Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 841017, 17 pages
doi:10.1155/2009/841017

Research Article

Robust Optimal Design of Beams Subject to Uncertain Loads

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Received 8 February 2009; Revised 29 June 2009; Accepted 15 October 2009

Recommended by Jerzy Warminski

Optimality conditions are derived for the robust optimal design of beams subject to a combination of uncertain and deterministic transverse and boundary loads using a variational min-max approach. The potential energy of the beam is maximized to compute the worst case loading and minimized to determine the optimal cross-sectional shape which results in coupled nonlinear differential equations for the unknown functions except for the case of a variable width beam. The uncertain component of the transverse load acting on the beam is not known a priori resulting in load uncertainty subject only to an norm constraint. Similarly the optimal area function is subject to a volume constraint leading to an isoperimetric variational problem. The min-max approach leads to robust optimal designs which are not susceptible to unexpected load variations as it occurs under operational conditions. The solution methodology is illustrated for the variable width beam by obtaining analytical results for several cases. The efficiency of the optimal designs is computed with respect to a uniform beam under worst case loading taking the maximum deflection as the quantity for comparison. It is observed that the optimal shapes are more than 70% efficient for the examples given in this study.

1. Introduction

Under operational conditions, a structure is usually subjected to uncertainties which may arise from fluctuation and scatter of external loads, environmental conditions, boundary conditions, and geometrical and material properties. However quite often, design uncertainties arise from incomplete knowledge and unpredictable nature of the load under operational conditions.

In the present study optimality conditions are derived for the optimal robust design of a beam subject to a transverse load which has unknown and known parts. Moreover uncertain moments and/or shear forces may be acting at the endpoints of the beam. In conventional design, it is common practice to neglect the load uncertainties when analyzing a structure and assessing the structural performance on the basis of a deterministic model. To compensate for performance variability caused by load variations, a safety factor is introduced magnitude of which correlates with the level of uncertainty with higher levels leading to larger safety factors. However, the safety factors specified may be either too conservative or too small to compensate for the lack of knowledge of operational loads. Efficiency and reliability of the structure can be improved by taking the load uncertainties into consideration in the design process leading to a design which is robust under load variations. This approach is equivalent to optimizing the design for worst case loading and leads to an optimization problem for the area function and to an antioptimization problem for the load function. The final design is robust in the sense that the sensitivity of the beam to load variations is substantially reduced. This is accomplished by maximizing its potential energy over loading while minimizing it with respect to its cross-sectional shape. Mathematically these results in a min-max optimal design problem can be studied by variational calculus.

Finding the worst case loading on a structure corresponds to an antioptimization problem the examples of which can be found [1, 2] where optimization under uncertain bending and buckling loads is studied. Designing robust structures to carry loads that are not known in advance is discussed in papers [3–5] where the authors proposed a minmax formulation to maximize the compliance with respect to loading and to minimize it with respect to design variables. These problems lead to a coupled optimization/antioptimization formulation where the objective is to compute the “best” design under “worst” case loading resulting in a robust design insensitive to load variations.

An alternative strategy to treat the uncertainties is convex modeling in which the uncertainties belong to a convex set [6–9]. This approach allows the designer to use not the averaged results but extremal properties of the system being modeled, according to the convex set chosen. The limitation of the convex modeling is that only small variations around a nominal value of the uncertain quantity can be considered and the model becomes less accurate as the variations become larger. Other methods of taking load uncertainties in the design process can be found in [10, 11]. The main objective of these techniques is to achieve robust designs which are not susceptible to failure under unexpected loads [12–14]. Recently the optimization under dimensional uncertainty was applied to the design of microbeams [15].

In the present work, a system of nonlinear differential equations is derived in terms of state, design and load variables using calculus of variations and the methods of Lagrange multipliers and slack variables. These expressions serve as the optimality and antioptimality conditions of the problem. The design variable is the cross-sectional shape of the beam and is subject to a volume constraint. The load variables comprise a combination of deterministic and uncertain transverse loads as well as uncertain moments and shear forces which may act at the boundaries. The only constraints on the unknown loads involve finite norms and an upper bound on the transverse load. The optimization method involves a minmax formulation where the objective is to minimize the compliance with respect to the cross-sectional shape and maximize it with respect to the unknown loads. The formulation ensures that the optimal designs correspond to the most unfavorable loading and, therefore, these designs are conservative for any other loading. The solution methodology is illustrated with several examples involving cases which allow the computation of closed-form solutions.

2. Problem Formulation and Design Constraints

The differential equation governing the deflection of a variable cross-section beam subject to a combination of uncertain and deterministic loads, as shown in Figure 1, can be expressed in nondimensional form as

$$\left(a^n(x)''(x)\right)'' = F(x) + G(x), \quad \text{for } x \in (0,1) \tag{2.1}$$

where $I(x) = a(x)^n$ is the stiffness function, $a(x) \in C^2[0,1]$ is the cross sectional area, $y(x) \in C^4[0,1]$ is the deflection, $F(x) \in C^0[0,1]$ is the unknown uncertain loading and $G(x) \in C^0[0,1]$ is the given deterministic loading. The primes denote the derivative with respect to $x \in [0,1]$. In (2.1), $n = 1,2,3$ is a constant where n corresponds to a variable width, a geometrically similar cross-section and a variable height, respectively. The beam is subject to two boundary conditions at the endpoints $x = 0$ and $x = 1$. The boundary conditions may also contain uncertain and deterministic parts in the form of moments and shear forces which can be expressed as

