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*Research Article*

# Maximal Regularity for Flexible Structural Systems in Lebesgue Spaces

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We study abstract equations of the form $\lambda u'''(t) + u''(t) = c^2 Au(t) + c^2\mu Au'(t) + f(t)$, $0 < \lambda < \mu$ which is motivated by the study of vibrations of flexible structures possessing internal material damping. We introduce the notion of $(\alpha;\beta;\gamma)$-regularized families, which is a particular case of $(a;k)$-regularized families, and characterize maximal regularity in $L^p$-spaces based on the technique of Fourier multipliers. Finally, an application with the Dirichlet-Laplacian in a bounded smooth domain is given.

## 1. Introduction

During the last few decades, the use of flexible structural systems had steadily increased importance. The study of a flexible aerospace structure involves problems of dynamical system theory governed by partial differential equations.

We consider here the problem of characterizing $L^p$-maximal regularity (or well-posedness) for a mathematical model of a flexible space structure like a thin uniform rectangular panel, for example, a solar cell array or a spacecraft with flexible attachments. This problem is motivated by both engineering and mathematical considerations.

The study of *vibrations* of flexible structures possessing internal material damping was first derived by Bose and Gorain [1]. The consideration of external forces leads to more general equations of the form

$$\alpha u'''(t) + u''(t) = \beta Au(t) + \gamma Au'(t) + f(t), \quad \alpha, \beta, \gamma \in \mathbb{R}, \tag{1.1}$$

where $A$ is a closed linear operator acting in a Banach space $X$ and $f$ is an $X$-valued function. We emphasize that the abstract Cauchy problem associated with (1.1) is in general ill posed; see, for example, [2]. Also it is well known that in order to analyze well-posedness, a direct approach leads to better results than those obtained by a reduction to a first-order equation.

Maximal regularity in Hölder spaces for (1.1) has been recently characterized in [3]. In case $\alpha = 0$, there are more literatures. For example, stability of the solution was studied by Gorain in [4]. In [5], Gorain and Bose studied exact controllability and boundary stabilization. More recently, Batkai and Piazzera [6, page 188] have obtained the exact decay rate. We note that well-posedness in Lebesgue spaces in the case of a damped wave equation has been only recently considered by Chill and Srivastava in [7], and in Hölder spaces by Poblete [8]. We note that the class studied in [8] includes equations with delay. In particular, well-posedness of the homogeneous abstract Cauchy problem has been observed in [9] for $\alpha = 0$ under certain assumptions on $A$.

This paper is organized as follows. Section 2, collects results essentially contained in [10] and standard literature on $R$-boundedness and maximal regularity (see [11] and [12]). In Section 3 we study, by an operator theoretical method, sufficient conditions for existence of solutions for (1.1). We obtain two results: a description of the solution by means of certain regularized families (Proposition 3.1) and the existence of such families in the particular case of positive self-adjoint operators (Theorem 3.2). In Section 4, we succeed in *characterizing* well-posedness of (1.1) in terms of $R$-boundedness of a resolvent set which involves $A$ (Theorem 4.2). This will be achieved in the Lebesgue spaces $L^p(\mathbb{R}, X)$, where $X$ is a UMD space (see below the definition). The methods to obtain this goal are those incorporated in [13] where a similar problem in case of the first-order abstract Cauchy problem has been studied. Our main result (Theorem 4.2) is a combination of the well-known (and deep) result due to Weis [14] stated in Theorem 2.8 and a direct calculation involving the parameters $\alpha$, $\beta$, and $\gamma$.

## 2. Preliminaries

Let $\alpha, \beta, \gamma > 0$ be given. In what follows we denote

$$\begin{gathered} k(t) = \frac{1}{\alpha}\int_0^t (t-s)e^{-s/\alpha}ds = -\alpha + t + \alpha e^{-t/\alpha}, \quad t \in \mathbb{R}_+, \\ a(t) = \beta k(t) + \frac{\gamma}{\alpha}\int_0^t e^{-s/\alpha}ds = -(\alpha\beta - \gamma) + \beta t + (\alpha\beta - \gamma)e^{-t/\alpha}, \quad t \in \mathbb{R}_+. \end{gathered} \tag{2.1}$$

In order to give an operator theoretical approach to (1.1) we introduce the following definition.

*Definition 2.1.* Let $A$ be a closed and linear operator with domain $D(A)$ defined on a Banach space $X$. One calls $A$ the generator of an $(\alpha, \beta, \gamma)$-regularized family $\{R(t)\}_{t \geqslant 0} \subset \mathcal{B}(X)$ if the following conditions are satisfied.

(R1) $R(t)$ is strongly continuous on $\mathbb{R}_+$ and $R(0) = 0$.

(R2) $R(t)D(A) \subset D(A)$ and $AR(t)x = R(t)Ax$ for all $x \in D(A), t \geqslant 0$.

(R3) The following equation holds:

$$R(t)x = k(t)x + \int_0^t a(t-s)R(s)Ax\,ds \tag{2.2}$$

for all $x \in D(A)$, $t \geqslant 0$. In this case, $R(t)$ is called the $(\alpha,\beta,\gamma)$-regularized family generated by $A$.

