Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 935030, 11 pages
doi:10.1155/2009/935030

Research Article

Analytical and Numerical Methods for the CMKdV-II Equation

Ömer Akin[1] and Ersin Özuğurlu[2]

[1] *Department of Mathematics, TOBB University of Economics and Technology, Sogutozu Street No: 43, Sogutozu, 06560 Ankara, Turkey*
[2] *Department of Mathematics and Computer Sciences, Faculty of Arts and Sciences, University of Bahcesehir, Besiktas, 34353 Istanbul, Turkey*

Correspondence should be addressed to Ersin Özuğurlu, ersin.ozugurlu@bahcesehir.edu.tr

Received 31 December 2008; Revised 3 April 2009; Accepted 9 April 2009

Recommended by Oleg Gendelman

Hirota's bilinear form for the Complex Modified Korteweg-de Vries-II equation (CMKdV-II) $U_t - 6|U|^2U_x + U_{xxx} = 0$ is derived. We obtain one- and two-soliton solutions analytically for the CMKdV-II. One-soliton solution of the CMKdV-II equation is obtained by using finite difference method by implementing an iterative method.

1. Introduction

It has been known that quasilinear parabolic equations or non-linear reaction-diffusion systems arise in physics, chemistry, biology, and other applied sciences. The following three equations are the examples of this type of partial differential equations (PDEs). First one is called the Korteweg-de Vries equation

$$U_t + cUU_x + U_{xxx} = 0, \tag{1.1}$$

and first encountered in the study of waters, Korteweg [1], denoted by KdV. The other is called the Complex Modified Korteweg-de Vries-I equation (CMKdV-I)

$$U_t + \alpha\left(|U|^2U\right)_x + \beta U_{xxx} = 0, \tag{1.2}$$

which arises both in the asymptotic investigation of electrostatic waves in a magnetized plasma and in the asymptotic investigation of one-dimensional plane-wave propagation in a micropolar medium, Erbay [2]. The last one is the Complex Modified Korteweg-de Vries-II equation (CMKdV-II)

$$U_t - 6|U|^2 U_x + U_{xxx} = 0, \tag{1.3}$$

which is another example for quasilinear parabolic equations or non-linear reaction-diffusion systems, Ablowitz [3]. Equation (1.2) does not hold the Painlevé property but the (1.1) and (1.3) do, Mohammad [4]. The equations which have Painlevé property may be solved by the method of Inverse Scattering Transformations (IST) and hence they are completely integrable [5, 6]. Sometimes it is not easy to solve IST problems [3], such as for CMKdV-II equation. Therefore the need for an easy and useful method which has to give soliton solutions for a given PDE is emerged. An important method is developed by Hirota for finding N-soliton solutions of non-linear PDE [5, 6].

In this paper, the Hirota's method is applied to the CMKdV-II equation. The Hirota's method generally requires the transformation of PDE into homogeneous bilinear forms of degree two. Only specific PDEs can be transformed in this way. This means, when a bilinear (form) equation can be solved, then N-parameter solution can be obtained as a series which self-truncates at finite length. These expansions that self-truncate in this way give automatically exact solutions. Self-truncation, however, does not occur for all bilinear equations; if it does, then the equation in question possesses multiple soliton solutions. The reason for this situation has never been adequately explained. In other words, self-truncation which is equivalent to complete integrability would require a connection with the conserved quantities of the original equation.

In this study, it is proven that the CMKdV-II equation has self-truncated Hirota expansions. It is shown that there is a direct equivalence between the N-soliton solutions of Hirota's bilinear form of CMKdV-II and the Backlund transformations proposed by Weiss, Tabor, and Carnevale [7, 8].