$$\mathcal{B}_{0i}(y)\big|_{x=0} = f_0 + g_0, \quad \mathcal{B}_{1i}(y)\big|_{x=1} = f_1 + g_1, \quad i = 1,2, \tag{2.2}$$

where $\mathcal{B}_{0i}$ and $\mathcal{B}_{1i}$ are boundary operators at $x = 0$ and $x = 1$, respectively, f_0 and f_1 are uncertain constants, and g_0 and g_1 are deterministic constants. In addition to the boundary conditions, the deflection, area and load functions should satisfy certain continuity conditions which arise as a result of discontinuities in the first derivative of the uncertain load function. Let the uncertain load function $F(x)$ act on the beam between the points s_1 and s_2, that is, it is applied on the interval $0 < s_1 \le x \le s_2 < 1$. In general $F(x)$ can be expressed in the form

$$F(x) = \begin{cases} 0, & \text{if } 0 < x \le s_1, \\ F_1(x) \le F_{\max}, & \text{if } s_1 \le x \le d_1, \\ F_{\max}, & \text{if } d_1 \le x \le d_2, \\ F_2(x) \le F_{\max}, & \text{if } d_2 \le x \le s_2, \\ 0, & \text{if } s_2 \le x < 1, \end{cases} \tag{2.3}$$

where d_1 and d_2 are unknown constants to be determined from the continuity conditions

$$F_1(d_1) = F_{\max}, \qquad F_2(d_2) = F_{\max}. \tag{2.4}$$

The optimal area function is computed for each interval and should be a continuous function of x which requires that

$$a_-(s_1) = a_+(s_1), \qquad a_-(s_2) = a_+(s_2), \qquad a_-(d_1) = a_+(d_1), \qquad a_-(d_2) = a_+(d_2), \tag{2.5}$$

where a_- and a_+ denote the values of the area function $a(x)$ to the left and right of the relevant point. Similarly the moment and shear force should be continuous along the beam in the absence of point loads or moments, and in particular at these points that can be expressed as

$$a^n y''\big|_{x=\xi_-} = a^n y''\big|_{x=\xi_+}, \qquad (a^n y'')'\Big|_{x=\xi_-} = (a^n y'')'\Big|_{x=\xi_+} \tag{2.6}$$

where ξ_- and ξ_+ are the points to the left and right of $\xi = s_1, s_2, d_1$, and d_2. In the sections on the implementation of the solution method, it is shown that the number of unknowns resulting from the integration of differential equations matches the number of boundary and continuity conditions leading to unique solutions.

Physically the stiffness of a beam has a finite value and in practice the volume of the beam is specified leading to a design constraint which can be expressed mathematically as

$$\int_0^1 a(x)dx = 1, \tag{2.7}$$

which simply constraints the volume of the beam available for optimization. Similarly the uncertain loads acting on the beam are required to have finite norms and an upper bound as would be the case under operating conditions even though these loadings are not known precisely. For the uncertain load $F(x)$, these constraints can be expressed as

$$\|F(x)\|_{L_p}^p = \int_0^1 F(x)^p dx = 1, \quad 1 < p < \infty, \quad \max_{0\le 0\le 1} F(x) \le F_{\max}, \tag{2.8}$$

that is, the L_p norm of the uncertain transverse load is constrained to have the value 1 and the uncertain transverse load is bounded from above. Finally, the uncertain moments and shear forces acting at the endpoints are of finite magnitude in practice that can be expressed as

$$m_0^p + m_1^p = \eta, \qquad v_0^p + v_1^p = \gamma, \tag{2.9}$$

where $\eta > 0$ and $\gamma > 0$ are given constants.

2.1. Objective Functional

The design optimization can be achieved by choosing a suitable performance index for the problem which serves as an objective functional of a minmax problem. In the present case a suitable objective functional is the potential energy of the beam which is given by

$$\begin{aligned}\mathrm{PE}(a, F, m, v; y) = \frac{1}{2}\int_0^1 a^n (y'')^2 dx - \int_0^1 (F + G) y dx \\ + \frac{1}{2}\big(y'(0)m(0) - y'(1)m(1) + y(0)v(0) - y(1)v(1)\big),\end{aligned} \tag{2.10}$$

the Euler-Lagrange equation of which yields the state equation (2.1) and the natural boundary conditions. Thus the functional $\mathrm{PE}(a, F, m, v; y)$ has the advantage of producing the deflection function $y(x)$ when minimized with respect to y. Another objective functional can be defined as the compliance of the beam given by

$$C(a, F, m, v; y) = \int_0^1 (F + G) y dx, \tag{2.11}$$

which is a measure of the relative stiffness of the beam under distributed loads. Substituting (2.1) into (2.11) and performing integration by parts, we obtain

$$\int_0^1 (F + G) y dx = \int_0^1 a^n (y'')^2 dx = 2\text{PE}(a, F, m, v; y) + 2C(a, F, m, v; y). \tag{2.12}$$

Equations (2.11) and (2.12) show that the potential energy and the beam compliance are related as

$$\text{PE}(a, F, m, v; y) = -\frac{1}{2} C(a, F, m, v; y). \tag{2.13}$$

As such the two objective functionals are closely related. In the present study the potential energy given by (2.10) is chosen as the performance index of the optimization problem.

