Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2010, Article ID 289678, 12 pages
doi:10.1155/2010/289678

Research Article

Invariant Sets of Hybrid Autonomous Systems with Disturbance

Li Jian-Qiang,[1] Zhu Zexuan,[1] Ji Zhen,[1] and Pei Hai-Long[2]

[1] *Shenzhen City Key Laboratory of Embedded System Design, College of Computer Science and Software Engineering, Shenzhen 518060, China*
[2] *Department of Automation, South China University of Technology, Guangzhou 510641, China*

Correspondence should be addressed to Li Jian-Qiang, lijq@szu.edu.cn and Zhu Zexuan, zhuzx@szu.edu.cn

Received 26 August 2010; Revised 24 October 2010; Accepted 4 November 2010

Academic Editor: G. Rega

The concept and model of hybrid systems are introduced. Invariant sets introduced by LaSalle are proposed, and the concept is extended to invariant sets in hybrid systems which include disturbance. It is shown that the existence of invariant sets by arbitrary transition in hybrid systems is determined by the existence of common Lyapunov function in the systems. Based on the Lyapunov function, an efficient transition method is proposed to ensure the existence of invariant sets. An algorithm is concluded to compute the transition mode, and the invariant set can also be computed as a convex problem. The efficiency and correctness of the transition algorithm are demonstrated by an example of hybrid systems.

1. Introduction

Hybrid systems and invariant sets are novel topics in recent years. Hybrid systems are systems which include discrete and continuous dynamics. In many applications, hybrid systems have multiple operating modes; each is described by a different dynamic equation [1–3]. Invariant set of hybrid system plays an important role in the many situations when dynamics system is constrained in some way [4, 5]. For hybrid systems, the transition is very important for constructing an invariant set for the systems. For some transitions, the states of the systems may get out of each subsystem's invariant set. In this paper, an efficient transition method which makes the states of the hybrid systems stay in the invariant sets is proposed. When the system is given an initial set, this method constructs transition regions for the hybrid system to ensure the invariant set existence. Then, the invariant set can be computed as ellipsoid set or polyhedral set [6–9].

Some studies have been done in the area of hybrid systems in recent years. Controller design and invariant set of hybrid systems have been studied [10, 11]. The lecture notes

[12] by Lygeros have talked about the basic theory of hybrid system including automata, existence, analysis and synthesis, model checking, and reachability. The set invariance in control provides a survey of the literature on invariant sets and their applications [13], and the stability of mode transitions has been studied in [14]. Research about invariant set of special hybrid systems has been done in [6, 7]. Polyhedral approximation computation of invariant set has been studied in [15].

This paper proposes an efficient approach to compute the transition regions. By the efficient transition based on the Lyapunov function theory, the system will stay in the invariant sets. This method is attractive as the invariant set is useful to design the switching strategies and the controller. The rest of the paper is organized as follows. Section 2 introduces the basic concept and model of the hybrid systems. The invariant sets, as well as the existence and computation of invariant sets are given in Section 3. Section 4 presents the invariant sets in the hybrid system and proposes an efficient transition method for the invariant set. In Section 5, an example demonstrating the efficacy of the proposed method is introduced. Section 6 summarizes the results presented in this paper and discusses future research.

2. Hybrid Systems

Roughly speaking, hybrid systems are dynamic systems that involve the interaction of different types of dynamics. Consider a hybrid system

$$\dot{x} = f_i(x(t)), \quad i = 1, 2, \ldots, m. \tag{2.1}$$

It can be modelled by a hybrid automaton [16].

Definition 2.1 (Hybrid Automaton). A Hybrid automaton H is the collection $H = (Q, X, U, f, \mathrm{Init}, D, E, G, R)$, where Q is a discrete state space, $X : X \subset R^n$ is a continuous state space, $U : U \subset R^\lambda$ is a set of inputs, $f : \mathrm{Q} \times X \rightarrow R^n$ is a vector field, Init is a set of initial states, $D : Q \rightarrow P(X)$ is a domain (the domain is sometimes called the invariant set in hybrid system), $E \subset Q \times Q$ is a set of edges, G is a guard condition, and $R : E \times X \rightarrow P(X)$ is a reset map.

