Research Article

Synchronization of Discrete-Time Chaotic Systems in Bandlimited Channels

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Over the last couple of decades, many methods for synchronizing chaotic systems have been proposed with communications applications in view. Yet their performance has proved disappointing in face of the nonideal character of usual channels linking transmitter and receiver, that is, due to both noise and signal propagation distortion. Here we consider a discrete-time master-slave system that synchronizes despite channel bandwidth limitations and an allied communication system. Synchronization is achieved introducing a digital filter that limits the spectral content of the feedback loop responsible for producing the transmitted signal.

1. Introduction

Since Pecora and Carroll's seminal work [1], much has been written about the potential usefulness of chaotic synchronization in communication systems (e.g., [2–8]). Much of the impetus for chaotic communications has been the rationale whereby both analog and digital chaotic modulations would have the same properties as conventional spread spectrum techniques [9].

The inherent wideband character of chaotic signals, however, becomes a problem when the communication channel imposes bandwidth limitations. Because of the receiver's nonlinear nature, if any spectral component is amiss, all spectral components at the receiver become affected. Even minute gain or phase changes are enough to fully hinder synchronism [6, 10].

Many papers have approached the problem by trying to eliminate channel distortion via channel inversion [11–13]. Reasonable results have only been attained when the

channel filters are represented by low-order filters. Furthermore, these methods usually unrealistically assume perfect channel frequency response knowledge at the receiver [6].

Rulkov and Tsimring [6] and Eisencraft and Gerken [14] independently proposed a method for synchronizing transmitter and receiver using chaotic signals under bandwidth limitations. The basic idea is to employ an identical filter at both the transmitter and the receiver so as to circumvent channel impairments. An analog circuit implementation was proposed by [6].

Here we extend this method to discrete-time dynamical systems [8, 15–17]. Much of the interest in this approach lies in the ease of employing Digital Signal Processors (DSPs) or microcontrollers for their implementation. These solutions have become ubiquitous in present day technology.

This paper is organized as follows. In Section 2, the Wu and Chua [4] synchronization method is extended to discrete-time systems. This is followed by the description of an allied communication system whose performance under bandwidth limitation is simulated. Means for overcoming the former limitations are proposed in Section 3. Finally our main conclusions are summed up in Section 4.

2. Wu and Chua's Synchronization of Discrete-Time Chaotic Systems

Wu and Chua's paper [4] addresses chaotic system synchronization differently from Pecora and Carroll's [1] seminal paper. Instead of using conditional Lyapunov exponents to check the asymptotic stability of the slave system and hence the possibility of synchronism, Wu and Chua propose that the master and slave equations be written in such a way that the dynamics of the synchronization error is simple enough to permit the direct verification of its convergence to zero.

In this section we adapt their method for discrete-time master-slave systems.

Consider two discrete-time systems defined by

$$\mathbf{x}(n+1) = A\mathbf{x}(n) + \mathbf{b} + \mathbf{f}(\mathbf{x}(n)), \tag{2.1}$$

$$\mathbf{y}(n+1) = A\mathbf{y}(n) + \mathbf{b} + \mathbf{f}(\mathbf{x}(n)), \tag{2.2}$$

where $n \in \mathbb{N}$, $\{\mathbf{x}(n), \mathbf{y}(n)\} \subset \mathbb{R}^K$, $\mathbf{x}(n) = [x_1(n), x_2(n), \ldots, x_K(n)]^T$, $\mathbf{y}(n) = [y_1(n), y_2(n), \ldots, y_K(n)]^T$. The real $A_{K \times K}$ matrix and vector $\mathbf{b}_{K \times 1}$ are constants. The function $\mathbf{f}(\cdot)$, $\mathbb{R}^K \to \mathbb{R}^K$ is nonlinear.

The system described by (2.1) is autonomous and is called *master*. The one described by (2.2) depends on $\mathbf{x}(n)$ and is called *slave*.