2.2. Robust Optimal Design Problem

Find the cross-sectional area $a(x)$ of the optimal beam for given values of n, p and the deterministic loading $G(x)$ such that the beam's potential energy $\text{PE}(a, F, m, v; y)$ given by (2.10) is minimized subject to the volume constraint given by (2.7) under the worst case loadings with respect to the uncertain load $F(x)$, uncertain end moments $m = (m_0, m_1)$, and shear forces $v = (v_0, v_1)$ where $F(x), m$, and s satisfy the constraints (2.8) and (2.9).

The robust design problem constitutes an optimization problem with respect to the area function $a(x)$, and an antioptimization problem with respect to the uncertain loads $F(x)$, m and s. Thus the solution of the following minmax problem is sought:

$$\min_{a(x)} \max_{F, m, v} \text{PE}(a, F, m, v; y), \tag{2.14}$$

subject to constraints (2.8) and (2.9) where $\text{PE}(a, F, m, v; y)$ is given by (2.10). The solution to the antioptimization problem yields the worst case loadings and the solution to the optimization problem yields the optimal cross-sectional area as a function of x. Solving this problem as a nested minmax problem leads to the optimal robust area function under worst case of loading and mathematically to a system of three nonlinear differential equations in three unknowns: $y(x)$, $a(x)$, and $F(x)$.

3. Optimality Conditions

The derivations of the optimality and antioptimality conditions are given next. In view of the presence of several constraints on the design problem, the Lagrange multiplier technique is implemented by introducing the Lagrangian given by

$$L(a,F,m,v;y) = \mathrm{PE}(a,F,m,v;y) + \mu_1\left(\int_0^1 F(x)^p dx - 1\right) + \mu_2\left(\int_0^1 a(x)dx - 1\right) + \mu_3\left(m_0^p + m_1^p - \eta\right) + \mu_4\left(v_0^p + v_1^p - \gamma\right) + \int_0^1 \mu_5(x)\left(F(x) - h(x)^2 - F_{\max}\right)dx, \tag{3.1}$$

where $\mu_i,\ i = 1,2,3,4$, and $\mu_5(x)$ are Lagrange multipliers and $h(x)$ is a slack variable. The variation of $L(a,F,m,v;y)$ with respect to y gives the differential equation and the boundary conditions. The variation of $L(a,F,m,v;y)$ with respect to $a(x)$ yields

$$\int_0^1 \left(na^{n-1}(y'')^2 - \mu_2\right)\delta a\, dx = 0, \tag{3.2}$$

where δa is arbitrary. Thus, from the fundamental theorem of calculus of variations, it follows that

$$na^{n-1}(y'')^2 - \mu_2 = 0. \tag{3.3}$$

The variation of $L(a,F,m,v;y)$ with respect to $F(x)$ yields

$$\int_0^1 \left(-y + \mu_1 pF^{p-1} + \mu_5(x)\right)\delta F(x)dx = 0, \tag{3.4}$$

where δF is arbitrary. Thus,

$$-y + \mu_1 pF^{p-1} + \mu_5(x) = 0. \tag{3.5}$$

The variation of $L(a,F,m,v;y)$ with respect to $h(x)$ yields

$$\mu_5(x)h(x) = 0. \tag{3.6}$$

This equation implies that $h(x)$ must be equal to zero when $\mu_5(x) \neq 0$, and $\mu_5(x)$ must be zero when $h(x) \neq 0$. As a result, $\mu_5(x)$ in equation (3.5) can be discarded since it is equal to zero when the second constraint in (2.8) is not active. Differentiation of $L(a,F,m,v;y)$ with respect to m_0, and m_1 as well as v_0, and v_1 yields

$$\begin{aligned} y'(0) + 2p\mu_3 m_0^{p-1} &= 0, \qquad -y'(1) + 2p\mu_3 m_1^{p-1} = 0, \\ y(0) + 2p\mu_4 v_0^{p-1} &= 0, \qquad -y(1) + 2p\mu_4 v_1^{p-1} = 0. \end{aligned} \tag{3.7}$$

Using equations (2.1) and (3.5) results in the following expressions for y'' and $a(x)$:

$$y'' = c_{11} a^{(1-n)/2}, \tag{3.8}$$

$$a(x) = c_{11}^{-1} \int_0^x (x-t)(F(t)+G(t))dt + a_1 x + a_2, \quad \text{for } n = 1 \tag{3.9}$$

$$= c_{12} \left(y''\right)^{2/(1-n)}, \quad \text{for } n = 2, 3, \tag{3.10}$$

where a_1, a_2, and c_{11} are arbitrary constants and $c_{12} = c_{11}^{2/(n-1)}$. Equations (3.8) and (3.9) are optimality conditions for the area function $a(x)$. Similarly, from (3.5) and (3.6), one obtains