*Remark 2.2.* It is proved in [10], in the more general context of $(a,k)$-regularized families, that an operator $A$ is the generator of an $(\alpha,\beta,\gamma)$-regularized family if and only if there exists $\omega \geqslant 0$ and a strongly continuous function $R : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{(\lambda^2+\alpha\lambda^3)/(\beta+\gamma\lambda) : \operatorname{Re}\lambda > \omega\} \subset \rho(A)$ and

$$H(\lambda)x := \frac{1}{\beta+\gamma\lambda}\left(\frac{\lambda^2+\alpha\lambda^3}{\beta+\gamma\lambda} - A\right)^{-1} x = \int_0^\infty e^{-\lambda t}R(t)x\,dt, \quad \operatorname{Re}\lambda > \omega,\ x \in X. \tag{2.3}$$

Because of the uniqueness of the Laplace transform, we note that an $(\alpha,\beta,\gamma)$-regularized family corresponds to an $(a,k)$-regularized family studied in [10]. In fact, we have

$$\widehat{a}(\lambda) = \frac{\beta+\gamma\lambda}{\lambda^2+\alpha\lambda^3}, \quad \widehat{k}(\lambda) = \frac{1}{\lambda^2+\alpha\lambda^3}, \quad \forall \operatorname{Re}\lambda > \omega. \tag{2.4}$$

As in the situation of $C_0$-semigroups, we have diverse relations of an $(\alpha,\beta,\gamma)$-regularized family and its generator. The following result is a direct consequence of [10, Proposition 3.1 and Lemma 2.2].

**Proposition 2.3.** *Let $R(t)$ be an $(\alpha,\beta,\gamma)$-regularized family on $X$ with generator $A$. Then the following hold.*

(a) *For all $x \in D(A)$ one has $R(\cdot)x \in C^3(\mathbb{R}_+; X)$.*

(b) *Let $x \in X$ and $t \geqslant 0$. Then $\int_0^t a(t-s)R(s)x\,ds \in D(A)$ and*

$$R(t)x = k(t)x + A\int_0^t a(t-s)R(s)x\,ds. \tag{2.5}$$

Results on perturbation, approximation, asymptotic behavior, representation, as well as ergodic-type theorems for $(\alpha,\beta,\gamma)$-regularized families can be also deduced from the more general context of $(a,k)$-regularized families (see [10, 15–18]).

We will need the following results on Laplace transform (see [19, Theorem 2.5.1 and Corollary 2.5.2] for a detailed proof).

**Lemma 2.4.** *Suppose that $q : \mathbb{C}_+ \to \mathbb{C}$ is holomorphic and satisfies $\sup_{\operatorname{Re}\lambda>0}|\lambda q(\lambda)| < \infty$ and let $b > 0$. Then there exists $f \in C(\mathbb{R}_+)$ with $\sup_{t>0}|e^{-\omega t}t^{-b}f(t)| < \infty$ such that $q(\lambda) = \lambda^b\int_0^\infty e^{-\lambda t}f(t)dt$ for all $\operatorname{Re}\lambda > 0$.*

**Lemma 2.5.** *Suppose that* $q : \mathbb{C}_+ \to \mathbb{C}$ *is holomorphic and satisfies* $|\lambda q(\lambda)| + |\lambda^2 q'(\lambda)| \leqslant M$ *for all* $\operatorname{Re}\lambda > 0$*. Then there exists a bounded function* $f \in C(\mathbb{R}_+)$ *such that* $q(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt$ *for all* $\operatorname{Re}\lambda > 0$.

We introduce the means

$$\|(x_1, \dots, x_n)\|_R := \frac{1}{2^n} \sum_{\epsilon_j \in \{-1,1\}^n} \left\| \sum_{j=1}^{n} \epsilon_j x_j \right\| \tag{2.6}$$

for $x_1, \dots, x_n \in X$.

*Definition 2.6.* Let $X, Y$ be Banach spaces. A subset $\mathcal{T}$ of $\mathcal{B}(X, Y)$ is called $R$-bounded if there exists a constant $c \geqslant 0$ such that

$$\|(T_1 x_1, \dots, T_n x_n)\|_R \leqslant c \|(x_1, \dots, x_n)\|_R \tag{2.7}$$

for all $T_1, \dots, T_n \in \mathcal{T}, x_1, \dots, x_n \in X, n \in \mathbb{N}$. The least $c$ such that (2.7) is satisfied is called the $R$-bound of $\mathcal{T}$ and is denoted as $R(\mathcal{T})$.

The notion of $R$-boundedness was implicitly introduced and used by Bourgain [20] and later on also by Zimmermann [21]. Explicitly it is due to Berkson and Gillespie [22] and to Clément et al. [23].

$R$-boundedness clearly implies boundedness. If $X = Y$, the notion of $R$-boundedness is strictly stronger than boundedness unless the underlying space is isomorphic to a Hilbert space [24, Proposition 1.17]. Some useful criteria for $R$-boundedness are provided in [11, 24].

*Remark 2.7.* (a) Let $\mathcal{S}, \mathcal{T} \subset \mathcal{B}(X, Y)$ be $R$-bounded sets, then $\mathcal{S} + \mathcal{T} := \{S + T : S \in \mathcal{S}, T \in \mathcal{T}\}$ is $R$-bounded.

(b) Let $\mathcal{T} \subset \mathcal{B}(X, Y)$ and $\mathcal{S} \subset \mathcal{B}(Y, Z)$ be $R$-bounded sets, then $\mathcal{S} \cdot \mathcal{T} := \{S \cdot T : S \in \mathcal{S}, T \in \mathcal{T}\} \subset \mathcal{B}(X, Z)$ is $R$-bounded and

$$R(\mathcal{S} \cdot \mathcal{T}) \leqslant R(\mathcal{S}) \cdot R(\mathcal{T}). \tag{2.8}$$

(c) Also, each subset $M \subset \mathcal{B}(X)$ of the form $M = \{\lambda I : \lambda \in \Omega\}$ is $R$-bounded whenever $\Omega \subset \mathbb{C}$ is bounded.