The synchronization error dynamics between the two systems $\mathbf{e}(n) = \mathbf{y}(n) - \mathbf{x}(n)$, in this case, is given by

$$\mathbf{e}(n+1) = A\mathbf{e}(n). \tag{2.3}$$

They are said *completely synchronized* if $\mathbf{e}(n) \to 0$ as n grows. Consequently, master and slave synchronize completely if the eigenvalues λ_i of A satisfy [18]

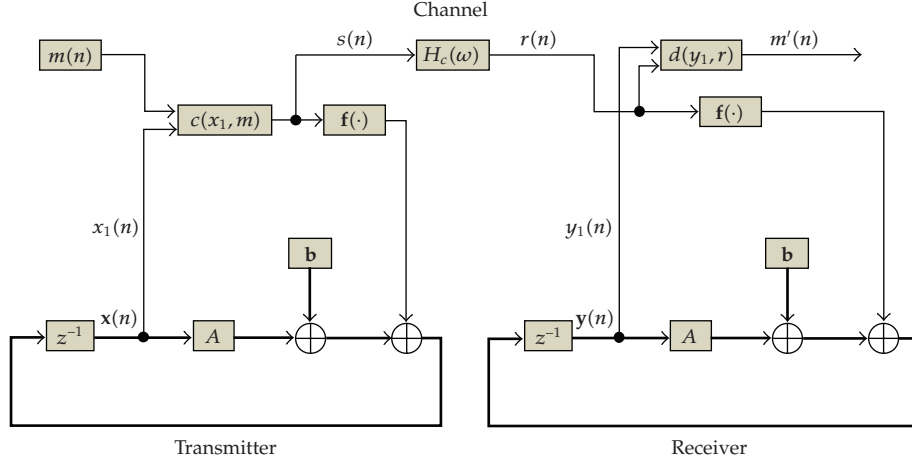$$|\lambda_i| < 1, \quad 1 \le i \le K. \tag{2.4}$$

Figure 1: Block Diagram of Wu and Chua's [4] proposal adapted to discrete-time systems.

Hence, if a system can be written as in (2.1) with A satisfying condition (2.4), it is easy to set up a slave system that synchronizes with it.

Using this synchronization method, Wu and Chua [4] proposed an information transmission system using chaotic signals that lead to no errors when channel conditions are ideal. The discrete-time version of this system is shown in Figure 1.

Here we consider that $\mathbf{f}(\cdot)$ depends solely on the component $x_1(n)$ of $\mathbf{x}(n)$, and that it can be written as

$$\mathbf{f}(\mathbf{x}(n)) = [f(x_1(n)); 0; 0; \ldots; 0]^T. \tag{2.5}$$

The communication channel is represented by a linear system with frequency response $H_C(\omega)$.

In this scheme, the information signal $m(n)$ is coded using the $x_1(n)$ component of the state vector $\mathbf{x}$ via a coding function

$$s(n) = c(x_1(n), m(n)), \tag{2.6}$$

so that the information signal can be decoded using the inverse function

$$m(n) = d(x_1(n), s(n)) = d(x_1(n), c(x_1(n), m(n))). \tag{2.7}$$

The equations governing the global system have the same form as (2.1) and (2.2). The only changes are the arguments of $f(\cdot)$:

$$\mathbf{x}(n+1) = A\mathbf{x}(n) + \mathbf{b} + \mathbf{f}(s(n)),$$

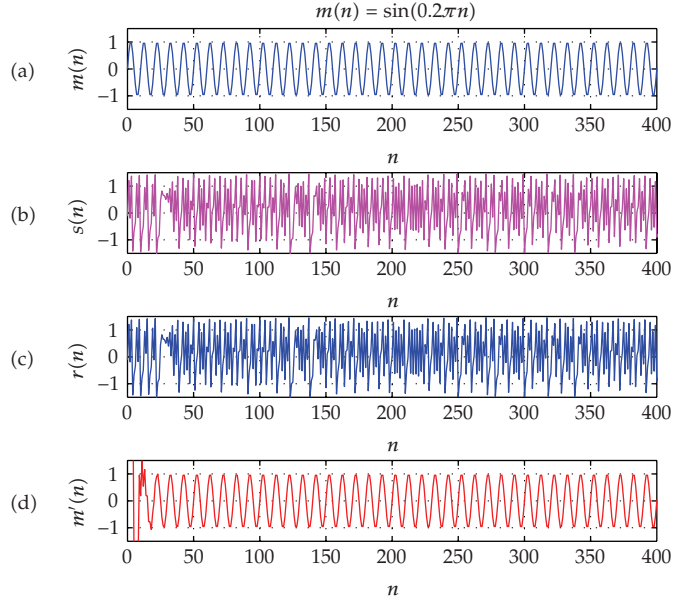$$\mathbf{y}(n+1) = A\mathbf{y}(n) + \mathbf{b} + \mathbf{f}(r(n)), \tag{2.8}$$

Figure 2: Simulations of the communication system shown in **Figure 1** for an ideal channel: (a) message $m(n)$; (b) transmitted signal $s(n)$; (c) received signal $r(n) = s(n)$; (d) recovered message $m'(n)$.

where $r(n)$ is the signal that the channel delivers to the receiver, that is,

$$r(n) = s(n) * h_c(n), \tag{2.9}$$

$h_c(n)$ is the impulse response of the channel and "$*$" represents linear convolution.

In principle, when choosing $c(\cdot, \cdot)$ and $d(\cdot, \cdot)$ one must seek to satisfy

$$s(n) \approx x_1(n) \tag{2.10}$$

for all admissible $m(n)$ because

(i) $s(n)$ is fed back in lieu of $x_1(n)$ at the transmitter. Condition (2.10) imposes that the dynamics is not significantly changed and consequently the system continues to generate chaotic signals;

(ii) also if secure communication is desired, condition (2.10) ensures that $m(n)$ is not immediately apparent from $s(n)$.

For an ideal channel, that is, $H_c(\omega) = 1$ and $s(n) = r(n)$, (2.8) become

$$\mathbf{x}(n + 1) = A\mathbf{x}(n) + \mathbf{b} + f(s(n)),$$

$$\mathbf{y}(n + 1) = A\mathbf{y}(n) + \mathbf{b} + f(s(n)). \tag{2.11}$$
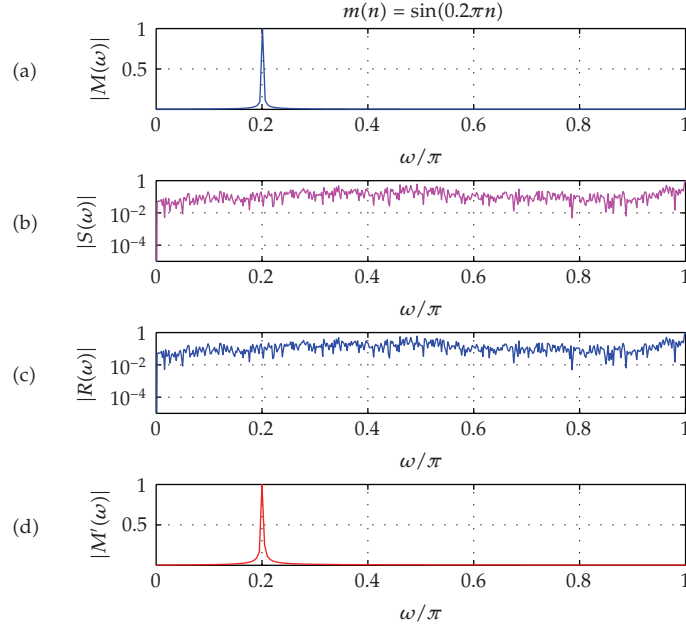
Figure 3: Normalized representation of the signals from Figure 2 in the frequency domain: (a) message; (b) transmitted signal; (c) received signal; (d) recovered message.