Now, the question here is where the soliton comes from. Firstly, J. S. Russel in 1834 recorded his observations of great solitary wave as a mean of developing the mathematical properties of a large class of solvable non-linear evolution equations. Solitary waves, solitons, Backlund transformations, conserved quantities and integrable evolutions which can be also named as completely integrable Hamiltonian systems are in the class of solvable non-linear evolution equations. The description of John Scott Russel has aroused among mathematicians and physicists one hundred and forty years later, Zabusky [9]. In their paper, they were the first ones who defined the solution for the following KdV equation:

$$U_t + 6UU_x + U_{xxx} = 0. \tag{1.4}$$

For the CMKdV-II Equation, the Hirota's bilinear form is given in Section 2 and the analytical one- and two-soliton solutions are presented in Section 3. The numerical procedure and results for one-soliton solution are outlined in Section 4.

2. Hirota's Bilinear Form of the CMKdV-II Equation

It is known that the equation

$$U_t - 6|U|^2 U_x + U_{xxx} = 0 \tag{2.1}$$

is the complex modified Korteweg de Vries II equation (CMKdV-II). Let g and f be the complex and real valued functions, respectively, satisfying

$$U = \frac{g}{f}, \qquad |g|^2 = -f f_{xx} + f_x^2. \tag{2.2}$$

By using the transformation above, CMKdV-II becomes

$$f g_t - f_t g + 3\left(f_{xx} g_x - f_x g_{xx}\right) + f g_{xxx} - g f_{xxx} = 0. \tag{2.3}$$

Let

$$D_x = \frac{\partial}{\partial x} - \frac{\partial}{\partial x'}, \qquad D_t = \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \tag{2.4}$$

then (2.3) becomes

$$\left[D_x^3 + D_t\right] g(x,t) f(x',t') \Big|_{x' = x, t' = t} = 0 \Longleftrightarrow \left(D_x^3 + D_t\right) gf = 0, \tag{2.5}$$

which is a homogeneous bilinear form and called Hirota's form of CMKdV-II equation. Let

$$U = \frac{U_0}{\Phi} + U_1 \tag{2.6}$$

be the truncated solution of CMKdV-II. Then

$$\begin{gathered}
|U_0|^2 = \Phi_x^2, \\
\overline{U_0} U_1 + \overline{U_1} U_0 = -\Phi_{xx}, \\
6U_{0x} |U_1|^2 - 6\Phi_{xx} U_{1x} = U_{0xxx} + U_{0t}, \\
6\Phi_x U_0 |U_1|^2 - 6\Phi_x^2 U_{1x} = 3(\Phi_x U_{0xx} - \Phi_{xx} U_{0x}) + U_0(\Phi_{xxx} + \Phi t), \\
U_{1t} - 6|U_1|^2 U_{1x} + U_{1xxx} = 0.
\end{gathered} \tag{2.7}$$

Here U and U_1 are both separated solutions of the CMKdV-II equation. Hence it is an onto Backlund transformation of CMKdV-II equation. The system of these equations are called the Painlevé relations. There is a relation between the soliton of the bilinear (2.5) and the function Φ of the Painlevé relations (2.7). Consider the solitons

$$U = \frac{g}{f},\qquad U_1 = \frac{g^{(1)}}{f^{(1)}} \tag{2.8}$$

having the properties

$$|U|^2 = (\log f)_{xx},\qquad |U|^2 = -\frac{\partial^2}{\partial x^2}\log\Phi + |U_1|^2. \tag{2.9}$$

It can be shown that

$$f = \Phi f^{(1)}, \qquad g = U_0 f^{(1)} + \Phi g^{(1)}. \tag{2.10}$$

Theorem 2.1. *If $f^{(n)}$ and $g^{(n)}$ satisfy* (2.5) *for all n, with*

$$\left|g^{(n)}\right| = f^{(n)} f^{(n)}_{xx} - \left(f^{(n)}_x\right)^2, \tag{2.11}$$

and if

$$f^{(n)} = \Phi_{(n-1)} f^{(n-1)},\qquad g^{(n)} = U_0 f^{(n-1)} + \Phi_{(n-1)} g^{(n-1)}, \tag{2.12}$$

then the resulting equations in Φ_{n-1},U_0 and $U^{(n-1)}$ are satisfied by the Painlevé relations (2.7)*. Furthermore*

$$f^{(n)} = \prod_{i=0}^{n-1}\Phi_i, \tag{2.13}$$

with

$$f^{(0)} = 1. \tag{2.14}$$

Proof. When substituting (2.12) into (2.5) and using the Painlevé relations when necessary yields the claim of the theorem. Using (2.12) succesively we obtain the relation (2.13) that completes the proof of the theorem [4]. □

