to the far-field fluid.

1. Introduction

After Stokes presented the significant paper [1] on pendulums in which the problems of the impulsive and oscillatory motions of a plane were studied, numerous studies based on his idea were subsequently carried out. In the field of fluid mechanics, flows driven by a plate of an impulsive or oscillatory motion are usually referred to the Stokes' problems. Using the integral transform, one can obtain the exact solution of the Stokes' problems which can pursue the fluid motion not only at larger times but also at small times [2–5]. In addition to the aforementioned papers dealing with the Newtonian fluid, lots of efforts were also made for the flows of the non-Newtonian fluids [6–19]. The effects of the mass influx (or efflux) on the velocity profiles were investigated as well [9, 14, 15, 17]. Though previous papers have made a comprehensive and in-depth contribution to the understanding of

Stokes' problems, studies on the flow driven by more complicated motions of boundaries are comparatively few. Usually, the main concern of changing boundary conditions is to simulate the practical flows occurred in natural environments. For this purpose, Zeng and Weinbaum [20] theoretically studied the Stokes' problems for moving half-planes. They provided the steady-state solutions which can be applied to many practical problems, for example, the flow induced by either earthquakes or fracture of ice sheets. In general, the exact solution consists of two parts, the steady-state solution and the transient solution. Though the transient part usually decays with time, the flow at the very early stage, however, cannot be precisely described by the steady-state solution only, especially for most of the earthquakes and other specific problems usually occurring in a very short period. Recently, Liu [21] reinvestigated the Stokes problems for the cases bounded by modified boundary conditions as well as the finite-depth cases.

Due to above descriptions, the viscous flow generated by relatively moving porous half-planes, which has not been studied yet, is analyzed in this paper. Mathematical formulation is firstly given in Section 2. The detailed derivation for the first and second problems is provided in Section 3. Results and discussions are given in Section 4. Finally, conclusions are made in Section 5.

2. Mathematical Formulation

The extended Stokes' problems for relatively moving porous planes are depicted in Figure 1. The fluid occupies the positive-y domain bounded by a porous plate located at $y = 0$. The fluid is initially at rest everywhere, and then is suddenly driven by the positive-z plate moving with either a constant speed (the first problem) or a harmonic motion (the second problem). The negative-z plate remains at rest for all times. The symbols η, u, and v denote the kinematic viscosity and the flow velocities in the x and y directions, respectively. A constant porous velocity on the plate is denoted as v_0. For this flow system, the governing equation is

$$\frac{\partial u}{\partial t} + v_0 \frac{\partial u}{\partial y} = \eta \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \tag{2.1}$$

and the boundary and initial conditions are

$$\begin{gathered} u(y = 0, z > 0, t) = f(t), \\ u(y = 0, z < 0, t) = 0, \\ u(y \longrightarrow \infty, z, t) = 0, \\ u(y, z \longrightarrow \pm\infty, t) \text{ is finite}, \\ u(y, z, t = 0) = 0, \end{gathered} \tag{2.2}$$

where

$$f(t) = u_0 \tag{2.3}$$

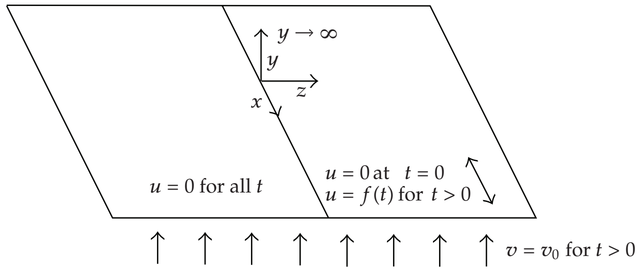

Figure 1: Diagram of the extended Stokes' problems.

is for the first problem, and

$$f(t) = u_0 \cos(\sigma t + \theta) \tag{2.4}$$

indicates the second problem. To solve this problem, if one employs the integral transforms which are often applied to solve the Stokes' problems, it will result in the solution in a quite inconvenient form because the inverse transform cannot be further simplified. Therefore, in this paper, an important mathematical technique which divides the original system equations (2.1) and (2.2) into two subsystems is introduced to overcome this situation

$$\begin{gathered}
\frac{\partial u_1}{\partial t} + v_0 \frac{\partial u_1}{\partial y} = \eta \frac{\partial^2 u_1}{\partial y^2}, \\
u_1(y = 0, t) = \frac{f(t)}{2}, \\
u_1(y \longrightarrow \infty, t) = 0, \\
u_1(y, t = 0) = 0,
\end{gathered} \tag{2.5}$$