Once more, the synchronization error dynamics is given by (2.3) and if condition (2.4) holds, then $\mathbf{y}(n) \rightarrow \mathbf{x}(n)$ and, in particular, $y_1(n) \rightarrow x_1(n)$. Hence, using (2.7),

$$m'(n) = d\big(y_1(n), s(n)\big) \longrightarrow d(x_1(n), s(n)) = m(n). \tag{2.12}$$

Therefore when transmitter and receiver parameters are perfectly matched over an ideal channel, the message is recovered at the receiver without degradation except for a synchronization transient.

To exemplify, consider the three-dimensional generalization of the Hénon map [19, 20]:

$$
\begin{aligned}
x_1(n+1) &= -ax_1^2(n) + x_3(n) + 1, \\
x_2(n+1) &= -bx_1(n), \\
x_3(n+1) &= bx_1(n) + x_2(n).
\end{aligned}
\tag{2.13}
$$

It can be immediately written in the form of (2.1)

$$
\mathbf{x}(n+1) = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ -b & 0 & 0 \\ b & 1 & 0 \end{bmatrix}}_{A} \mathbf{x}(n) + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{b}} + \underbrace{\begin{bmatrix} -ax_1^2(n) \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{f}(\mathbf{x}(n))}. \tag{2.14}
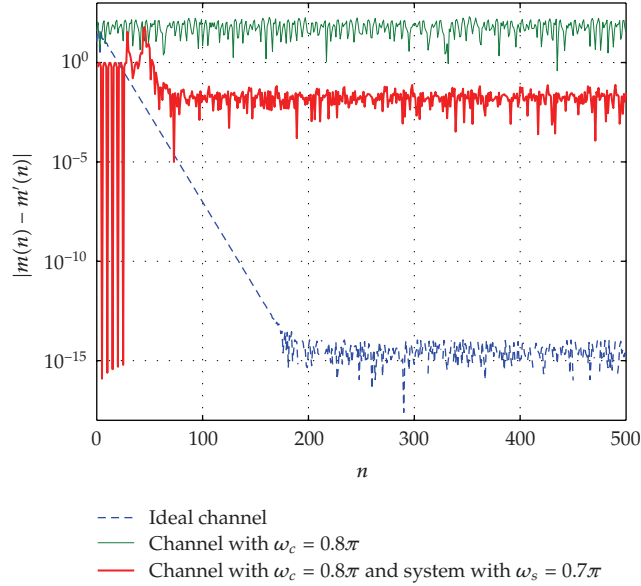$$

Figure 4: Demodulation error $|m(t) - m'(t)|$ for the simulated systems.

Taking $a = 1.07$ and $b = 0.3$ [19], simulations show that, for almost all initial conditions in the unity sphere, (2.13) generates chaotic orbits whose largest Lyapunov exponent is $h_1 \approx 0.23$.

In this case, the eigenvalues of A are

$$\begin{aligned}
\lambda_1 &= 0.4084 + 0.4477i, \\
\lambda_2 &= 0.4084 - 0.4477i, \\
\lambda_3 &= -0.8169,
\end{aligned} \tag{2.15}$$

ensuring the satisfaction of the stability condition in (2.4). This allows assembling the communications system depicted in Figure 1.