Hybrid automaton defines possible evolution for their states. Starting from an initial value $(x_0, q_0) \in \mathrm{Init}$, the continuous states x flow according to the vector field $f(q_0, \cdot)$ in the discrete state q_0 and remain in $D(q_0)$. If the states reach a guard $G(q_0, q_i)$, the discrete state will change to q_i. At the same time the states will reset to $R(q_0, q_i, x)$ if the system has the reset condition. The discrete state q_i is called mode in hybrid system. So in the hybrid automaton, the system starts from initial mode q_0 to any mode q_i. The process can be repeated.

By the hybrid automaton, a temperature controller can be described as follows.

In mode q_0, u_d and u are the inputs, the temperature x flows according to $f(q_0, u_d, u)$, and state x stays in domain $D(q_0)$. If the temperature reaches $G(q_0, q_1) = \{x > 18\}$, the mode will change to q_1. It is the same process in mode q_1 as shown in Figure 1. By the hybrid controller, the temperature will stay in a domain which is called invariant set.

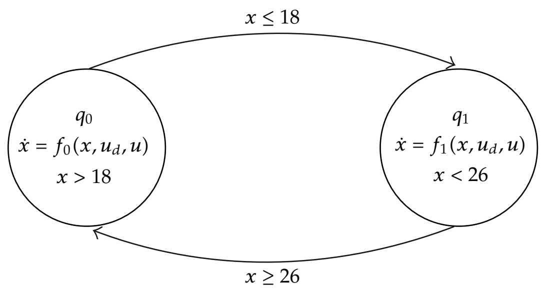

Figure 1: Temperature controller automaton.

3. Invariant Set

An (positive) invariant set of a dynamic system is a subset of the state space such that once the state enters this set, it will remain in it for all the future time [6].

That is,

$$x(0) \in M \longrightarrow x(t) \in M, \quad t > 0, \tag{3.1}$$

where $x(t)$ is the subset of the dynamic system at time t and X is a subset of the state space. Consider the continuous dynamic system

$$\dot{x}(t) = f(x(t)), \tag{3.2}$$

where f is a continuous Lipschitz function with x. A sufficient and necessary condition for X in the invariant set is that the differential equation is directed into the set at every point of the boundary.

Definition 3.1 (Bouligand, 1932). Given a closed set $\kappa \subset \mathbb{R}^n$, the tangent cone to κ in x_0 is the set

$$\kappa_{x_0} = \left\{ z \in \mathbb{R}^n : \lim_{h \to 0} \inf \frac{\operatorname{dist}(x_0 + hz, \kappa)}{h} = 0 \right\}. \tag{3.3}$$

The definition is due to Bouligand. As shown in Figure 2, if $x_0 \in \operatorname{int}\{\kappa\}$, the interior of κ, then $\kappa_{x_0} = \mathbb{R}^n$; if $x_0 \notin \operatorname{int}\{\kappa\}$, then $\kappa_{x_0} = \phi$, and $x_0 \in \partial\kappa$. The tangent cone is a cone which contains all the vectors directing into the set κ. According to Nagumo theorem, the closed set is a positively invariant set for the system (3.2) if and only if for all $x \in \kappa$,

$$f(x) \in \kappa_x. \tag{3.4}$$

Every point on the boundary is directed into the set. This can be expressed as follows:

$$n_\kappa(x)^T f(x) \le 0, \quad \forall x \in \kappa, \tag{3.5}$$

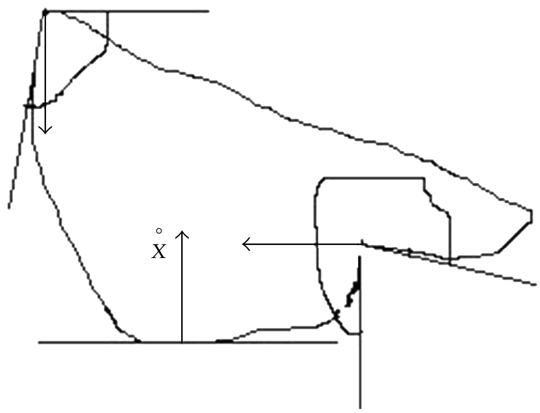

Figure 2: Tangent cone.