and

$$\begin{gathered}
\frac{\partial u_2}{\partial t} = \eta \left(\frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_2}{\partial z^2} \right), \\
u_2(y = 0, z > 0, t) = \frac{f(t)}{2}, \\
u_2(y = 0, z < 0, t) = -\frac{f(t)}{2}, \\
u_2(y \longrightarrow \infty, z, t) = 0, \\
u_2(y, z, t = 0) = 0,
\end{gathered} \tag{2.6}$$

where the total solution is $u = u_1 + u_2$. Note that for the latter subsystem, since the velocity u_2 is antisymmetrical with respect to $z = 0$, an additional condition is required

$$u_2(z = 0) = 0. \tag{2.7}$$

3. Derivation of the Extended Stokes' Problems

The detailed derivation for solutions to the flow systems governed by (2.5), (2.6), and (2.7) will be presented in this section. For the first subsystem, that is, the traditional Stokes' problem associated with the porous effect on the boundary, Liu et al. [5] provided the exact solution

$$u_1 = \frac{u_0}{4}\left[\operatorname{erfc}\left(\frac{y}{2\sqrt{\eta t}} - \frac{v_0\sqrt{t}}{2\sqrt{\eta}}\right) + e^{v_0 y/\eta}\cdot\operatorname{erfc}\left(\frac{y}{2\sqrt{\eta t}} + \frac{v_0\sqrt{t}}{2\sqrt{\eta}}\right)\right], \tag{3.1}$$

for the first problem. They demonstrated that the mass injection on the plate hastens the velocity profile to be uniform with the plate speed. On the contrary, the mass suction retards the energy transferring from the plate to the fluid. Accordingly, one can conclude that the mass inflow is advantageous to the energy transfer and the mass outflow hinders the development of the flow. As for the second problem, the solution is [5]

$$\begin{aligned} u_1 = \frac{u_0}{4}e^{v_0 y/2\eta}\cdot\Re\Bigg\{ & e^{i(\sigma t+\theta+\Pi^-\sqrt{(\sigma/\eta)}y)+\Pi^+\sqrt{(\sigma/\eta)}y}\cdot\operatorname{erfc}\left[\frac{y}{2\sqrt{\eta t}} + \left(\Pi^+ + i\Pi^-\right)\sqrt{\frac{\sigma t}{2}}\right] \\ & + e^{i(\sigma t+\theta-\Pi^-\sqrt{(\sigma/\eta)}y)-\Pi^+\sqrt{(\sigma/\eta)}y}\cdot\operatorname{erfc}\left[\frac{y}{2\sqrt{\eta t}} - \left(\Pi^+ + i\Pi^-\right)\sqrt{\frac{\sigma t}{2}}\right]\Bigg\}, \end{aligned} \tag{3.2}$$

where

$$\Pi^{\pm} = \left(\sqrt{\left(\frac{v_0^2}{4\sigma\eta}\right)^2 + 1} \pm \frac{v_0^2}{4\sigma\eta}\right)^{0.5}. \tag{3.3}$$

It is also found that the effects of mass inflow and outflow on the energy transferred from the plate to the far-field fluid are similar to the phenomena occurred in the first problems.

To solve the second subsystem governed by (2.6) and (2.7), one only needs to solve the solution of the positive-z domain since the flow is antisymmetrical with respect to $z = 0$. By applying both the Laplace transform and the Fourier Sine transform respectively defined as

$$\begin{aligned} \hat{u}(s) &= \int_0^\infty u_2(t)\cdot e^{-st}\,\mathrm{d}s, \\ \tilde{u}(\omega) &= \int_0^\infty \hat{u}(y)\cdot\sin(\omega y)\,\mathrm{d}y, \end{aligned} \tag{3.4}$$

to this system, it yields

$$\tilde{u}_{zz} - \left(\omega^2 + \frac{s}{\eta}\right)\tilde{u} = -\omega\cdot\hat{f}(s). \tag{3.5}$$

From the boundedness of $\tilde{u}$ as z approaches infinity and the boundary condition at $z = 0$, the solutions to $\tilde{u}$ is

$$\tilde{u} = \frac{\omega\eta}{(\omega^2\eta + s)}\left[1 - \exp\left(-\sqrt{\omega^2 + s/\eta} \cdot z\right)\right]\hat{f}(s). \tag{3.6}$$