We recall that those Banach spaces $X$ for which the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$, for some $p \in (1, \infty)$, are called UMD spaces. For more information and details on the Hilbert transform and the UMD Banach spaces we refer to [12]. Examples of UMD spaces include Hilbert spaces, Sobolev spaces $W_p^s(\Omega)$, $1 < p < \infty$ (see [25]), Lebesgue spaces $L^p(\Omega, \mu)$, $1 < p < \infty$, $L^p(\Omega, \mu; X)$, $1 < p < \infty$, when $X$ is a UMD space, and the Schatten-von Neumann classes $C_p(H)$, $1 < p < \infty$ of operators on Hilbert spaces.

After these preliminaries, we state the following operator-valued Fourier multiplier theorem. It is fundamental in our treatment. A proof can be founded in [11].

**Theorem 2.8.** *Suppose that $X$ is a UMD space and let $1 < p < \infty$. Let $M \in C^1(\mathbb{R} \setminus \{0\}; \mathcal{B}(X))$ be such that the following conditions are satisfied.*

(i) *The set $\{M(\rho)\}_{\rho \in \mathbb{R}\setminus\{0\}}$ is R-bounded.*

(ii) *The set $\{\rho M'(\rho)\}_{\rho \in \mathbb{R}\setminus\{0\}}$ is R-bounded.*

*Then the operator $T$ defined by*

$$Tf = \left(M(\cdot)\left[\widehat{f}(\cdot)\right]\right)^{\vee} \quad \text{where } f \in \mathcal{S}(X) \tag{2.9}$$

*extends to a bounded operator from $L^p(\mathbb{R}, X)$ to $L^p(\mathbb{R}, X)$.*

## 3. Existence of Solutions

Let $\alpha, \beta, \gamma \in (0, \infty)$. Consider the equation

$$u''(t) + \alpha u'''(t) = \beta A u(t) + \gamma A u'(t) + f(t), \tag{3.1}$$

with initial conditions $u(0) = u'(0) = u''(0) = 0$, where $A$ is the generator of an $(\alpha, \beta, \gamma)$-regularized family $R(t)$. By a solution of (3.1) we understand a function $u \in C(\mathbb{R}_+; D(A)) \cap C^3(\mathbb{R}_+; X)$ such that $u' \in C(\mathbb{R}_+; D(A))$ and verify (3.1).

**Proposition 3.1.** *Let $R(t)$ be an $(\alpha, \beta, \gamma)$-regularized family on $X$ with generator $A$. If $f \in L^1_{\text{loc}}(\mathbb{R}_+, D(A^2))$, then $u(t)$ given by*

$$u(t) = \int_0^t R(t-s) f(s)\, ds, \quad t \geqslant 0 \tag{3.2}$$

*is a solution of* (3.1).

*Proof.* Given that $x \in D(A)$, we obtain from Proposition 2.3 that $R(\cdot)x$, and hence $u$, is of class $C^3(\mathbb{R}_+, X)$. For all $x \in D(A)$, we have

$$R'(t)x = \left(1 - e^{-t/\alpha}\right)x + \int_0^t \left[\beta + (\gamma/\alpha - \beta)e^{-(t-s)/\alpha}\right] R(s) A x\, ds. \tag{3.3}$$

If $x \in D(A^2)$, then $R'(t)x \in D(A)$. Moreover,

$$\begin{aligned} R''(t)x &= \frac{1}{\alpha} e^{-t/\alpha} x + \frac{\gamma}{\alpha} R(t) A x + \int_0^t \left(\frac{\beta}{\alpha} - \frac{\gamma}{\alpha^2}\right) e^{-(t-s)/\alpha} A R(s) x\, ds, \\ R'''(t)x &= -\frac{1}{\alpha^2} e^{-t/\alpha} + \frac{\gamma}{\alpha} R'(t) A x + \frac{\beta}{\alpha} R(t) A x - \frac{\gamma}{\alpha^2} A R(t) x \\ &\quad + \int_0^t \left(\frac{\gamma}{\alpha^3} - \frac{\beta}{\alpha^2}\right) e^{-(t-s)/\alpha} A R(s) x\, ds. \end{aligned} \tag{3.4}$$

Since $f \in L^1_{\text{loc}}(\mathbb{R}_+, D(A^2))$, from (3.2), we have that $u(t), u'(t) \in D(A)$ and

$$u'(t) = \int_0^t R'(t-s)f(s)ds,$$
$$u''(t) = \int_0^t R''(t-s)f(s)ds, \tag{3.5}$$
$$u'''(t) = R''(0)f(t) + \int_0^t R'''(t-s)f(s)ds.$$

Hence,

$$\begin{aligned} &u''(t) + \alpha u'''(t) - \beta Au(t) - \gamma Au'(t) \\ &\quad = \int_0^t R''(t-s)f(s)ds + f(t) + \alpha \int_0^t R'''(t-s)f(s)ds \\ &\quad - \beta A \int_0^t R(t-s)f(s)ds - \gamma A \int_0^t R'(t-s)f(s)ds. \end{aligned} \tag{3.6}$$