where $n_k(x)$ denotes a normal to $\partial\kappa$ at x. The invariant set is described by an inequality

$$\kappa = \{x \in \mathbb{R}^n \mid V(x) \le b\}. \tag{3.6}$$

$V(x)$ which defines the invariant set is a function of x. Considering the existence of a quadratic Lyapunov function for a system and the fact that the level sets of Lyapunov functions are invariant sets, a lemma can be concluded as follows.

Lemma 3.2. *A system $\dot{x} = Ax$, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^n$, exists as invariant set if all eigenvalues of A lies in the open left half plane.*

Given a set of initial states, if there exists invariant set in the system from Lemma 3.2, the invariant set can be computed as invariant ellipsoid or invariant polyhedron.

Computed as an invariant polyhedron, the invariant set can be represented in the following form [11]:

$$\delta = \left\{x : Fx \le \overline{1}\right\}, \tag{3.7}$$

where F is a $r \times n$ matrix and $\overline{1} \in R^r$ denotes vector of the form

$$\overline{1} = [1 \;\; 1 \cdots 1]^T. \tag{3.8}$$

There are mainly two computation methods [17] which include eigenstructure analysis/assignment method [13] and the iterative method as follows.

Algorithm 3.3. One has the following.

Initialization: $k = 0$ and $\delta_0 = \delta$

Repeat: $\delta_{k+1} = \{x \in \delta : \exists u(x) \in U : Ax + Bu(x) + Ew \in \delta_k, \text{ for all } w \in W\}$

$$\delta_\infty \doteq \bigcap_{i=0}^{\infty} \delta_i \tag{3.9}$$

Until $\delta_{i+1} = \delta_i$, then $\delta_\infty = \delta_i$.

At each step, the algorithm computes the set of states for which all solutions of the system stay in the δ_{k+1}. The invariant set of the system is the intersection of δ_i until $\delta_{i+1} = \delta_i$.

As an invariant ellipsoid, the invariant set can be represented as the following ellipsoid:

$$\delta = \left\{x \in \mathbb{R}^n \mid (x - x_a)^T Q^{-1}(x - x_a) \leq 1\right\}, \tag{3.10}$$

where $Q^T = Q \in \mathbb{R}^n$ and $Q \succ 0$. Given an initial set, the maximal invariant set in it can be computed as a convex optimization problem. x_a is the center of the ellipsoid. If a set of initial states δ_0 is given, it can be described as a polytope by linear inequalities first:

$$\delta_0 = \left\{x \in \mathbb{R}^n \mid v_k^T x \leq 1, \ k = 1, \ldots, q\right\}. \tag{3.11}$$

Then this invariant set can be computed as a convex problem (CP), and it is more convenient to compute by LMI toolbox:

$$\begin{aligned} \text{Minimize} \quad & \log\det Q^{-1} \\ \text{Subject to} \quad & QA^T + AQ \leq 0, \quad P \succ 0, \\ & v_k^T Q^{-1} v_k \leq 1. \end{aligned} \tag{3.12}$$

4. Invariant Sets in Hybrid System

The computation for the existence of invariant sets in linear systems presented in the previous section can be used for computation for the existence of invariant set for the subsystems in the hybrid systems. In hybrid systems, the existence of invariant set can be determined by the common Lyapunov function first.

4.1. Common Lyapunov Function

In some cases, it is possible to prove the existence of invariant set. Based on the stability theorem [18, 19], the computation is based on the sufficient conditions.

Theorem 4.1. *A hybrid system q_i, $f_i(x) = A_i x$ exists as invariant sets by arbitrarily transition if a common Lyapunov function exists in each subsystem.*

By the common Lyapunov function, if there exists $A_i^T P A_i \leq 0$, $P \succ 0$, the states of the systems will stay in an invariant set for a given initial set.