The solutions to u_2 can be obtained by taking the inverse transforms twice. For the first problem, the solution is

$$u_2 = \frac{u_o}{2}\operatorname{erfc}\left(\frac{y}{2\sqrt{\eta t}}\right) + \frac{u_0}{2\sqrt{\pi}}\int_{t'=0}^{t'=t}\operatorname{erfc}\left(\frac{z}{2\sqrt{\eta t'}}\right)\exp\left(-\frac{y^2}{4\eta t'}\right)\mathrm{d}\left(\frac{y}{\sqrt{\eta t'}}\right). \tag{3.7}$$

For the second problem, it reads

$$\begin{aligned} u_2 &= \frac{u_0\sqrt{\sigma}y}{4\sqrt{\pi\eta}}\int_0^{\sigma t}\alpha^{-1.5}\cos(\sigma t - \alpha + \theta)\exp\left(-\frac{\sigma y^2}{4\alpha\eta}\right)\mathrm{d}\alpha \\ &\quad + \frac{u_0\sqrt{\sigma}z}{2\sqrt{\pi^3\eta}}\int_0^{\infty}\int_0^{\sigma t}\sin\left(\frac{\sqrt{\sigma}\omega y}{\sqrt{\eta}}\right)\cdot G_1(\omega, \sigma t - \alpha)\cdot G_2\left(\omega, \alpha, \sqrt{\frac{\sigma}{\eta}}z\right)\mathrm{d}\alpha\,\mathrm{d}\omega, \end{aligned} \tag{3.8}$$

where

$$\begin{gathered} G_1(\omega, \beta) = \frac{\omega}{\omega^4 + 1}\left[\sin\theta\left(e^{-\omega^2\beta} - \cos\beta + \omega^2\sin\beta\right) - \cos\theta\left(\sin\beta + \omega^2\cos\beta - \omega^2 e^{-\omega^2\beta}\right)\right], \\ G_2(\omega, \beta, Z) = \beta^{-1.5}\exp\left(-\omega^2\beta - \frac{Z^2}{4\beta}\right). \end{gathered} \tag{3.9}$$

4. Results

Before analyzing the solutions derived in the previous section, dimensionless parameters are required to enhance the understanding of the flow. Hence, for the first problem, applying the following dimensionless parameters:

$$U = \frac{u}{u_0}, \qquad Y = \frac{u_0}{\eta}y, \qquad Z = \frac{u_0}{\eta}z, \qquad V_w = \frac{v_0}{u_0}, \qquad T = \frac{u_0^2}{\eta}t, \tag{4.1}$$

and combining (3.1) and (3.7) together, the total solution is

$$\begin{aligned} U &= \frac{1}{4}\left[\operatorname{erfc}\left(\frac{Y}{2\sqrt{T}} - \frac{V_w\sqrt{T}}{2}\right) + e^{V_w Y}\cdot\operatorname{erfc}\left(\frac{Y}{2\sqrt{T}} + \frac{V_w\sqrt{T}}{2}\right)\right] \\ &\quad + \frac{1}{2\sqrt{\pi}}\int_{Y/\sqrt{T}}^{\infty}\operatorname{erf}\left(\frac{Z\alpha}{2Y}\right)\exp\left(-\frac{\alpha^2}{4}\right)\mathrm{d}\alpha \quad \forall Z. \end{aligned} \tag{4.2}$$

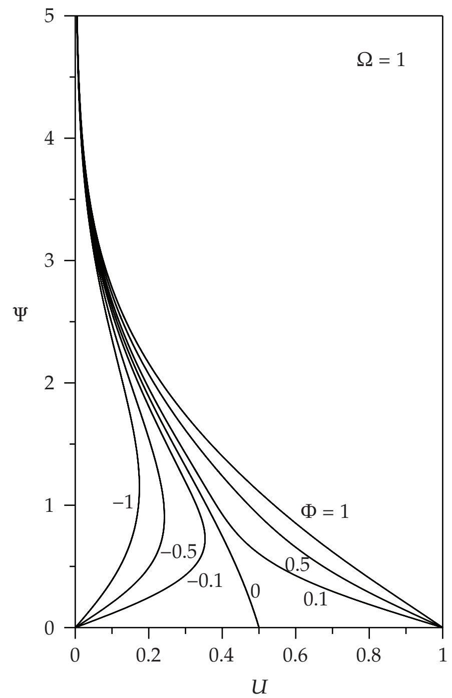

Figure 2: Velocity profiles at various values of Φ for the case Ω = 1 (the first problem).