In the simulations we chose

$$s(n) = c(x_1(n), m(n)) = x_1(n) + 0.01m(n), \tag{2.16}$$

so that

$$m'(n) = d(y_1(n), r(n)) = 100(r(n) - y_1(n)). \tag{2.17}$$

Furthermore, the transmitted message was taken as

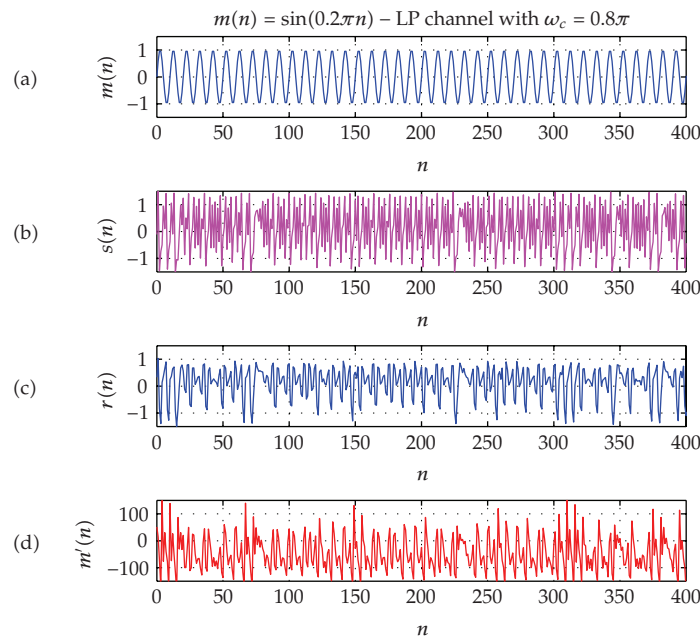$$m(n) = \sin(0.2\pi n). \tag{2.18}$$

Figure 5: Simulations of the communication system shown in **Figure** 1 for an LP FIR channel with cutoff frequency at $\omega_c = 0.8\pi$: (a) message $m(n)$; (b) transmitted signal $s(n)$; (c) received signal $r(n)$; (d) recovered message $m'(n)$. Note that the scale used in (d) is different from the previous ones.
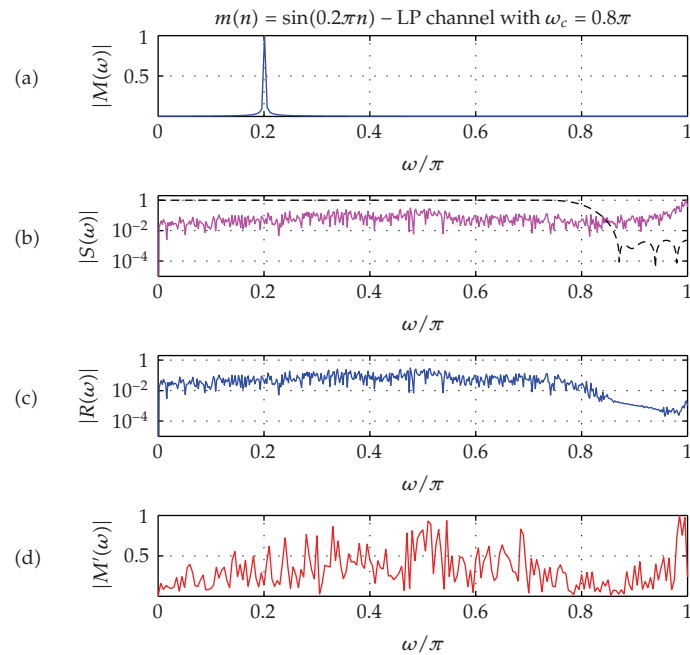