Proof. (1) Take an arbitrary mode q_i, f_i, and define $v(t) := V(x(t))$

$$\dot{v} = \frac{\partial v}{\partial x}(x)\dot{x} = \frac{\partial v}{\partial x} f_i \le 0. \tag{4.1}$$

(2) There is

$$v(t) := V(x(t)) \le v(0) := V(x(0)). \tag{4.2}$$

So, if the hybrid system is given an initial set which can be found as $V(x(0)) \le b$ and $V(x(t)) \le b$, the invariant set of the hybrid system exists.

In some cases, it is needed to verify that no common Lyapunov function in Theorem 4.1 exists. If the systems satisfy the following condition, as matrices R_i, $R_i \succ 0$ can be found to satisfy

$$\sum_{q_i \in Q} A_i^T R_i + R_i A_i > 0, \tag{4.3}$$

then there is not a solution $P = P^T$ existing in Theorem 4.1. □

4.2. Problem Formulation

In a hybrid system, there may not exist an invariant set by arbitrary transition if common Lyapunov function does not exist, although an invariant set exists in each subsystem according to Lemma 3.2. It can be shown as the example below.

Example 4.2. One has the following:

$$\begin{gathered} q_1 : f_1(x) = A_1 x, \qquad q_2 : f_2(x) = A_2 x, \\ A_1 = \begin{pmatrix} -0.1 & 1 \\ -10 & -0.1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -0.1 & 10 \\ -1 & -0.1 \end{pmatrix}. \end{gathered} \tag{4.4}$$

This is a simple transition system. In mode q_1, it is easy to compute that all eigenvalues of A_1 lie in the open left half plane $\max\{\mathrm{Re}(\lambda(A_1))\} = -0.1 < 0$, $\max\{\mathrm{Re}(\lambda(A_2))\} = -0.1 < 0$. So an invariant set exists in subsystem 1 and subsystem 2 if given an initial set as Figure 3.

However, after arbitrary transitions between the two modes, the states of the system may not shrink to an invariant set. As Figure 4 shows, the states get out of the initial set after transitions. There is no invariant set existing.

4.3. Efficient Transition

As the example in Section 4.2 shows, the invariant set does not exist by arbitrary transitions although it exists in each subsystem. If there is not an invariant set in the subsystems, how to design the transition strategy to construct the invariant set for the hybrid systems is very

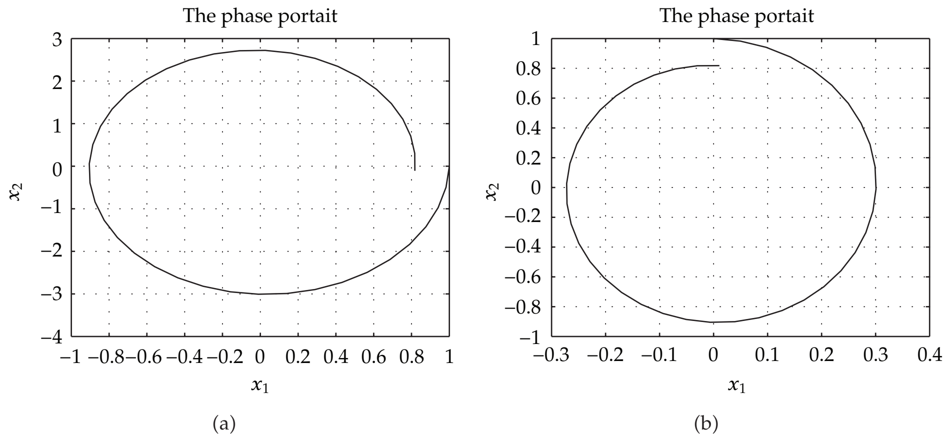

Figure 3: Trajectory of system $\dot{x} = f_1(x)$, $\dot{x} = f_2(x)$.