By the other side, for all $x \in D(A^2)$, we obtain

$$\begin{aligned} &R''(t)x + \alpha R'''(t)x - \beta AR(t)x - \gamma AR'(t)x \\ &\quad = \frac{1}{\alpha}e^{-t/\alpha}x + \frac{\gamma}{\alpha}AR(t)x + \int_0^t \left(\frac{\beta}{\alpha} - \frac{\gamma}{\alpha^2}\right)e^{-(t-s)/\alpha}AR(s)x\,ds - \frac{1}{\alpha}e^{-t/\alpha} + \gamma AR'(t)x \\ &\quad + \beta AR(t)x - \frac{\gamma}{\alpha}AR(t)x + \int_0^t \left(\frac{\gamma}{\alpha^2} - \frac{\beta}{\alpha}\right)e^{-(t-s)/\alpha}AR(s)x\,ds - \beta AR(t)x - \gamma AR'(t)x \\ &\quad = 0. \end{aligned} \tag{3.7}$$

Since $f(t) \in D(A^2)$ and $A$ is closed, from (3.6) we conclude that $u(t)$ verify (3.1). □

The following remarkable result provides a wide class of generators of $(\alpha, \beta, \gamma)$-regularized families. In what follows we denote

$$\varphi(\lambda) := \frac{1}{\hat{a}(\lambda)} = \frac{\lambda^2(1 + \alpha\lambda)}{\beta + \gamma\lambda}. \tag{3.8}$$

**Theorem 3.2.** *Let $-B$ be a positive self-adjoint operator on a Hilbert space $H$ such that*

$$\alpha\beta \leqslant \gamma. \tag{3.9}$$

*Then $B$ is the generator of a bounded $(\alpha, \beta, \gamma)$-regularized family on $H$.*

*Proof.* Since $-B$ is a positive self-adjoint operator in $H$, the spectrum $\sigma(B)$ is a subset of the negative real axis and the resolvent operator $(\mu - B)^{-1}$ is defined at least for all negative non real $\mu$. Let $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > 0$. If $\operatorname{Im}\varphi(\lambda) \neq 0$, then clearly $\varphi(\lambda) \in \rho(B)$. If $\operatorname{Im}\varphi(\lambda) = 0$, then we claim that $\operatorname{Re}\varphi(\lambda) > 0$. In fact, for $\lambda = a + bi \in \mathbb{C},\ a > 0$, with a direct computation we obtain

$$\operatorname{Re}\varphi(\lambda) = \frac{(a^2 - b^2)(1+\alpha a)(\beta+\gamma a) - 2ab^2\alpha(\beta+\gamma a) + \alpha\gamma b^2(a^2-b^2) + 2ab^2(1+\alpha a)}{(\beta+\gamma a)^2 + \gamma^2 b^2},$$
$$\operatorname{Im}\varphi(\lambda) = \frac{\alpha b(a^2-b^2)(\beta+\gamma a) + 2ab(1+\alpha a)(\beta+\gamma a) - \gamma b(a^2-b^2)(1+\alpha a) + 2ab^3\alpha\gamma}{(\beta+\gamma a)^2+\gamma^2 b^2}. \tag{3.10}$$

Note that $Im\varphi(\lambda) = 0$ if and only if $b = 0$ or $\alpha(a^2 - b^2)(\beta+\gamma a) + 2a(1+\alpha a)(\beta+\gamma a) - \gamma(a^2-b^2)(1+\alpha a) + 2ab^2\alpha\gamma = 0$.

Since $\alpha\beta \leqslant \gamma$, we have that

$$\begin{aligned} &\alpha\left(a^2-b^2\right)(\beta+\gamma a) + 2a(1+\alpha a)(\beta+\gamma a) - \gamma\left(a^2-b^2\right)(1+\alpha a) + 2ab^2\alpha\gamma \\ &\quad = 2\alpha\gamma a b^2 + b^2(\gamma - \alpha\beta) + \gamma a^2 + 3\alpha\beta a^2 + 2\beta a + 2\alpha\gamma a^2 \\ &\quad > 0. \end{aligned} \tag{3.11}$$

Hence, $Im\varphi(\lambda) = 0$ if and only if $b = 0$. Since $a > 0$, a direct calculation gives

$$\operatorname{Re}\varphi(\lambda) = \frac{a^2(1+\alpha a)}{\beta+\gamma a} > 0, \tag{3.12}$$

proving the claim. We conclude that $\varphi(\lambda) \in \rho(B)$ for all $\operatorname{Re}\lambda > 0$. Hence (see Kato [26, Section V.3.5]),

$$\left\|(\varphi(\lambda) - B)^{-1}\right\| = \frac{1}{\operatorname{dist}(\varphi(\lambda), \sigma(B))} \quad \forall \operatorname{Re}\lambda > 0. \tag{3.13}$$

Note that

$$\sup_{\operatorname{Re}\lambda>0}\left(\frac{|\lambda|^2+1}{\operatorname{dist}(\varphi(\lambda),\sigma(B))}\right) < M, \tag{3.14}$$

since $\operatorname{dist}(\varphi(\lambda), \sigma(B))$ has order $|\lambda|^2$. Define $Q(\lambda) = (1/(\beta+\gamma\lambda))\,(\varphi(\lambda) - B)^{-1}$. We have by (3.14) and (3.13) that for all $\operatorname{Re}\lambda > 0$

$$\|\lambda Q(\lambda)\| = \left\|\frac{\lambda}{(\beta+\gamma\lambda)}(\varphi(\lambda)-B)^{-1}\right\| \leqslant \frac{|\lambda|}{|\beta+\gamma\lambda|}\ \frac{1}{\operatorname{dist}(\varphi(\lambda),\sigma(B))} < M. \tag{3.15}$$