Figure 6: Normalized representation of the spectral components absolute value of signals in **Figure** 5: (a) message; (b) transmitted signal and amplitude frequency response of the channel (dashed line); (c) received signal; (d) recovered message.
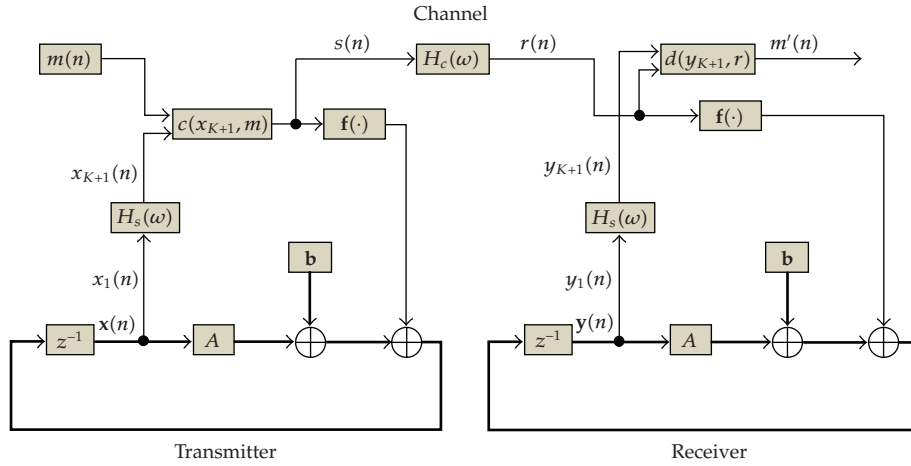
Channel

Figure 7: Proposed discrete-time communication system for bandlimited channels.
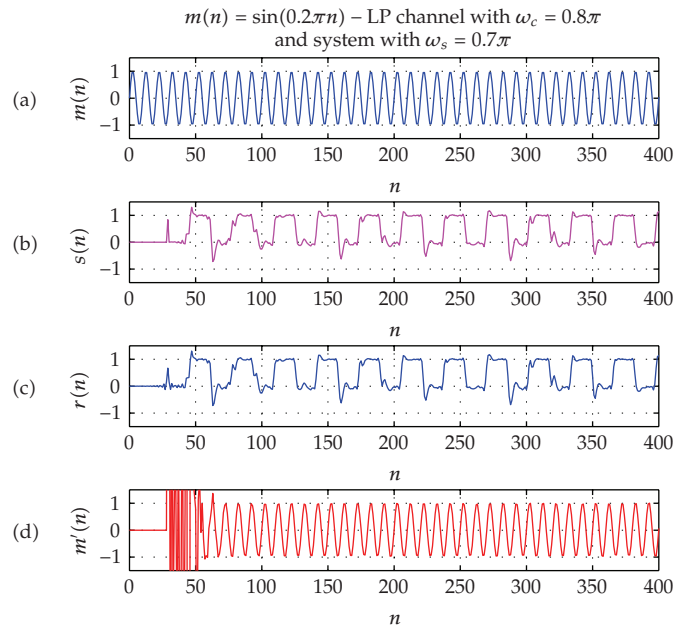

Figure 8: Simulations of the proposed communication system for an LP FIR channel with cutoff frequency at $\omega_c = 0.8\pi$ and LP FIR feedback filter $H_S(\omega)$ with cutoff frequency at $\omega_S = 0.7\pi$: (a) message $m(n)$; (b) transmitted signal $s(n)$; (c) received signal $r(n)$; (d) recovered message $m'(n)$.

Figures 2 and 3 illustrate a realization of the involved signals both in the time and in the frequency domains for an ideal channel. The spectra are normalized relatively to their maximum value.

Perfect recovery of the message once the synchronization is reached can be clearly seen in Figure 2(d) and the spectral spreading due to the chaotic signal modulation is shown in Figure 3(b). Note that the presence of $m(n)$ in the transmitted signal is not apparent neither in the time nor in the frequency domain.

The demodulation error $|m(n) - m'(n)|$ for the ideal channel case is shown by the dashed line in Figure 4. It quickly falls and remains at the numerical precision level of the computer simulation.

This situation changes completely when the channel is not ideal, that is, $H_c(\omega) \neq 1$, $\forall \omega$. In this case, $r(n) \neq s(n)$ in (2.8) and the synchronism is affected so that $m'(n) \nrightarrow m(n)$. As an illustrative example, consider that $H_c(\omega)$ is a linear-phase Low Pass (LP) Finite Impulse Response (FIR) filter of order $N_C = 50$ with cutoff frequency at $\omega_c = 0.8\pi$ [21]. Figures 5 and 6 show a realization of the signals involved in this case. The amplitude frequency response $|H_c(\omega)|$ is shown by the dashed line in Figure 6(b). Just about 20% of the spectrum of the transmitted signal is affected, yet the message is completely lost as seen in Figures 5(d) and 6(d). Note that the bandwidth associated with the original message $m(n)$, concentrated at 0.2π, is practically unaffected.