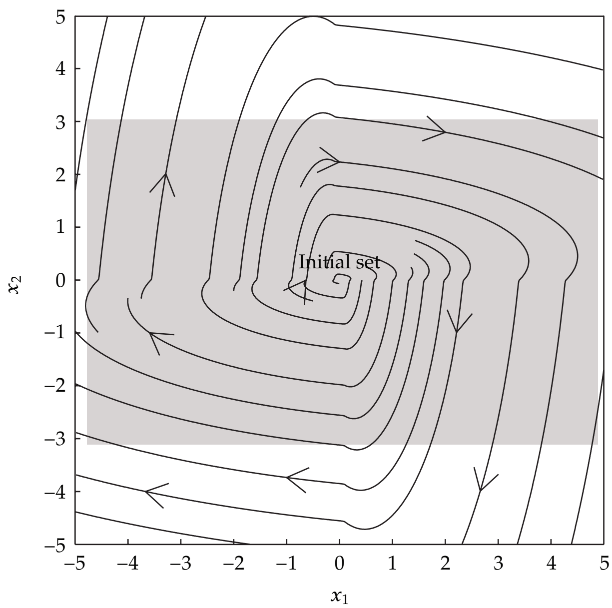

Figure 4: No invariant set exists.

important such as the temperature controller in Section 1. There is not an invariant set existing in each mode of the temperature controller, but the temperature stays in a domain after transitions between the two modes. Based on the stability theorem by Li [20], the transition method constructing for an invariant set can also be established as follows.

Theorem 4.3. *A hybrid system*

$$\dot{x}(t) = A_i x(t) + E_i(t)x(t), \quad \|E_i(t)\| \le \eta_i, \quad i \in Q \tag{4.5}$$

if there exist a positive matrix $P(P = P^T)$ and positive number β satisfying the matrix inequalities below:

$$\left(\sum_{i=1}^{m}\delta_i A_i\right)^T P + P\left(\sum_{i=1}^{m}\delta_i A_i\right) + P^2 + \beta I \leq 0, \qquad \left(\sum_{i=1}^{m}\delta_i\right) = 1, \tag{4.6}$$

$$\eta \leq \sqrt{\frac{\beta}{2}}. \tag{4.7}$$

Then, there exist invariant sets in this hybrid system by efficient transition.

In Theorem 4.3, $(\sum_{i=1}^{m}\delta_i) = 1$, $E_i(t)x(t)$ is the disturbance of each system i, and $\|E_i(t)\| \leq \eta_i$, η_i is a constant, $\eta = \max(\eta_i)$. The domain of each system can be computed as follows:

$$D_i = \left\{x(t) \mid x^T(t)\left(A_i^T P + PA_i + P^2 + \beta I\right)x(t) \leq 0\right\}. \tag{4.8}$$

Proof. Construct the Lyapunov function as follows:

$$V(x(t)) = x^T(t)Px(t) + \gamma\int_0^t x^T(\tau)x(\tau)d(\tau), \quad \gamma > 0. \tag{4.9}$$

so

$$\begin{aligned}\dot{V}(x(t)) &= x^T(t)\left(A^T P + PA\right)x(t) + 2x^T(t)PE(t)x(t) + \gamma x^T(t)x(t)\\ &\leq x^T(t)\left(A^T P + PA\right)x(t) + x^T(t)P^2x(t) + x^T(t)E^T(t)E(t)x(t) + \gamma x^T(t)x(t)\\ &= x^T(t)\left(A^T P + PA + P^2 + 2\gamma I\right)x(t) + x^T(t)\left(\eta^2 - \gamma\right)x(t).\end{aligned} \tag{4.10}$$

From (4.3),

$$\delta_1\left(A_1^T P + PA_1\right) + \delta_2\left(A_2^T P + PA_2\right) + \cdots + \delta_m\left(A_m^T P + PA_m\right) + P^2 + \beta I \leq 0. \tag{4.11}$$

Considering the condition $\delta_1 + \delta_2 + \cdots + \delta_m = 1$, the inequality can also be written as

$$\delta_1\left(A_1^T P + PA_1 + P^2 + \beta I\right) + \delta_2\left(A_2^T P + PA_2 + P^2 + \beta I\right) + \cdots + \delta_m\left(A_m^T P + PA_m + P^2 + \beta I\right) \leq 0. \tag{4.12}$$