3. Solitons for the CMKdV-II Equation

By using the usual perturbation method, N-parameter exact solitary wave solutions of (7) can be obtained, Nayfeh [10]. The power series of g and f which are given in Hirota [11] in a small parameter ε are:

$$g = \varepsilon g^{(1)} + \varepsilon^3 g^{(3)} + \varepsilon^5 g^{(5)} + \cdots$$
$$f = 1 + \varepsilon^2 f^{(2)} + \varepsilon^4 f^{(4)} + \cdots , \tag{3.1}$$

where g and f are the solutions of the (2.5). Then considering the increasing powers of ε from (2.5) it is clear that

$$g^{(1)}_{xxx} + g^{(1)}_t = 0, \tag{3.2}$$

$$g^{(3)}_{xxx} + g^{(3)}_t = -\left(D^3_x + D_t\right) f^{(2)} g^{(1)}, \tag{3.3}$$

$$g^{(5)}_{xxx} + g^{(5)}_t = -\left(D^3_x + D_t\right) \left[f^{(4)} g^{(1)} + f^{(2)} g^{(3)}\right], \tag{3.4}$$

$$g^{(5)}_{xxx} + g^{(5)}_t = -\left(D^3_x + D_t\right) \left[f^{(6)} g^{(1)} + f^{(4)} g^{(3)} + f^{(2)} g^{(5)}\right] \tag{3.5}$$

and (2.2) yields

$$f^{(2)}_{xx} = -g^{(1)} g^{(1)*}, \tag{3.6}$$

$$f^{(4)}_{xx} = -g^{(1)} g^{(3)*} - g^{(1)*} g^{(3)} - f^{(2)}_x f^{(2)}_x - f^{(2)} f^{(2)}_{xx}, \tag{3.7}$$

$$f^{(6)}_{xx} = -g^{(3)} g^{(3)*} - g^{(1)*} g^{(5)} - g^{(1)} g^{(5)*} - 2 f^{(2)}_x f^{(4)}_x - f^{(4)} f^{(2)}_{xx} - f^{(2)} f^{(4)}_{xx}, \tag{3.8}$$

where $*$ stands for the complex conjugate.

3.1. One-Soliton Solution for the CMKdV-II Equation

To obtain one-soliton solution of the CMKdV-II equation, let's take

$$g^{(1)} = e^{\Phi + I\Psi}; \quad \Phi + I\Psi = a_1 x + b_1 t + c_1. \tag{3.9}$$

Hence

$$\begin{aligned}
a_1 &= \mathrm{re}(\mathrm{a}_1) + I \cdot \mathrm{im}(a_1),\\
b_1 &= \mathrm{re}(b_1) + I \cdot \mathrm{im}(b_1),\\
c_1 &= \mathrm{re}(c_1) + I \cdot \mathrm{im}(c_1),\\
\mathrm{re}(b_1) &= -(\mathrm{re}(a_1))^3 + 3 * \mathrm{re}(a_1) - (\mathrm{im}(a_1))^2,\\
\mathrm{im}(b_1) &= (\mathrm{re}(a_1))^3 - 3 * \mathrm{im}(a_1) - (\mathrm{re}(a_1))^2,\\
f^{(2)} &= \frac{e^{2\Phi}}{4(\mathrm{re}(a_1))^2},
\end{aligned} \tag{3.10}$$

and $f^{(2n)} = 0$, $g^{(2n-1)} = 0$ for all $n \geq 2$ where I stands for the complex number i. Therefore

$$f(t,x) = 1 + \frac{\varepsilon^2}{4(\mathrm{re}(a_1))^2} e^{2\Phi}, \qquad g(x,t) = \varepsilon e^{\Phi + I\Psi} \tag{3.11}$$

is a solution of Hirota bilinear form (2.5) and

$$U = \frac{g(x,t)}{f(x,t)} \tag{3.12}$$

is the corresponding one-soliton solution of the CMKdV-II equation.