If one further takes the new set of parameters

$$(\Psi, \Phi, \Omega) = \left(\frac{Y}{\sqrt{T}}, \frac{Z}{\sqrt{T}}, V_w\sqrt{T}\right), \tag{4.3}$$

into account, the total solution equation (4.2) becomes

$$U = \frac{1}{4}\left[\operatorname{erfc}\left(\frac{\Psi - \Omega}{2}\right) + e^{\Psi \cdot \Omega} \cdot \operatorname{erfc}\left(\frac{\Psi + \Omega}{2}\right)\right] + \frac{1}{2\sqrt{\pi}}\int_{\Psi}^{\infty} \operatorname{erf}\left(\frac{\Phi}{2\Psi}\alpha\right)\exp\left(-\frac{\alpha^2}{4}\right)d\alpha. \tag{4.4}$$

For viscous flows, spatial and time variables in the solution are often coupled in certain forms. Such a solution is also know as a similarity solution in which the solution profile with respect to the spatial coordinate is similar at all times. In the present problem, the similarity not only appears in the Y and Z variables but also in the porous velocity V_w. With the idea of the similarity; four variables (Y, Z, T, V_w) in (4.2) can be reduced to three dependent variables (Ψ, Φ, Ω), as shown in (4.4). Figure 2 shows the velocity profiles for various values of Φ under the condition $\Omega = 1$. When the flow develops (i.e., T goes large), the velocity distribution at the far end of $Z \gg 0$ will gradually approach that of the traditional Stokes' first problem. This implies that the effects of the still plate on the velocity distribution will become weaker

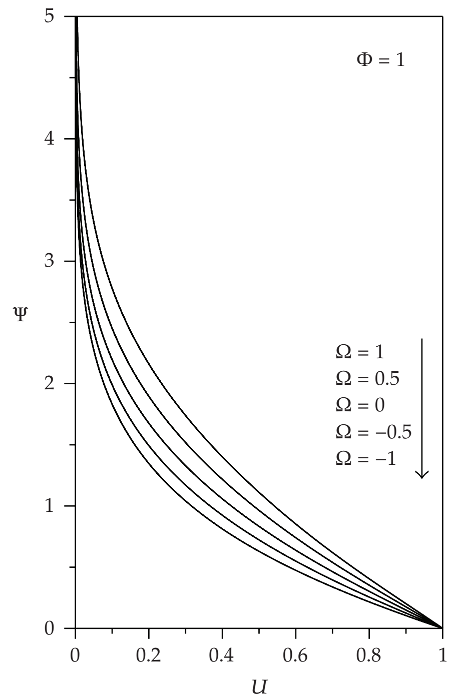

Figure 3: Velocity profiles of various values of Ω for the case $\Phi = 1$ (the first problem).

for larger values of Φ. Similarly, the velocity profile will approach zero at the other far end ($Z \ll 0$), where the influence from the moving plate is weak as well. The porous effects on the velocity profiles are shown in Figure 3. The effects of different porous velocities on the flow for $\Phi = 1$ are plotted. It is found that the mass influx (positive Ω) hastens the development of the flow, while the mass efflux (negative Ω) retards the energy transferred from the moving plate to the far-field fluid.

For the second problem, the total dimensionless solution is obtained by combining (3.2) and (3.8) together

$$\begin{aligned} U = {} & \frac{1}{4} e^{(V_w/2)Y} \cdot \Re \Bigg\{ e^{i(T+\theta+\overline{\Pi}^{-}Y)+\overline{\Pi}^{+}Y} \cdot \operatorname{erfc}\left[\frac{Y}{2\sqrt{T}} + \left(\overline{\Pi}^{+} + i\overline{\Pi}^{-}\right)\sqrt{\frac{T}{2}} \right] \\ & \qquad + e^{i(T+\theta-\overline{\Pi}^{-}Y)-\overline{\Pi}^{+}Y} \cdot \operatorname{erfc}\left[\frac{Y}{2\sqrt{T}} - \left(\overline{\Pi}^{+} + i\overline{\Pi}^{-}\right)\sqrt{\frac{T}{2}} \right] \Bigg\} \\ & \mp \frac{Y}{4\sqrt{\pi}} \int_0^T \alpha^{-1.5} \cos(T - \alpha + \theta) \exp\left(-\frac{Y^2}{4\alpha} \right) \mathrm{d}\alpha \\ & + \frac{Z}{2\sqrt{\pi^3}} \int_0^\infty \int_0^T \sin(\omega Y) \cdot G_1(\omega, T-\alpha) \cdot G_2(\omega, \alpha, Z) \mathrm{d}\alpha \, \mathrm{d}\omega \quad \text{for the} \pm Z \text{ domain,} \end{aligned} \tag{4.5}$$