$$y = \begin{cases} c_{21} F^{p-1}, & \text{for } F < F_{\max}, \\ c_{21} F_{\max}^{p-1}, & \text{for } F \geq F_{\max}, \end{cases}$$
$$F(x) = \begin{cases} c_{22} y^{1/(p-1)}, & \text{for } y < \left(\dfrac{F_{\max}}{c_{22}}\right)^{p-1}, \\ F_{\max}, & \text{for } y \geq \left(\dfrac{F_{\max}}{c_{22}}\right)^{p-1}, \end{cases} \tag{3.11}$$

where c_{21} is an arbitrary constant and $c_{22} = c_{21}^{1/(1-p)}$. Equation (3.11) is the antioptimality condition for the worst case loading $F(x)$. Substituting $a(x)$ and $F(x)$ from (3.9) and (3.11), respectively, into (2.1), we obtain a fourth-order nonlinear differential equation for $y(x)$ given by

$$\left(c_{12}^n \left(y''\right)^{(1+n)/(1-n)}\right)'' = c_{22} y^{1/(p-1)} + G(x) \quad \text{for } y < \left(\frac{F_{\max}}{c_{22}}\right)^{p-1}, \tag{3.12}$$

$$\left(c_{12}^n \left(y''\right)^{(1+n)/(1-n)}\right)'' = F_{\max} + G(x) \quad \text{for } y \geq \left(\frac{F_{\max}}{c_{22}}\right)^{p-1}, \tag{3.13}$$

where $n = 2, 3$. It is noted that for the beam with a variable width, that is, $n = 1$, the optimality condition (3.8) uncouples the functions $y(x)$ and $a(x)$ that lead to linear differential equations for the unknown functions. This case is treated in more detail in the examples given to illustrate the solution methodology. Alternatively, (3.12) can be expressed as an integrodifferential equation by integrating it twice, namely,

$$\left(c_{12}^n \left(y''\right)^{(1+n)/(1-n)}\right)'' = \int_0^x (x-t)\left(c_{22} y^{1/(p-1)} + G(t)\right)dt + c_3 x + c_4, \tag{3.14}$$

where c_3 and c_4 are integration constants which are to be determined from the boundary conditions (2.2). In the presence of uncertain moments and shear forces, the boundary conditions are obtained by noting that

$$m_0 = c_5\left(-y'(0)\right)^{1/(p-1)}, \qquad m_1 = c_5\left(y'(1)\right)^{1/(p-1)}, \tag{3.15}$$

$$v_0 = c_6\left(-y(0)\right)^{1/(p-1)}, \qquad v_1 = c_6\left(y(0)\right)^{1/(p-1)}, \tag{3.16}$$

where $c_5 = (2p\mu_3)^{1/(1-p)}$ and $c_6 = (2p\mu_4)^{1/(1-p)}$. If the boundary $x = 0$ is subject to uncertain and deterministic moments $m(0)$, then from (2.2) and (3.15), it follows that

$$a(0)^n y''(0) - c_5\left(-y'(0)\right)^{1/(p-1)} = m_{0d}. \tag{3.17}$$

Similarly, if uncertain and deterministic shear force $v(1)$ is applied at $x = 1$, then

$$a(1)^n y''(1) - c_6\left(-y'(1)\right)^{1/(p-1)} = v_{1d}, \tag{3.18}$$

for which (2.2) and (3.16) are used. Once the solution for $y(x)$ is computed, the optimal area function $a_{\text{opt}}(x)$ and the worst case loading F_{worst} are obtained from (3.9)–(3.11). A nonlinear differential equation in terms of $F(x)$ can also be obtained by noting that

$$a^{(1-n)/2} = \left(\frac{c_{21}}{c_{11}}\right)\left(F^{p-1}\right)'', \quad n = 2, 3, \tag{3.19}$$

which follows from equations (3.10) and (3.11). Substituting (3.8) and (3.19) into (2.1), we obtain

$$c_7\left(\left(\left(F^{p-1}\right)''\right)^{2n/(1-n)}\left(\left(F^{p-1}\right)''\right)^{-1}\right)'' = F(x) + G(x), \tag{3.20}$$

where $c_7 = c_{11}^{2n/(n-1)} c_{21}^{(n+1)/(1-n)}$ is an arbitrary constant.

4. Analytical Solutions

The theoretical framework developed in Sections 2 and 3 to solve optimal design problems under load uncertainty, in general, requires the use of numerical methods for its implementation. However, the special case of $n = 1$ (variable width beam) and $p = 2$ (L_2 norm of the transverse load) can be studied using closed-form solutions. These solutions, in turn, can be used to illustrate the method and investigate the efficiency of the robust designs. Moreover explicit solutions for the optimal beam shape and the worst case loadings serve as benchmark results for other cases where the solutions can be obtained numerically. In this section, the problem to be solved is summarized briefly and closed-form solutions are obtained for the optimal area function and the uncertain loads.