On the other hand,

$$\lambda^2 Q'(\lambda) = \frac{-\gamma\lambda}{\beta+\gamma\lambda}[\lambda Q(\lambda)] + [\lambda Q(\lambda)]\left[\lambda^2(\varphi(\lambda)-B)^{-1}\right]\left[\lambda\frac{\widehat{a}(\lambda)'}{\widehat{a}(\lambda)}\right]\frac{1}{\lambda^2\widehat{a}(\lambda)}, \tag{3.16}$$

where

$$\frac{1}{\lambda^2\widehat{a}(\lambda)} = \frac{1+\alpha\lambda}{\beta+\gamma\lambda}, \qquad \lambda\frac{\widehat{a}(\lambda)'}{\widehat{a}(\lambda)} = -\frac{2\alpha\gamma\lambda^2+(\gamma+3\alpha\beta)\lambda+2\beta}{(1+\alpha\lambda)(\beta+\gamma\lambda)} \tag{3.17}$$

and, by (3.14),

$$\left\|\lambda^2(\varphi(\lambda)-B)^{-1}\right\| \leqslant \frac{|\lambda^2|}{\operatorname{dist}(\varphi(\lambda),\sigma(B))} < M \tag{3.18}$$

for all $\operatorname{Re}\lambda > 0$. We conclude that $\sup_{\operatorname{Re}\lambda>0}\|\lambda^2 Q'(\lambda)\| < \infty$.

By Lemma 2.5 there exists a strongly continuous family $R(t)$ such that $\|R(t)\| \leqslant K$ and $Q(\lambda) = \widehat{R}(\lambda)$ for $\operatorname{Re}\lambda > 0$. In consequence, for all $\operatorname{Re}\lambda > 0$ we have

$$\widehat{R}(\lambda) = \frac{\varphi(\lambda)}{\lambda^2(1+\alpha\lambda)}(\varphi(\lambda)-B)^{-1} = \frac{1}{\beta+\gamma\lambda}\left(\frac{\lambda^2+\alpha\lambda^3}{\beta+\gamma\lambda}-B\right)^{-1}, \tag{3.19}$$

and, by Remark 2.2, it shows that $R(t)$ is a bounded $(\alpha,\beta,\gamma)$-regularized family generated by $B$. □

Since it is a known fact that the Dirichlet-Laplacian operator is a self-adjoint operator on $L^2(\Omega)$ and $\sigma(\Delta) \subset (-\infty, 0)$, we obtain the following corollary.

**Corollary 3.3.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and assume that $\alpha\beta \leqslant \gamma$. Then the Dirichlet-Laplacian operator $\Delta$ with domain $H^2(\Omega)\cap H_0^1(\Omega)$ is the generator of an $(\alpha,\beta,\gamma)$-regularized family on $X = L^2(\Omega)$.*

*Remark 3.4.* In Theorem 3.2 the condition $\alpha\beta \leqslant \gamma$ is fundamental to have $\varphi(\lambda) \in \rho(B)$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$, which is the key in the proof. Figure 1 is the typical situation, where we have mapped by $\varphi$ the lines $\operatorname{Re}(\lambda) = 1, 2$, and 3 with $\alpha = 3$, $\beta = 1$, and $\gamma = 4$.
Note that in case $\alpha\beta > \gamma$ it can happen that $\varphi(\lambda) \in \sigma(B)$. For example, taking $\alpha = 1$, $\beta = 5$, and $\gamma = 1$, we obtain Figure 2 of $\varphi(\lambda)$ for $\operatorname{Im}(\lambda) \in \mathbb{R}$ and $\operatorname{Re}(\lambda) = 1$

## 4. $L^p$-Well-Posedness

Having presented preliminary material on $R$-boundedness and Fourier multipliers, we will now show how these tools can be used to handle (3.1). Our main result give concrete conditions on the operator $A$ under which (3.1) has $L^p$-maximal regularity.

The definition of $L^p$-maximal regularity which we investigate in this section is given as follows.

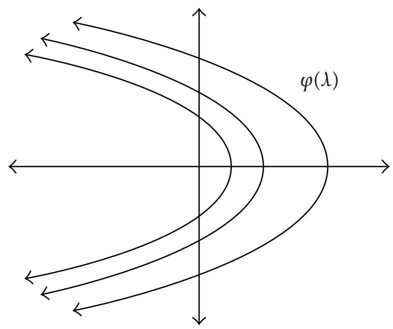


**Figure** 1

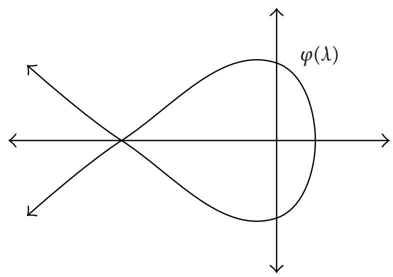


**Figure** 2

*Definition 4.1.* One says that (3.1) has $L^p$-maximal regularity (or is $L^p$-well posed) on $\mathbb{R}_+$ if for each $f \in L^p(\mathbb{R}_+, X)$ there is a unique function $u \in W^{3,p}(\mathbb{R}_+, X) \cap W^{1,p}(\mathbb{R}_+, [D(A)]) \cap W^p(\mathbb{R}_+, [D(A)])$ such that (3.1) holds a.e.

The following is the main abstract result of this section. It completely characterizes the maximal regularity of solutions for (3.1) in Lebesgue spaces.