The error in obtaining $m'(n)$ in this case is shown by the solid thinner line in Figure 4.

This example shows that the system in Figure 1 fails when channel conditions are not ideal. This makes its use impractical, in the same way as it happens with its continuous counterpart, as pointed out by [6, 14].

A mean to circumvent these difficulties is to adjust the spectrum of the transmitted chaotic signal so that it becomes less vulnerable in the presence of a given channel. This idea, proposed for continuous time systems in [6, 14], is extended for the discrete-time case in the next section.

3. Synchronizing Discrete-Time Systems in Band-Limited Channels

A way to combat the harmful effects of a bandlimited channel on the communication system of Figure 1 is to insert a filter $H_S(\omega)$ in the feedback loops of both the transmitter and the receiver so that the total transmitted signal power is contained within the channel bandwidth.

A block diagram of the proposed system is shown in Figure 7.

Considering that the filter $H_S(\omega)$ is an FIR filter of order N_S, its output $x_{K+1}(n)$ can be expressed as

$$x_{K+1}(n) = c_1 x_1(n) + c_2 x_1(n-1) + \cdots + c_{N_S+1} x_1(n - N_S), \tag{3.1}$$

where $c_1, c_2, \ldots, c_{N_S+1}$ are the filter coefficients.

Consequently, the dimensions of the difference equation systems that describes the master and the slave subsystems now are of order $K + N_S$ instead of K.

As an example, consider that the channel $H_C(\omega)$ is an LP FIR filter of order $N_C = 50$ and with cutoff frequency at $\omega_c = 0.8\pi$, as in the previous section. To combat the channel effects, we used linear-phase LP FIR filters $H_S(\omega)$ with cutoff frequency at $\omega_s = 0.7\pi$ of order $N_S = 30$. Figures 8 and 9 show a realization of the signals involved.

The information signal is unaffected by these filters and the message is fully recovered as can be seen from Figures 8(d) and 9(d). The transient time is larger than that of the previous examples due to the additional filtering effects.

The demodulation error in this case is shown by the thicker solid line in Figure 4. After the synchronization transient, the mean error is about 1%.

An important question that must be addressed is whether the generated signals remain chaotic with the introduction of the feedback filter.
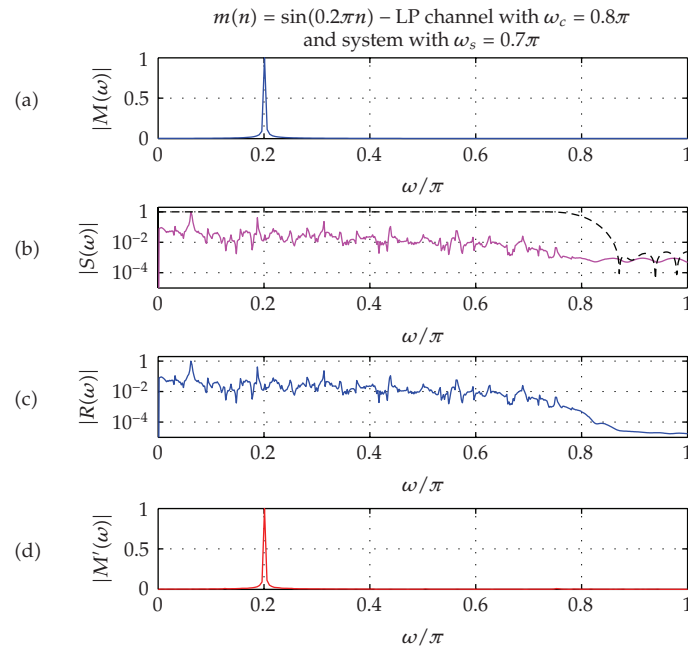
$m(n) = \sin(0.2\pi n)$ – LP channel with $\omega_c = 0.8\pi$
and system with $\omega_s = 0.7\pi$