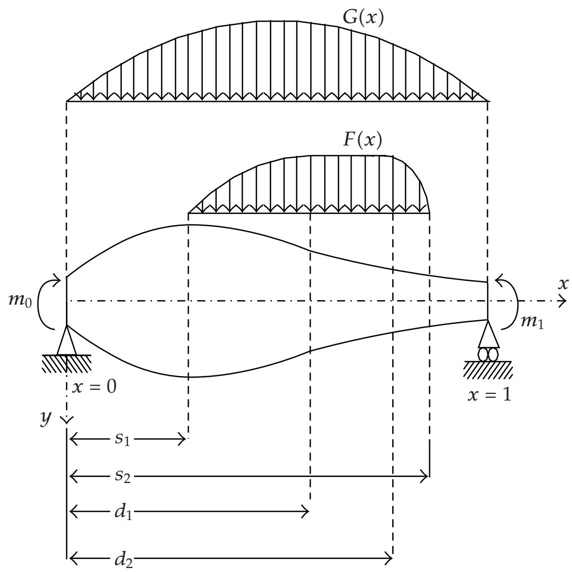

Figure 1: Beam diagram with external forces.

We consider a simply supported beam subject to an uncertain load, $F(x),\ 0 \le x \le 1$, which may be acting on part of the beam and may have an upper limit as shown in Figure 1 where $G(x)$ is the deterministic component of the transverse load. The uncertain load can be defined as

$$F(x) = \begin{cases} 0, & \text{if } 0 \le x \le s_1, \\ f(x), & \text{if } s_1 \le x \le s_2, \\ 0, & \text{if } s_2 \le x \le 1, \end{cases} \tag{4.1}$$

where s_1 and s_2 are given parameters and $f(x) \in C^0[s_1, s_2]$ is an unknown continuous function. In addition the beam is subjected to uncertain moments m_0 and m_1 at the boundaries $x = 0$ and $x = 1$, respectively (Figure 1).

Governing equation of the deflection of the beam given by (2.1) becomes

$$\left(a(x)y(x)''\right)'' = F(x) + G(x), \tag{4.2}$$

for $n = 1$. The constraints given by (2.7)–(2.9) become

$$\begin{gathered} \int_0^1 a(x)dx = 1, \qquad \|F(x)\|_{L_2}^2 \equiv \int_0^1 F(x)^2 dx = 1, \\ \max_{0 \le x \le 1} F(x) \le f_{\max}, \qquad m_0^2 + m_1^2 = \eta. \end{gathered} \tag{4.3}$$

For a simply supported beam, the boundary conditions are given by

$$\begin{aligned} y(0) &= 0, \qquad m(0) = a(0) \qquad y''(0) = m_0, \\ y(1) &= 0, \qquad m(1) = a(1) \qquad y''(1) = -m_1, \end{aligned} \tag{4.4}$$

where $m(0)$ and $m(1)$ are moments at the boundary points $x = 0$ and $x = 1$. The design problem involves the minimization of the potential energy of the beam under worst case of loading and as such involves optimization with respect to the area function $a(x)$ and antioptimization with respect to the loading functions $F(x)$, m_0 and m_1 subject to the constraints (4.3). This problem can be expressed as a minmax problem,namely,

$$\min_{a(x)} \max_{F(x),m} \mathrm{PE}\big(a(x), F(x), m; y\big), \tag{4.5}$$

where PE is the performance index (potential energy) given by

$$\begin{aligned} \mathrm{PE}\big(a, F, m; y\big) &= \mathrm{PE}\big(a(x), F(x), m; y\big) \\ &= \frac{1}{2}\int_0^1 a(x)\left(y''\right)^2 dx - \int_0^1 (F(x) + G(x)) y dx \\ &\quad + \frac{1}{2} m_0 y'(0) - \frac{1}{2} m_1 y'(1), \end{aligned} \tag{4.6}$$

and m denotes the vector $m = (m_0, m_1)$. In (4.5), the first term is the strain energy and the second, third and fourth terms make up the potential energy of the external loadings.

For the computation of the optimal area function $a(x)$ and the worst case loading $F(x), m_0$ and m_1 subject to the constraints (4.3), the Lagrangian given by (3.1) becomes

$$\begin{aligned} L\big(a, F, m; y\big) &= \mathrm{PE}\big(a, F, m; y\big) + \mu_1\left(\int_0^1 F(x)^2 dx - 1\right) \\ &\quad + \mu_2\left(\int_0^1 a(x) dx - 1\right) + \mu_3\left(m_0^2 + m_1^2 - \eta\right). \end{aligned} \tag{4.7}$$

The variation of $L(a, F, m; y)$ with respect to y gives the differential equation (4.2) and the boundary conditions (4.4). The variation of $L(a, F, m; y)$ with respect to $a(x)$ yields

$$\int_0^1 \left(y''\right)^2 \delta a dx + \mu_2 \int_0^1 \delta a dx = 0, \tag{4.8}$$

where δa is arbitrary. Thus, from the fundamental theorem of calculus of variations, it follows that

$$\left(y''\right)^2 + \mu_2 = 0. \tag{4.9}$$

Similarly, the variations of $L(a, F, m; y)$ with respect to F and m yield

$$-y + 4\mu_1 f(x) = 0, \quad \text{for } x \in [s_1, s_2], \tag{4.10}$$

$$y'(0) + 4\mu_3 m_0 = 0, \quad -y'(1) + 2\mu_3 m_1 = 0. \tag{4.11}$$

Thus, the optimality condition of the problem is given by

$$y'' = \text{constant} = \beta. \tag{4.12}$$

Similarly, the antioptimization conditions can be expressed as

$$f(x) = \begin{cases} \dfrac{y}{2\mu_1}, & \text{for } y > 2\mu_1 f_{\max}, \\ f_{\max}, & \text{for } y < 2\mu_1 f_{\max}, \end{cases} \quad \text{where } s_1 \le x \le s_2, \tag{4.13a}$$