**Theorem 4.2.** *Let $X$ be a UMD space, $1 < p < \infty$, and let $A$ be the generator of a bounded $(\alpha, \beta, \gamma)$-regularized family $R(t)$. The following statements are equivalent.*

(i) *Equation* (3.1) *has $L^p$-maximal regularity on $\mathbb{R}_+$.*

(ii) $b(\rho) := -\rho^2((1 + i\alpha\rho)/(\beta + i\gamma\rho)) \in \rho(A)$ *for all $\rho \in \mathbb{R} \setminus \{0\}$ and the set*

$$\left\{ \frac{\rho^3}{\beta + i\gamma\rho} R(b(\rho), A) \right\}_{\rho \in \mathbb{R}\setminus\{0\}} \quad \text{is } R\text{-bounded}. \tag{4.1}$$

*Proof.* (i) ⇒ (ii). By (3.1) and Definition 4.1 together with Proposition 3.1, the convolution operator with kernel

$$K_4(t) := R'''(t)\chi_{(0,\infty)}(t), \quad t \in \mathbb{R}, \tag{4.2}$$

is a bounded operator from $L^p(\mathbb{R}, X)$ to $L^p(\mathbb{R}, X)$. Note that the Fourier transform $\widetilde{R}(\rho)$ exists for $\rho \neq 0$ because $R(t)$ is bounded and $\widehat{R}(\lambda)(\operatorname{Re}\lambda > 0)$ can be analytically extended from $\operatorname{Re}\lambda > 0$ to the imaginary axis. Then the symbol of this convolution operator is given by

$$M(\rho) = \frac{\rho^3}{\beta + i\gamma\rho} R(b(\rho), A), \quad \rho \in \mathbb{R} \setminus \{0\}, \tag{4.3}$$

and the conclusion follows from [11, Proposition 3.17].
(ii) $\Rightarrow$ (i). Define $N(\rho) := (1/(\beta + i\gamma\rho))R(b(\rho), A)$ and

$$N_1(\rho) := AN(\rho). \tag{4.4}$$

We check that the set $\{N_1(\rho)\}_{\rho\in\mathbb{R}\setminus\{0\}}$ is $R$-bounded.
Since $(b(\rho) - A)R(b(\rho), A) = I$, we have that $AR(b(\rho), A) = b(\rho)R(b(\rho), A) - I$. Replacing in (4.4)

$$N_1(\rho) = \frac{b(\rho)}{\beta + i\gamma\rho} R(b(\rho), A) - \frac{1}{\beta + i\gamma\rho} I = -\frac{1 + i\alpha\rho}{\beta + i\gamma\rho}\rho^2 N(\rho) - \frac{1}{\beta + i\gamma\rho} I. \tag{4.5}$$

Note that

$$\begin{gathered} \left|\frac{1 + i\alpha\rho}{\beta + i\gamma\rho}\right|^2 = \frac{1 + \alpha^2\rho^2}{\beta^2 + \gamma^2\rho^2} < \frac{1}{\beta^2} + \frac{\alpha^2}{\gamma^2}, \\ \left|\frac{1}{\beta + i\gamma\rho}\right|^2 = \frac{1}{\beta^2 + \gamma^2\rho^2} < \frac{1}{\beta^2}. \end{gathered} \tag{4.6}$$

Since the sum of $R$-bounded sets is $R$-bounded, see [11], we obtain that $\{N_1(\rho)\}$ is $R$-bounded.

We now check that the set $\{\rho N_1'(\rho)\}_{\rho\in\mathbb{R}\setminus\{0\}}$ is $R$-bounded. With a direct computation, we obtain

$$\begin{aligned} N_1'(\rho) &= b'(\rho)N(\rho) + b(\rho)N'(\rho) + \frac{i\gamma}{(\beta + i\gamma\rho)^2} I \\ &= \frac{2\alpha\gamma}{(\beta + i\gamma\rho)^2}\rho^3 N(\rho) - \frac{\gamma + 3\alpha\beta}{(\beta + i\gamma\rho)^2} i\rho^2 N(\rho) - \frac{2\beta}{(\beta + i\gamma\rho)^2}\rho N(\rho) \\ &\quad - \frac{i\gamma}{\beta + i\gamma\rho} b(\rho)N(\rho) - \frac{2\alpha\gamma\rho^2 - (\gamma + 3\alpha\beta)i\rho - 2\beta}{\beta + i\gamma\rho}\rho b(\rho)N(\rho)N(\rho) + \frac{i\gamma}{(\beta + i\gamma\rho)^2} I. \end{aligned} \tag{4.7}$$