(a)

(b)

(c)

(d)

Figure 9: Normalized representation of signals in Figure 8 in the frequency domain: (a) message; (b) transmitted signal and frequency response of the channel (dashed line); (c) received signal; (d) recovered message.
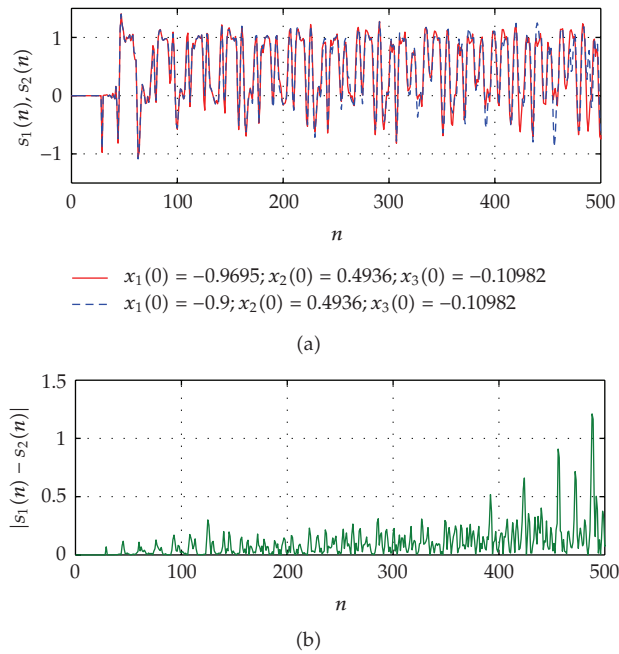

$x_1(0) = -0.9695; x_2(0) = 0.4936; x_3(0) = -0.10982$
$x_1(0) = -0.9; x_2(0) = 0.4936; x_3(0) = -0.10982$

(a)

(b)

Figure 10: (a) Transmitted signals $s(n)$ for two slightly different initial conditions; (b) absolute value of the difference between the signals in (a).

In the example, the transmitted signal $s(n)$ shown in Figure 8(b) presents aperiodic behavior and sensitivity to initial conditions. This last property is illustrated in Figure 10 where signals generated using two slightly different initial conditions diverge from one another, which is evidence of chaotic signal behavior.

In a more general framework, analysis shows that when we write the system equations as in (2.8) including the feedback filter, the $K+N_S$ eigenvalues of A do not depend on the filter coefficients so that synchronism remains unaffected. However, the conditions for the signals to remain chaotic require further investigation. For low-order filters, inadequate coefficient choices may transform an unstable fixed point of the original system into a stable one and thereby destroy the chaotic attractor. These results are the subject of current investigation and lie beyond our intended scope.

4. Conclusions

In this paper, we proposed a discrete-time communication system using chaotic signals for bandlimited channels. Our system is based on the one proposed by [6, 14] who have developed their ideas within a continuous-time framework. Using discrete-time systems based on difference equations makes their implementation quite natural via today's ubiquitously available DSPs and microcontrollers.

We have shown numerical examples based on a three-dimensional generalization of the Hénon map and on linear-phase FIR filters.

It is important to note that introducing filters into feedback loops can change the dynamics of both the master and the slave systems in relevant ways. The conditions on filter coefficients for which the systems remain chaotic signal generators have the status of a nontrivial open problem. Work is in progress to simulate the use of this system to transmit digital messages and to assess its symbol error rate performance for different channels and feedback filter bandwidths.

Acknowledgments

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