$$m_0 = -\frac{y'(0)}{4\mu_3}, \qquad m_0 = \frac{y'(1)}{4\mu_3}. \tag{4.13b}$$

Substituting the optimality and antioptimality conditions into the differential equation (4.10), we obtain

$$a''(x) = \begin{cases} \dfrac{G(x)}{\beta}, & \text{for } 0 \le x \le s_1, \\ \dfrac{f(x) + G(x)}{\beta}, & \text{for } s_1 \le x \le s_2, \\ \dfrac{G(x)}{\beta}, & \text{for } s_2 \le x \le 1, \end{cases} \tag{4.14}$$

where $f(x)$ is given by (4.13a). A system of linear differential equations in $y(x)$ and $a(x)$ given by (4.12), (4.13a), and (4.14) can be solved simultaneously. In the present case it is possible to find an analytical solution for $y(x)$ satisfying the boundary conditions, namely,

$$y = \frac{\beta}{2} x(x-1), \quad \text{for } 0 \le x \le 1. \tag{4.15}$$

Similarly, the optimal area function is given by

$$a_{\text{opt}}(x) = \begin{cases} \dfrac{G(x)}{\beta} + c_1 x + c_2, & \text{for } 0 \le x \le s_1, \\ \dfrac{x^3}{48\mu_1}(x-2) + \dfrac{1}{\beta} G(x) + c_3 x + c_4, & \text{for } s_1 \le x \le s_2, \\ G(x)\beta + c_5 x + c_6, & \text{for } s_2 \le x \le 1, \end{cases} \tag{4.16}$$

when $y > 2\mu_1 f_{\max}$ where $G(x)$ is the second indefinite integral of $G(x)$ and $c_i,\ i = 1,\ldots,6$ are integration constants to be determined from the boundary conditions (4.4), and continuity conditions

$$a_-(s_1) = a_+(s_1), \quad a_-(s_2) = a_+(s_2), \tag{4.17}$$

where a_- and a_+ denote the area function to the left and right of the points s_1 and s_2, respectively. Furthermore in the absence of concentrated loads as required by the continuity of the uncertain and deterministic loads, the shear force $V(x) = (a(x)y''(x))'$ on the beam will also be continuous. From the optimality condition (4.12), it follows that $V(x) = \beta a'(x)$. Thus we have the further continuity conditions

$$V(s_1) = a'_-(s_1) = a'_+(s_1), \qquad V(s_2) = a'_-(s_2) = a'_+(s_2), \tag{4.18}$$

where a'_- and a'_+ denote the derivatives of the area function to the left and right of the points s_1 and s_2 , respectively. The case when $y(x) < 2\mu_1 f_{\max}$ for x in a finite interval will be solved in the example problems. The uncertain functions $f(x), m_0$ and m_1 can be computed from equations (4.3), (4.10), (4.11) and (4.14). In particular the uncertain loading $f(x), s_1 \le s \le s_2$ is given by

$$f(x) = \begin{cases} \dfrac{\beta}{4\mu_1} x(x-1), & \text{for } s_1 \le x \le d_1, \\ f_{\max}, & \text{for } d_1 \le x \le d_2, \\ \dfrac{\beta}{4\mu_1} x(x-1), & \text{for } d_2 \le x \le s_2, \end{cases} \tag{4.19}$$

where d_1 and d_2 are unknown locations to be determined from the continuity conditions

$$f_-(d_1) = f_+(d_1) = f_{\max}, \qquad f_-(d_2) = f_+(d_2) = f_{\max}, \tag{4.20}$$

where $f(x)_-$ and $f(x)_+$ denote the uncertain load functions to the left and right of the points d_1 and d_2, respectively. From equations (4.3), (4.11) and (4.14), it follows that $m_0 = m_1 = \eta/\sqrt{2}$.

It is noted that the number of unknowns equals the number of equations resulting in unique solutions. This aspect the method of solution will be illustrated in the next section by applying the technique to several problems of practical interest.

To assess the efficiency of the optimal designs, comparisons are made with uniform beams under uncertain loads for which $a(x) = 1$ for $0 \le x \le 1$. The antioptimality condition

(4.10) applies to this case also and consequently the differential equation for a uniform beam under worst case loading becomes

$$\frac{d^4 y}{dx^4} = \begin{cases} G(x), & \text{for } 0 \le x \le s_1, \\ \dfrac{y}{2\mu_1} + G(x), & \text{for } s_1 \le x \le d_1, \\ f_{\max} + G(x), & \text{for } d_1 \le x \le d_2, \\ \dfrac{y}{2\mu_1} + G(x), & \text{for } d_2 \le x \le s_2, \\ G(x), & \text{for } s_2 \le x \le 1. \end{cases} \tag{4.21}$$

The solution of the differential equation (4.21) subject to the boundary conditions (4.4) and the constraints (4.3) gives the deflection $y_{\text{un}}(x)$ of a uniform beam under worst case loading. The efficiency of the design can be determined by comparing the maximum deflections of the uniform and optimal beams, namely,

$$I_{\text{eff}} = \frac{y_{\max}}{y_{\text{un}}} \times 100\%, \tag{4.22}$$

where I_{eff} is the efficiency index in percentage, y_{un} and $y_{\max}$ are the maximum deflections of the uniform and optimal beams under worst case of loadings.