3.2. Two-Soliton Solution for the CMKdV-II Equation

To find two-soliton solution, let's take

$$g^{(1)} = e^{\Phi_1 + I\Psi_1} + e^{\Phi_2 + I\Psi_2}, \tag{3.13}$$

where

$$\begin{aligned}
\Phi_1[t,x] &= \mathrm{re}(a_1) * x + \mathrm{re}(b_1) * t + \mathrm{re}(c_1)\\
\Psi_1[t,x] &= \mathrm{im}(a_1) * x + \mathrm{im}(b_1) * t + \mathrm{im}(c_1)\\
\Phi_2[t,x] &= \mathrm{re}(a_2) * x + \mathrm{re}(b_2) * t + \mathrm{re}(c_2)\\
\Psi_2[t,x] &= \mathrm{im}(a_2) * x + \mathrm{im}(b_2) * t + \mathrm{im}(c_2)
\end{aligned} \tag{3.14}$$

with

$$
\begin{aligned}
\operatorname{re}(b_1) &= -(\operatorname{re}(a_1))^3 + 3 * \operatorname{re}(a_1) - (\operatorname{im}(a_1))^2 \\
\operatorname{im}(b_1) &= (\operatorname{re}(a_1))^3 - 3 * \operatorname{im}(a_1) - (\operatorname{re}(a_1))^2 \\
\operatorname{re}(b_2) &= -(\operatorname{re}(a_2))^3 + 3 * \operatorname{re}(a_2) - (\operatorname{im}(a_2))^2 \\
\operatorname{im}(b_2) &= (\operatorname{re}(a_2))^3 - 3 * \operatorname{im}(a_2) - (\operatorname{re}(a_2))^2.
\end{aligned} \tag{3.15}
$$

Hence from (3.8), it can immediately be found that

$$
f^{(2)} = \frac{e^{2\Phi_1}}{4(\operatorname{re}(a_1))^2} + \frac{e^{2\Phi_2}}{4(\operatorname{re}(a_2))^2}. \tag{3.16}
$$

Since the right hand side of (3.3) is not zero, a special solution for $g^{(3)}$ can be found by the method of undetermined coefficients. Considering (3.7), $f_{xx}^{(4)}$ is a nonzero real function. The right hand side of (3.4) is not zero, and we can take $g^{(5)} = 0$. From (3.8), it is seen that $f_{xx}^{(6)}$ is zero and that $f^{(6)} = 0$ is taken for convenience. Further computations have shown that