Hence

$$\begin{aligned}
\rho N_1'(\rho) &= \frac{2\alpha\gamma\rho}{(\beta+i\gamma\rho)^2}\rho^3 N(\rho) - \frac{\gamma+3\alpha\beta}{(\beta+i\gamma\rho)^2} i\rho^3 N(\rho) - \frac{2\beta}{(\beta+i\gamma\rho)^2}\rho^2 N(\rho) + \frac{i\gamma-\alpha\gamma\rho}{(\beta+i\gamma\rho)^2}\rho^3 N(\rho) \\
&\quad + \left(2\alpha\gamma\rho^2 - (\gamma+3\alpha\beta)i\rho - 2\beta\right)\frac{1+i\alpha\rho}{(\beta+i\gamma\rho)^2}\rho^4 N(\rho)N(\rho) + \frac{i\gamma\rho}{(\beta+i\gamma\rho)^2} I \\
&= \frac{2\alpha\gamma\rho}{(\beta+i\gamma\rho)^2}\rho^3 N(\rho) - \frac{\gamma+3\alpha\beta}{(\beta+i\gamma\rho)^2} i\rho^3 N(\rho) - \frac{2\beta}{(\beta+i\gamma\rho)^2}\rho^2 N(\rho) + \frac{i\gamma-\alpha\gamma\rho}{(\beta+i\gamma\rho)^2}\rho^3 N(\rho) \\
&\quad + 2\alpha\gamma\frac{1+i\alpha\rho}{(\beta+i\gamma\rho)^2}\rho^3 N(\rho)\rho^3 N(\rho) - (\gamma+3\alpha\beta)\frac{i-\alpha\rho}{(\beta+i\gamma\rho)^2}\rho^3 N(\rho)\rho^2 N(\rho) \\
&\quad - 2\beta\frac{1+i\alpha\rho}{\beta+i\gamma\rho}\rho^3 N(\rho)\rho N(\rho) + \frac{i\gamma\rho}{(\beta+i\gamma\rho)^2} I \\
&= \frac{\alpha\gamma\rho-3\alpha\beta i}{(\beta+i\gamma\rho)^2}\rho^3 N(\rho) - \frac{2\beta}{(\beta+i\gamma\rho)^2}\rho^2 N(\rho) + 2\alpha\gamma\frac{1+i\alpha\rho}{(\beta+i\gamma\rho)^2}\rho^3 N(\rho)\rho^3 N(\rho) \\
&\quad - (\gamma+3\alpha\beta)\frac{i-\alpha\rho}{(\beta+i\gamma\rho)^2}\rho^3 N(\rho)\rho^2 N(\rho) - 2\beta\frac{1+i\alpha\rho}{\beta+i\gamma\rho}\rho^3 N(\rho)\rho N(\rho) + \frac{i\gamma\rho}{(\beta+i\gamma\rho)^2} I.
\end{aligned} \tag{4.8}$$

Since the set $\{\rho^3 N(\rho)\}$ is $R$-bounded and the complex functions appearing in the above equality are bounded, we obtain the claim from the fact that the sum of $R$-bounded sets is again $R$-bounded. We employ now Theorem 2.8 to conclude that the operator $T_1$ defined by

$$T_1 f = \left(N_1(\cdot)\left[\hat{f}(\cdot)\right]\right)^{\vee} \quad \text{where } f \in \mathcal{S}(X) \tag{4.9}$$

extends to a bounded operator from $L^p(\mathbb{R}, X)$ to $L^p(\mathbb{R}, X)$.

Define

$$N_2(\rho) := \frac{\rho}{\beta+i\gamma\rho} AR(b(\rho), A). \tag{4.10}$$

We will prove that the sets $\{N_2(\rho)\}_{\rho\in\mathbb{R}\setminus\{0\}}$ and $\{\rho N_2'(\rho)\}_{\rho\in\mathbb{R}\setminus\{0\}}$ are $R$-bounded.

In fact, note that $N_2(\rho) = \rho\ N_1(\rho) = -((1+i\alpha\rho)/(\beta+i\gamma\rho))\rho^3 N(\rho) - (\rho/(\beta+i\gamma\rho))I$. Hence the set $\{N_2(\rho)\}$ is $R$-bounded. Moreover, we have

$$\begin{aligned}\rho N_2'(\rho) &= \rho^2 N_1'(\rho) + \rho N_1(\rho)\\ &= \frac{\alpha\gamma\rho^2 - 3\alpha\beta\rho i}{(\beta+i\gamma\rho)^2}\rho^3 N(\rho) - \frac{2\beta}{(\beta+i\gamma\rho)^2}\rho^3 N(\rho) + 2\alpha\gamma\frac{\rho+i\alpha\rho^2}{(\beta+i\gamma\rho)^2}\rho^3 N(\rho)\rho^3 N(\rho)\\ &\quad - (\gamma+3\alpha\beta)\frac{i-\alpha\rho}{(\beta+i\gamma\rho)^2}\rho^3 N(\rho)\rho^3 N(\rho) - 2\beta\frac{1+i\alpha\rho}{\beta+i\gamma\rho}\rho^3 N(\rho)\rho^2 N(\rho)\\ &\quad + \frac{i\gamma\rho^2}{(\beta+i\gamma\rho)^2} I + N_2(\rho),\end{aligned} \tag{4.11}$$

obtaining that the set $\{\rho N_2'(\rho)\}_{\rho\in\mathbb{R}\setminus\{0\}}$ is $R$-bounded. By Theorem 2.8 we conclude that the operator $T_2$ defined by

$$T_2 f = \left(N_2(\cdot)\left[\widehat{f}(\cdot)\right]\right)^{\vee} \quad \text{where } f \in \mathcal{S}(X) \tag{4.12}$$

extends to a bounded operator from $L^p(\mathbb{R}, X)$ to $L^p(\mathbb{R}, X)$.