5. Applications of Method

Example 5.1 (unconstrained $F(x)$ with $0 < s_1 < s_2 < 1$). Let the beam be subjected to only the uncertain transverse load $F(x)$ given by equation (4.1) with $0 < s_1 < s_2 < 1$, that is, no uncertain moments are applied on the boundaries so that $m_0 = m_1 = \eta = 0$, and there is no deterministic load applied, that is, $G(x) = 0$. Moreover it is set equal to $f_{\max} = \infty$. For this case the optimal area function satisfying the moment boundary conditions in equation (4.4) can be computed from equations (4.16) as

$$a(x) = \begin{cases} c_1 x, & \text{for } 0 \le x \le s_1, \\ \dfrac{1}{48\mu_1} x^3 (x-2) + c_2 x + c_3, & \text{for } s_1 \le x \le s_2, \\ c_4 (1-x), & \text{for } s_2 \le x \le 1. \end{cases} \tag{5.1}$$

Equations (4.20) for $a(x)$ contain six unknowns $\beta, \mu_1, c_1, c_2, c_3$, and c_4 which are computed from six equations for the volume and L_2 norm constraints (4.3), and the continuity conditions (4.17) and (4.18). These constants in terms of s_1 and s_2 are given by

$$\begin{gathered}
\mu_1 = \frac{1}{40}s_2^5 - \frac{1}{40}s_1^5 - \frac{1}{16}s_2^4 + \frac{1}{16}s_1^4 + \frac{1}{24}s_2^3 - \frac{1}{24}s_1^3, \\
c_1 = \frac{5}{L}\begin{pmatrix} 3s_1^3 - 8s_1^2 + 3s_2 s_1^2 + 6s_1 - 8s_2 s_1 + 3s_2^2 s_1 \\ +6s_2 - 8s_2^2 + 3s_2^3 \end{pmatrix}, \\
c_2 = \frac{5\left(3s_1^4 - 4s_1^3 + 8s_2^3 - 6s_2^2 - 3s_2^4\right)}{K}, \\
c_3 = \frac{-5s_1^3(3s_1 - 4)}{K}, \\
c_4 = \frac{-5\left(3s_1^3 - 4s_1^2 + 3s_2 s_1^2 - 4s_2 s_1 + 3s_2^2 s_1 - 4s_2^2 + 3s_2^3\right)}{L}, \\
\beta = \pm\frac{\sqrt{P}}{2},
\end{gathered} \tag{5.2}$$

where

$$\begin{aligned}
L &= 6s_1^4 - 15s_1^3 + 6s_1^3 s_2 + 10s_1^2 - 15s_2 s_1^2 + 6s_2^2 s_1^2 + 10s_2 s_1 \\
&\quad - 15s_2^2 s_1 + 6s_1 s_2^3 + 10s_2^2 - 15s_2^3 + 6s_2^4, \\
K &= -6s_2^5 + 6s_1^5 + 15s_2^4 - 15s_1^4 - 10s_2^3 + 10s_1^3, \\
P &= \frac{1}{5}s_2^5 - \frac{1}{5}s_1^5 - \frac{1}{2}s_2^4 + \frac{1}{2}s_1^4 + \frac{1}{3}s_2^3 - \frac{1}{3}s_1^3.
\end{aligned} \tag{5.3}$$

A numerical example is given for the case $s_1 = 0.2$ and $s_2 = 0.8$ for which $\beta = -0.08584$, $\mu_1 = 0.003684$, $c_1 = 4.479$, $c_2 = 5.655$, $c_3 = -0.1538$, and $c_4 = 4.479$. The optimal area function $a(x)$ and the antioptimal $F(x)$ (worst case loading) are shown in Figure 2.

In the case of a uniform beam, the worst case loading is given by $f(x) = 5.825x(1 - x)$ for $s_1 \le x \le s_2$ and the corresponding deflection is $y = 0.04292x(1 - x)$. In this case $y_{\text{un}} = 0.01452$, $y_{\max} = 0.01073$ and the efficiency is 74% as determined by the efficiency index given by equation (4.22).

Example 5.2 (unconstrained $F(x)$ with $s_1 = 0, s_2 = 1$ and deterministic loading). Let the beam subject to only the uncertain transverse load $F(x)$ given by equation (4.1) with $s_1 = 0, s_2 = 1$, and the deterministic load $G(x) > 0$. Moreover, $f_{\max} = \infty$. Solution for the worst case loading is given by (4.19) with $s_1 = d_1 = 0, s_2 = d_2 = 1$, that is, $F(x) = (\beta/4\mu_1)x(x - 1)$ for $0 \le x \le 1$. The differential equation for the area function takes the form

$$a''(x) = \frac{1}{4\mu_1}\left(x^2 - x\right) + \frac{1}{\beta}G(x), \quad 0 \le x \le 1, \tag{5.4}$$