$$
f^{(2n)} = 0, \quad g^{(2n-1)} = 0, \quad n \geq 3. \tag{3.17}
$$

Hence

$$
\begin{aligned}
f[t,x] = {} & 1 - \frac{1}{4(\operatorname{re}(a_1))^2} e^{2\Phi_1[t,x]} - \frac{1}{4(\operatorname{re}(a_2))^2} e^{2\Phi_2[t,x]} \\
& + \frac{\left((\operatorname{im}(a_1) - \operatorname{im}(a_2))^2 + (\operatorname{re}(a_1) - \operatorname{re}(a_2))^2\right)^2}{16(\operatorname{re}(a_1))^2(\operatorname{re}(a_2))^2\left[(\operatorname{im}(a_1) - \operatorname{im}(a_2))^2 + (\operatorname{re}(a_1) - \operatorname{re}(a_2))^2\right]} e^{2(\Phi_1[t,x]+\Phi_2[t,x])} \\
& - \frac{1}{[\operatorname{re}(a_1) + I \cdot (\operatorname{im}(a_1) - \operatorname{im}(a_2)) + \operatorname{re}(a_2)]^2} e^{\Phi_1[t,x]+\Phi_2[t,x]+I\cdot\Psi_1[t,x]-I\cdot\Psi_2[t,x]} \\
& - \frac{1}{[\operatorname{re}(a_1) + I \cdot (\operatorname{im}(a_1) - \operatorname{im}(a_2)) + \operatorname{re}(a_2)]^2} e^{\Phi_1[t,x]+\Phi_2[t,x]-I\cdot\Psi_1[t,x]+I\cdot\Psi_2[t,x]}, \\
g[t,x] = {} & e^{\Phi_1[t,x]+\Phi_2[t,x]+I\cdot\Psi_1[t,x]+I\cdot\Psi_2[t,x]} \\
& + \frac{(-I \cdot \operatorname{im}(a_1) - \operatorname{re}(a_1) + I \cdot \operatorname{im}(a_2) + \operatorname{re}(a_2))^2}{[2\operatorname{re}(a_2)(\operatorname{im}(a_1) - I \cdot \operatorname{re}(a_1) - \operatorname{im}(a_2) + I \cdot \operatorname{re}(a_2))]^2} e^{\Phi_1[t,x]+2\Phi_2[t,x]+I\cdot\Psi_1[t,x]} \\
& + \frac{(-I \cdot \operatorname{im}(a_1) - \operatorname{re}(a_1) + I \cdot \operatorname{im}(a_2) + \operatorname{re}(a_2))^2}{[2\operatorname{re}(a_2)(\operatorname{im}(a_1) - I \cdot \operatorname{re}(a_1) - \operatorname{im}(a_2) + I \cdot \operatorname{re}(a_2))]^2} e^{2\Phi_1[t,x]+\Phi_2[t,x]+I\cdot\Psi_2[t,x]}
\end{aligned} \tag{3.18}
$$

Table 1: The difference of two consecutive solution values ($\mathbf{N} = 256$, $-128 \le \mathbf{x} \le 128$, $\mathbf{M} = 500$, $0 \le \mathbf{t} \le 5$).

Δx	Δt	L_2	L_∞
1.0	0.01	0.1410966549	0.6322912259
1.0	0.01	0.0033087030	0.0105278461
1.0	0.01	0.0000823911	0.0002140195
1.0	0.01	0.0000012444	0.0000028444

is a solution of the Hirota's bilinear equation (2.5) and

$$w = \frac{g(x,t)}{f(x,t)} \tag{3.19}$$

is the corresponding two-soliton solution of the CMKdV-II (1.3). Similarly, it is possible to find N solitary wave solutions by taking

$$g^{(N)} = \sum_{i=1}^{n} e^{a_i x + b_i t + c_i}, \tag{3.20}$$

where

$$b_i = -{a_i}^3 \tag{3.21}$$

but the computations are very tedious for $i > 3$.

4. Numerical results

4.1. Iterative Methods Using Finite Difference Schemes

Previously many researchers have used the finite difference methods to solve the KdV equation, Feng [12]. In the last decade, the CMKdV-II type equations were solved numerically by using split-step Fourier method [13–15]. Also parallel implementation of the split-step Fourier method using Fast Fourier Transform (FFT) has been studied by Taha [16] (see references therein). Here, in this work, the one-soliton solution of the CMKdV-II equation is considered. A finite interval for our numerical purposes is subjected, namely, $[a, b]$. The constants a and b can be chosen sufficiently large so that the boundaries do not affect the propagation of solitons. For the CMKdV-II (1.3), a numerical (finite difference) method of solution using iterative method is introduced. U_t is approximated by using forward time difference scheme, U_x and U_{xxx} by the central-space difference scheme using four-points. Equation (1.3) becomes

$$\left(\frac{z_m^{n+1} - z_m^n}{\Delta t}\right) - 6\left|z_m^{n-1}\right|^2 \left[\frac{-z_{m+2}^{n+1} + 8z_{m+1}^{n+1} - z_{m+1}^{n+1} + z_{m-2}^{n+1}}{12\Delta x}\right] + \left[\frac{z_{m+2}^{n+1} - 2z_{m+1}^{n+1} + 2z_{m-1}^{n+1} - z_{m-2}^{n+1}}{2(\Delta x)^3}\right] = 0, \tag{4.1}$$

where $z_m^n = z(t_n, x_m) = z(nk, mh)$, $k = \Delta t$, $h = \Delta x$.