Suppose that

$$D_i = \left\{x(t) \mid x^T(t)\left(A_i^T P + PA_i + P^2 + \beta I\right)x(t) \leq 0\right\}. \tag{4.13}$$

Taking $\beta = 2\gamma$, when $x(t) \in D_i$,

$$\dot{V}(x(t)) \leq x^T(t)\left(A_i^T P + PA_i + P^2 + \beta I\right)x(t) + x^T(t)\left(\eta^2 - \frac{\beta}{2}\right)x(t). \tag{4.14}$$

Then in conditions (4.6) and (4.7), $\dot{V}(x(t)) \leq 0$, the states stay in the level set of Lynapunov function $V(x)$. So, an invariant set exists by the efficient transition. From (4.7), we know that η determines the bound of $E(t)$, so η is bigger and the robustness of the system is better. □

From common Lyapunov function and the concept of Theorem 4.3, the computation for the efficient transition can be concluded as follows.

(1) Compute whether there exists common Lyapunov function by Theorem 4.1 and inequality (4.3). If it exists, invariant sets exist in the hybrid systems by arbitrary transitions in a given set. Otherwise go to step (3.1).

(2) Compute $\beta_{\max}$ and matrix P in (4.6), and inequality (4.6) is equivalent to the linear matrix inequalities (LMI) as below:

$$P > 0, \quad \beta > 0, \quad \begin{pmatrix} \left(\sum_{i=1}^{m} \delta_i A_i\right)^T P + P\left(\sum_{i=1}^{m} \delta_i A_i\right) + P^2 + \beta I & P \\ P & -I \end{pmatrix} < 0. \tag{4.15}$$

$\beta_{\max}$ and P can be computed by LMI toolbox in Matlab.

(3) Compute whether $\eta \leq \sqrt{\beta_{\max}/2}$; if the inequality is true, the invariant set exists in the hybrid system (4.5).

(4) Compute the transition domain by inequality (4.8), and the computation of invariant set in hybrid systems can be extended to the computation methods in Section 3.

5. Example

In this sector, the hybrid system which includes two continuous dynamics in $\mathbb{R}^2$ is chosen as the trajectories and sets can be easily visualized.

Consider a class of hybrid system as follows.

Example 5.1. One has the following:

$$\dot{x}(t) = A_i x(t) + E_i(t)x(t), \tag{5.1}$$

$$\begin{aligned} A_1 &= \begin{pmatrix} -3 & 0 \\ -2 & 2 \end{pmatrix}, \qquad E_1 = \begin{pmatrix} 0.6\sin t & 0 \\ 0 & -0.3\cos t \end{pmatrix}, \\ A_2 &= \begin{pmatrix} 1 & 6 \\ 0 & -6 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} -0.1\sin t & 0 \\ 0 & 0.5\sin t \end{pmatrix}. \end{aligned} \tag{5.2}$$

For the system, we know that

$$\eta = \max\{\eta_1, \eta_2\} = 0.6. \tag{5.3}$$

The transition domain can be computed by the computation algorithm in Section 4.3.

(1) Common Lyapunov function. As in the modes q_1 and q_2,

$$\begin{aligned} \max\{\operatorname{Re}(\lambda(A_1))\} &= 2, \\ \max\{\operatorname{Re}(\lambda(A_2))\} &= 1. \end{aligned} \tag{5.4}$$

There exists no Lyapunov function in the subsystems and no common Lyapunov function in the hybrid system, so an invariant set exists neither in each subsystem nor in the hybrid system by arbitrary transitions.

(2) Compute $\beta_{\max}$ and matrix P.
Fist, δ_i can be selected as $\delta_1 = 0.3$, $\delta_2 = 0.7$. Then, compute the linear matrix inequalities (4.7) by LMI toolbox in Matlab. The variables can be computed as below:

$$\beta_{\max} = 0.993, \qquad P = \begin{bmatrix} 0.6174 & 0.7978 \\ 0.7978 & 2.8776 \end{bmatrix}. \tag{5.5}$$

(3) Compute the maximum range of the disturbance, $\sqrt{\beta_{\max}/2} = 0.7069$, $\eta \le \sqrt{\beta_{\max}/2}$, and the disturbance is inside the bound. So, there exists an invariant set in the hybrid system.