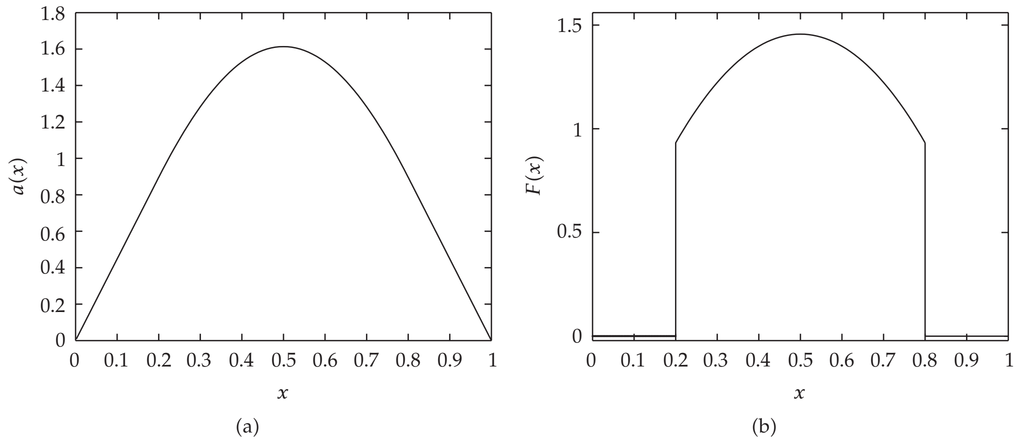

Figure 2: Curves of optimal $a(x)$ and worst case loading $F(x)$ are plotted against x for $s_1 = 0.2$ and $s_2 = 0.8$ (Example 5.1).

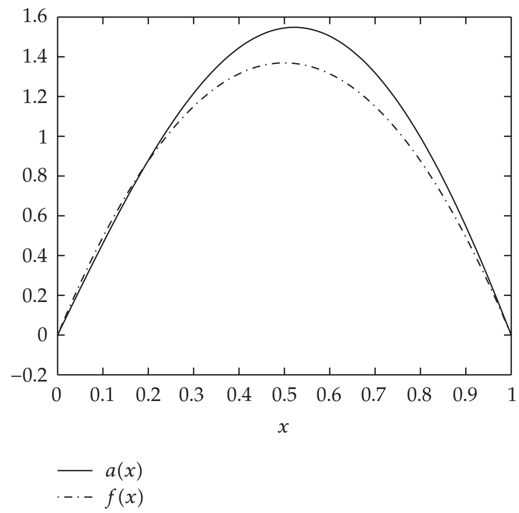

Figure 3: Curves of optimal $a(x)$ (solid line) and $F(x)$ (broken line) are plotted against x (Example 5.2).

where the optimality condition (4.12) is used. Let $G(x)$ be specified as $G(x) = \sin((\pi/2)x)$. Then the solution of (5.4) satisfying the volume constraint (4.3) and the boundary conditions (4.4) is given as

$$a_{\text{opt}}(x) = \frac{1}{48\mu_1}x^3(x-2) - \frac{4}{\beta\pi^2}\sin\left(\frac{\pi}{2}x\right) + \left(\frac{4}{\beta\pi^2} + \frac{1}{48\mu_1}\right)x, \tag{5.5}$$

where $\beta = -0.1467$ and $\mu_1 = -0.006694$. The optimal area function $a_{\text{opt}}(x)$ and the worst case loading $F(x)$ are shown in Figure 3.

In the case of a uniform beam, the worst case loading is given by $f(x) = 5.479x(1-x)$ for $0 \leq x \leq 1$, and the corresponding deflection is $y = 0.07335x(1-x)$. In this case

$y_{un} = 0.02321$, $y_{max} = 0.01833$, and the efficiency is 79% as determined by the efficiency index given by equation (4.22).

6. Conclusions

The optimality conditions were derived for robust shape design of beams subject to unknown loads with the moment of inertia related to the area function as $I = a^n$, $n = 1, 2, 3$, and the L_p, $1 < p < \infty$ norm of the transverse load subject to an equality constraint. The potential energy of the beam was specified as the performance index which was minimized with respect to area function (optimization) and maximized with respect to load function (antioptimization). Lagrange multiplier method was used to take various constraints into account and the slack variable method was used to take the upper bound on the transverse load into account. In general the optimality conditions are expressed as a system of coupled nonlinear differential equations in terms of deflection, area and load functions necessitating the use of numerical methods. However, the special case of $n = 1$ and $p = 2$ was solved explicitly to illustrate the method, to provide benchmark solutions and to assess the efficiency of the designs.

Analytical solutions were obtained for the special case under consideration and several example problems were solved involving various cases of loadings. Numerical results were given for the optimal area and antioptimal load functions. The efficiencies of the optimal designs were computed in terms of the maximum deflections of the optimal beam and the uniform beam under least favorable loading. It was shown that for the cases studied the design efficiency can exceed 70%. Load uncertainties arise due to the unpredictable conditions occurring under operational conditions and are usually the cause of unexpected failures. This situation indicates the importance of a robust design since a design optimized for a known load is strong only for this load, but it will be weak if the loading conditions happen to change.

Acknowledgment

The first author (IK) would like to acknowledge the financial support provided by American University of Sharjah and University of KwaZulu-Natal (UKZN) to enable him to spend a research leave at UKZN in 2006.

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