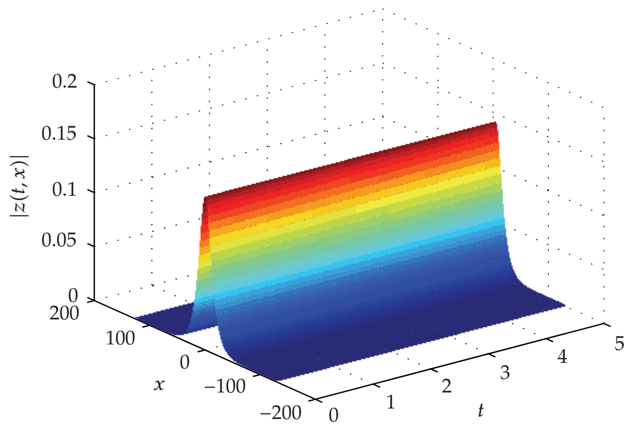

Figure 1: The modulus of the one-soliton numerical solution for the CMKdV-II equation with $N = 256$, $-128 \le x \le 128$, $M = 100$, $0 \le t \le 500$.

Multiplying both sides by $2(\Delta x)^3$ and rearranging the terms we get

$$\begin{aligned} & z_{m+2}^{n+1}\left[1+6(\Delta x)^2\left(\left|z_m^{n-1}\right|^2\right)\right]+z_{m+1}^{n+1}\left[-2-48(\Delta x)^2\left|z_m^{n-1}\right|^2\right] \\ & \quad +z_m^{n+1}\left[\frac{2(\Delta x)^2}{\Delta t}\right]+z_{m-1}^{n+1}\left[2+48(\Delta x)^2\left(\left|z_m^{n-1}\right|^2\right)\right] \\ & \quad +z_{m-2}^{n+1}\left[-1-6(\Delta x)^2\left(\left|z_m^{n-1}\right|^2\right)\right]=\frac{2(\Delta x)^3}{\Delta t} z_m^n \end{aligned} \tag{4.2}$$

for $3 \le m \le N-2$. For $m = N-1$ and $m = N$, the backward difference scheme is chosen in U_x and U_{xxx}. Three more equations come from the boundary conditions, namely, $U(a) = 0$, $U(b) = 0$, and $\partial U/\partial x = 0$ at $x = a$. Thus, N unknowns, namely, z_i^{n+1}, $i = 1, \ldots, N$ and N equations are obtained. Since the value of the non-linear term is known here, a system of linear equations is obtained. The initial guess is taken as $U(x,0) = \sqrt{2c/\alpha}\,\text{sech}[\sqrt{c}(x - x_0)]e^{(i\theta_0)}$ which represents a solitary wave initially at x_0 moving to the right with velocity c and θ_0 is the polarization angle. The main idea is to assume that the non-linear term $|z_m^{n-1}|^2$ is zero first and then solve the problem for whole time domain. Afterwards, this solution is taken and substituted for the non-linear term and solved again iteratively. The following two norms, namely, L_∞ and L_2 are used to measure the accuracy of the approximate solutions for stopping criteria. These norms are defined as following:

$$L_\infty = \max_n(||\tilde{z}_n| - |z_n||), \qquad L_2 = \sqrt{\sum_n (|\tilde{z}_n| - |z_n|)^2}, \tag{4.3}$$

where $\tilde{z}_n$ and z_n are the two consecutive new and old approximate solutions, respectively, at point $(n\Delta x, T)$ for all n, where T is the final or terminating time. FORTRAN and MATLAB are used to obtain the results and figures, respectively. The graph of one-soliton numerical solution is shown in Figure 1.