(4) Compute the transition domain of the hybrid system by the inequality(4.6)

$$\begin{aligned} D_1 &= \left\{x(t) \mid x^T(t)\left(A_1^T P + P A_1 + P^2 + \beta I\right)x(t) < 0\right\} \\ &= \left\{x(t) \mid x^T \begin{pmatrix} -4.8786 & -3.7647 \\ -3.7647 & 21.4268 \end{pmatrix} x < 0\right\} \\ &= \left\{(x_1, x_2)^T \mid \left(-4.8786x_1^2 - 7.5294x_1x_2 + 21.4268x_2^2 < 0\right)\right\}, \\ D_2 &= \left\{x(t) \mid x^T(t)\left(A_2^T P + P A_2 + P^2 + \alpha I\right)x(t) < 0\right\} \\ &= \left\{x(t) \mid x^T \begin{pmatrix} 3.2518 & 2.5037 \\ 2.5037 & -15.0412 \end{pmatrix} x < 0\right\} \\ &= \left\{(x_1, x_2)^T \mid \left(3.2518x_1^2 - 5.0074x_1x_2 - 15.0412x_2^2 < 0\right)\right\}. \end{aligned} \tag{5.6}$$

By the transition method, there exist invariant sets in the system (5.1). The simulation of the system by efficient transition is shown as Figure 5.

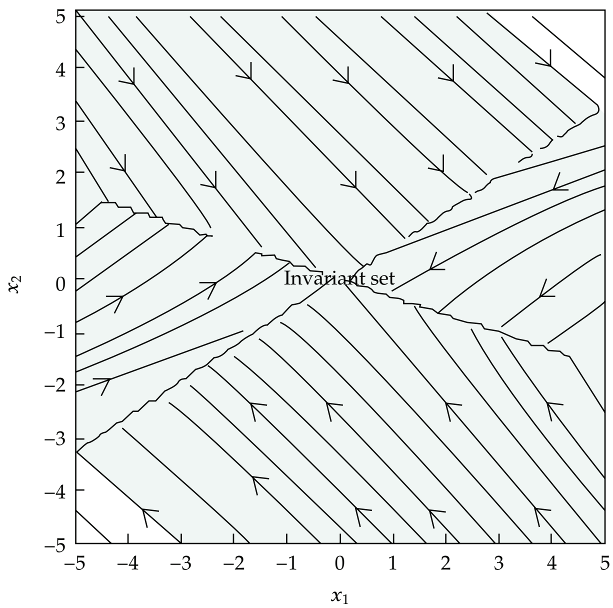

Figure 5: Invariant set exists by efficient transitions.

In Figure 5, it is shown that there exists an invariant set for a given initial set (rectangle: 5 × 5) by the efficient transition which has been computed. The invariant set here is easy to visualize, as shown in Figure 5; the invariant set is a part of polygon shadow. The invariant set also can be computed as an invariant ellipsoid in (5.1) by LMI toolbox.

6. Conclusions

The concepts of hybrid systems and invariant set have been introduced in this paper, and the computation of invariant set is also introduced. The concept is extended to invariant set in hybrid systems in this paper. It is shown that the existence of invariant sets by arbitrary transition in hybrid systems is determined by whether there exists a common Lyapunov function in the systems. Based on the Lyapunov function, an efficient transition method is proposed to ensure the existence of an invariant set. Given an initial set, an algorithm is proposed to compute the transition mode, and the invariant set is also computed as an invariant ellipsoid.

Since the invariant sets and the efficient transition mode can be computed efficiently, the proposed invariant sets make it possible to model predictive control, protection, and decision for mode transitions. An example of hybrid systems is given in this paper. The applications to the realistic problem are currently being studied.