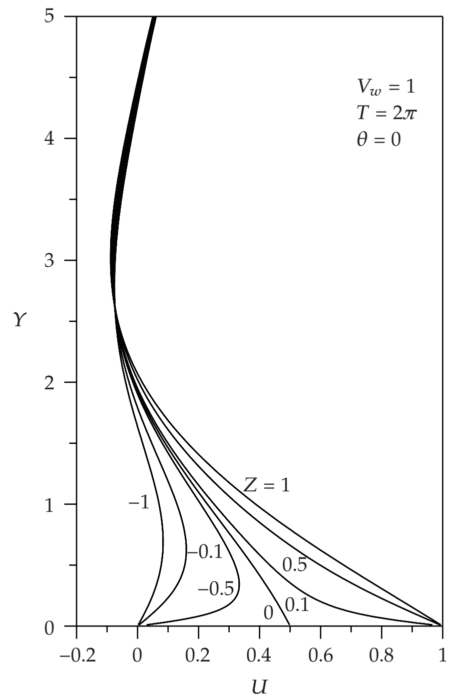

Figure 4: Velocity profiles at various Z sections for the case $V_w = 1$, $T = 2\pi$, and $\theta = 0$ (the second problem).

where

$$U = \frac{u}{u_0}, \qquad Y = \sqrt{\frac{\sigma}{\eta}}y, \qquad Z = \sqrt{\frac{\sigma}{\eta}}z, \qquad T = \sigma t,$$
$$V_w = \frac{v_0}{\sqrt{\sigma\eta}}, \qquad \overline{\Pi}^{\pm} = \left(\sqrt{\frac{V_w^4}{16} + 1} \pm \frac{V_w^2}{4}\right)^{0.5}, \tag{4.6}$$

and G_1 and G_2 are given in (3.9). From (4.5), it is noted that there exists no similarity solution for the second problem. Figure 4 shows the velocity profiles at various Z sections at $T = 2\pi$ for the cosine oscillation ($\theta = 0$) with the porous velocity $V_w = 1$. Similar to the results of the first problem, the velocity distribution at the far end in the positive-Z direction will gradually approach the solution of the traditional Stokes' second problem when T becomes large. To investigate the porous effects, Figure 5 displays the velocity profiles affected by different porous velocities at $Z = 1$ at $T = 2\pi$ for the cosine oscillation. It is also obvious that the larger influx velocity leads to the faster development of the flow.

5. Concluding Remarks

A still fluid suddenly driven by relatively moving porous half-planes is theoretically analyzed in this paper. In addition to the integral transforms, it is impossible to acquire the exact solution without using an important technique which divides the original problem into two

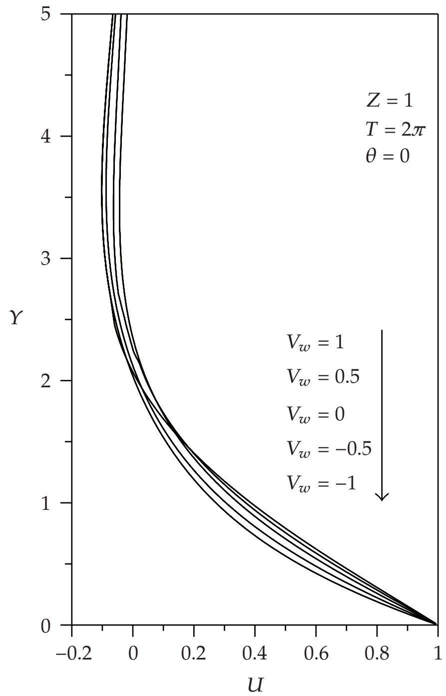

Figure 5: Velocity profiles of various values of V_w for the case $Z = 1$, $T = 2\pi$, and $\theta = 0$ (the second problem).

subsystems. The solution to the first subsystem is equivalent to half of the solution of the traditional Stokes' problem. As for the second subsystem, the velocity profiles in the whole domain can be obtained by solving the flow in the positive-z domain since the flow is antisymmetrical to $z = 0$. Based on present solutions, it is found that the mass influx on the boundary hastens the development of the flow, and the mass efflux retards the energy transferred from the plate to the far-field fluid.

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