5. Conclusions

In this study, Hirota's bilinear form for the complex modified Korteweg-de Vries-II equation is derived. One- and two-soliton solutions of the CMKdV-II equation are obtained analytically. One-soliton solution of the CMKdV-II equation is obtained by using finite difference method by implementing an iterative method. The computational cost is due to only finding the inverse of the matrix. The difference of two consecutive solution values according to the formula which is given in (4.3) is shown in Table 1 result. The convergence rate in the method presented above is quadratic as it can be seen in Table 1 result. It would be interesting to see what happens if this numerical scheme for the interaction of two-soliton waves for the CMKdV-II equation is applied. The numerical scheme deserves further study according to its application to the CMKdV-II equation.

Acknowledgments

The authors would like to thank the referees for the various suggestions to improve the paper, and would like also to thank Professor Allaberen Ashyralyev for pointing out useful numerical algorithms.

References

[1] D. J. Korteweg and G. de Vries, "On the change form of long waves advancing in a rectangular canal, and on a new type of long stationary waves," *Philosophical Magazine Series*, vol. 5, no. 62, pp. 422–443, 1895.

[2] S. Erbay and E. S. Şuhubi, "Nonlinear wave propagation in micropolar media—I: the general theory," *International Journal of Engineering Science*, vol. 27, no. 8, pp. 895–914, 1989.

[3] M. J. Ablowitz, A. Ramani, and H. Segur, "A connection between nonlinear evolution equations and ordinary differential equations of P-type—I," *Journal of Mathematical Physics*, vol. 21, no. 4, pp. 715–721, 1980.

[4] A. A. Mohammad, *Classical lie symmetries and similarity reductions of the CMKdV-II equation*, Ph.D. thesis, Istanbul Technical University, Istanbul, Turkey, 1996.

[5] M. D. Kruskal, A. Ramani, and B. Grammatikos, "Singularity analysis and its relation to complete, partial and non integrability," in *Physics*, R. Conte and N. Boccara, Eds., pp. 321–372, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.

[6] V. E. Zakharov and L. D. Faddeev, "KdV equation: a completely integrable Hamiltonian system," *Functional Analysis and Its Applications*, vol. 5, no. 4, pp. 280–287, 1971.

[7] J. D. Gibbon, P. Radmore, M. Tabor, and D. Wood, "The Painlevé property and Hirota's method," *Studies in Applied Mathematics*, vol. 72, no. 1, pp. 39–63, 1985.

[8] J. Weiss, M. Tabor, and G. Carnevale, "The Painlevé property for partial differential equations," *Journal of Mathematical Physics*, vol. 24, no. 3, pp. 522–526, 1983.

[9] N. J. Zabusky and M. D. Kruskal, "Interaction of solitons in a collisionless plasma and the recurrence of initial states," *Physical Review Letters*, vol. 15, no. 6, pp. 240–243, 1965.

[10] A. H. Nayfeh, *Introduction to Perturbation Techniques*, Wiley-Interscience, New York, NY, USA, 1981.

[11] R. Hirota, "Direct methods in soliton theory," in *Solitons*, R. K. Bullough, P. J. Caudrey, et al., Eds., pp. 157–176, Springer, Berlin, Germany, 1980.

[12] B.-F. Feng and T. Mitsui, "A finite difference method for the Korteweg-de Vries and the Kadomtsev-Petviashvili equations," *Journal of Computational and Applied Mathematics*, vol. 90, no. 1, pp. 95–116, 1998.

[13] G. M. Muslu and H. A. Erbay, "A split-step Fourier method for the complex modified Korteweg-de Vries equation," *Computers & Mathematics with Applications*, vol. 45, no. 1–3, pp. 503–514, 2003.

[14] S. Yu, S. Zhao, and G. W. Wei, "Local spectral time splitting method for first- and second-order partial differential equations," *Journal of Computational Physics*, vol. 206, no. 2, pp. 727–780, 2005.

[15] G. M. Muslu and H. A. Erbay, "Higher-order split-step Fourier schemes for the generalized nonlinear Schrödinger equation," *Mathematics and Computers in Simulation*, vol. 67, no. 6, pp. 581–595, 2005.

[16] T. R. Taha and X. M. Xu, "Parallel split-step Fourier methods for the coupled non-linear Schröedinger type equations," *The Journal of Supercomputing*, vol. 32, no. 1, pp. 5–23, 2005.