Finally, define

$$N_3(\rho) := \frac{\rho^2}{\beta+i\gamma\rho} R(b(\rho), A) = \rho^2 N(\rho). \tag{4.13}$$

The set $\{N_3(\rho)\}_{\rho\in\mathbb{R}\setminus\{0\}}$ is $R$-bounded from hypothesis and also note that the set $\{\rho N_3'(\rho)\}_{\rho\in\mathbb{R}\setminus\{0\}}$ is $R$-bounded, since

$$\begin{aligned}\rho N_3'(\rho) &= 2\rho^2 N(\rho) - \frac{i\gamma}{\beta+i\gamma\rho}\rho^3 N(\rho) - \frac{2\alpha\gamma}{\beta+i\gamma\rho}\rho^3 N(\rho)\rho^3 N(\rho)\\ &\quad + \frac{\gamma+3\alpha\beta}{\beta+i\gamma\rho} i\rho^3 N(\rho)\rho^2 N(\rho) + \frac{2\beta}{\beta+i\gamma\rho}\rho N(\rho)\rho^3 N(\rho).\end{aligned} \tag{4.14}$$

Again by Theorem 2.8 we conclude that the operator $T_3$ defined by

$$T_3 f = \left(N_3(\cdot)\left[\widehat{f}(\cdot)\right]\right)^{\vee} \quad \text{where } f \in \mathcal{S}(X) \tag{4.15}$$

extends to a bounded operator from $L^p(\mathbb{R}, X)$ to $L^p(\mathbb{R}, X)$. From (4.9), (4.12), and (4.15) and since it is clear that (3.1) has $L^p$-maximal regularity if the convolution operator with each one of the kernels

$$K_1(t) := AR(t)\chi_{(0,\infty)}(t), \quad K_2(t) := AR'(t)\chi_{(0,\infty)}(t), \quad K_3(t) := R''(t)\chi_{(0,\infty)}(t), \quad t \in \mathbb{R}, \tag{4.16}$$

is a bounded operator from $L^p(\mathbb{R}, X)$ to $L^p(\mathbb{R}, X)$ (see [11]), we conclude (i) and the proof is complete. □

Of course, $R$-boundedness in (4.1) can be replaced by boundedness in case $X = H$ is a Hilbert space.

**Corollary 4.3.** *The solution $u$ of* (3.1)*, under the conditions given by Theorem 4.2, satisfies the following maximal regularity property: $u, u' \in L^p(\mathbb{R}_+; [D(A)])$ and $Au, Au', u'', u''' \in L^p(\mathbb{R}_+; X)$. Moreover, there exists a constant $C > 0$ independent of $f \in L^p(\mathbb{R}_+; X)$ such that*

$$\|u\|_p + \|u'\|_p + \|u''\|_p + \|u'''\|_p + \|Au\|_p + \|Au'\|_p + \leqslant C\|f\|_p. \tag{4.17}$$

The proof follows by the closed-graph theorem.

As an example, we consider for $A = \Delta$ the vibration equation subject to the action of an external force. Explicitly, we consider

$$\begin{gathered}
v_{tt}(t,x) + \lambda v_{ttt}(t,x) = c^2\left(\Delta v(t,x) + \mu \Delta v_t(t,x)\right) + f(t,x) \text{ in } ]0,T] \times \Omega, \\
v(t,x) = 0 \text{ on } ]0,T] \times \Omega, \\
v(0,x) = 0 \text{ in } \Omega, \\
v_t(0,x) = 0 \text{ in } \Omega, \\
v_{tt}(0,x) = 0 \text{ in } \Omega
\end{gathered} \tag{4.18}$$

in a smooth bounded region $\Omega \subset \mathbb{R}^n$. Also, we assume that $f \in L^2(\mathbb{R}; L^2(\mathbb{R}^n))$. We have the following application in the Hilbert space setting.

**Theorem 4.4.** *Let $\Omega$ be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$. Suppose that $0 < \lambda < \mu$. Then the initial value problem* (4.18) *defined on $L^2(\Omega)$ with Dirichlet boundary conditions has $L^2$-maximal regularity on $\mathbb{R}_+$.*

*Proof.* Let $\alpha = \lambda$, $\beta = c^2$, and $\gamma = c^2\mu$ and note that $\alpha\beta < \gamma$ if and only if $\lambda < \mu$. By Corollary 3.3 we conclude that $\Delta$ generates a bounded $(\alpha, \beta, \gamma)$-regularized family on $L^2(\Omega)$.

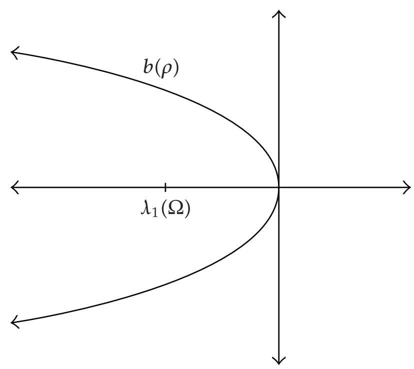


**Figure** 3

Note that we have $b(\rho) = -\rho^2((1+i\alpha\rho)/(\beta+i\gamma\rho)) \in \rho(\Delta)$ and there exists a constant $C > 0$ such that

$$\begin{aligned}\left\|\frac{\rho^3}{\beta+i\gamma\rho}R(b(\rho),\Delta)\right\| &= \left\|\rho\frac{b(\rho)}{1+i\alpha\rho}(b(\rho)-\Delta)^{-1}\right\| \\ &= \frac{|\rho|}{|1+i\alpha\rho|}\frac{|b(\rho)|}{\operatorname{dist}(b(\rho),\lambda_1(\Omega))} \leqslant C,\end{aligned} \tag{4.19}$$

for all $\rho \in \mathbb{R}$. Here $\lambda_1(\Omega)$ is the first eigenvalue of the Dirichlet-Laplacian. Hence, by Theorem 4.2 the assertion follows. □

*Remark 4.5.* In Figure 3, we show $b(\rho)$ in case $\lambda = 3$, $\mu = 4$, and $c^2 = 1$.

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