After the efficient transition in hybrid system, the computation for complex systems may be difficult as it has large event sets. More efficient methods for the invariant set are investigated in the next step. Controller design is important in the invariant set. By the effect of the control, the system stays in the invariant set, and the application of the invariant set in hybrid systems will be studied in the future. The method can be used to the realistic application such as unmanned helicopter control [20] and power system.

Acknowledgments

The authors gratefully acknowledge the contribution of the National Science Foundation of China [61001185][60903114][61003271], the Specialized Research Fund for the Doctoral Program of Higher Education [20040561031], and the National Natural Science Foundation of Guangdong Province [031407].

References

[1] Z. Sun and S. S. Ge, *Switched Linear Systems—Control and Deign*, Springer, London, UK, 2005.

[2] E. C. Kerrigan, *Robust constraint satisfaction: invariant sets and predictive control*, Ph.D. thesis, ontrol Group Department of Engineering University of Cambridge, 2000.

[3] S. Di Cairano, A. Bemporad, and J. Júlvez, "Event-driven optimization-based control of hybrid systems with integral continuous-time dynamics," *Automatica*, vol. 45, no. 5, pp. 1243–1251, 2009.

[4] B. D. O'Dell and E. A. Misawa, "Semi-ellipsoidal controlled invariant sets for constrained linear systems," *Journal of Dynamic Systems, Measurement and Control*, vol. 124, no. 1, pp. 98–103, 2002.

[5] J. Li, H. L. Pei, S. Zhang, and B. Liu, "Efficient transition for Invariant set in a class of hybrid systems," in *Proceedings of the IEEE International Conference on Control and Automation (ICCA '07)*, pp. 1722–1726, June 2007.

[6] M. Jirstrand, "Invariant sets for a class of hybrid systems," in *Proceedings of the 37th IEEE Conference on Decision and Control (CDC '98)*, pp. 3699–3704, December 1998.

[7] A. Girard, C. Le Guernic, and O. Maler, "Efficient computation of reachable sets of linear time-invariant systems with inputs," in *Proceedings of the International Conference on Hybrid Systems: Computa tion and Control (HSCC '06)*, pp. 257–271, 2006.

[8] S. Boyd, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, Pa, USA, 1994.

[9] S. V. Raković, E. C. Kerrigan, K. I. Kouramas, and D. Q. Mayne, "Invariant approximations of the minimal robust positively invariant set," *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 406–410, 2005.

[10] M. Baotić, F. J. Christophersen, and M. Morari, "Constrained optimal control of hybrid systems with a linear performance index," *IEEE Transactions on Automatic Control*, vol. 51, no. 12, pp. 1903–1919, 2006.

[11] S. Di Cairano, A. Bemporad, and J. Júlvez, "Event-driven optimization-based control of hybrid systems with integral continuous-time dynamics," *Automatica*, vol. 45, no. 5, pp. 1243–1251, 2009.

[12] J. Lygeros, *Lecture Notes on Hybrid Systems*, Department of Electrical and Computer Engineering, University of Patras, 2004.

[13] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, no. 11, pp. 1747–1767, 1999.

[14] H. L. Pei and B. H. Krogh, "Stability regions for systems with mode transitions," in *Proceedings of the American Control Conference*, pp. 4834–4839, June 2001.

[15] A. Chutinan and B. H. Krogh, "Computing polyhedral approximations to flow pipes for dynamic systems," in *Proceedings of the 37th IEEE Conference on Decision and Control (CDC '98)*, pp. 2089–2094, December 1998.

[16] H. Johansson, J. Lygeros, and S. Sastry, *Modeling of Hybrid Systems*, UNESCO Encyclopedia of Life Support Systems, 2002.

[17] J. Q. Li, H. L. Pei, and S. X. Zhang, "Efficient computation of invariant sets for mode transitions systems," in *Proceedings of the International Conference on Computational Intelligence and Security*, 2006.

[18] M. S. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 475–482, 1998.

[19] R. A. Decarlo, M. S. Branicky, S. Pettersson, and B. Lennartson, "Perspectives and results on the stability and stabilizability of hybrid systems," *Proceedings of the IEEE*, vol. 88, no. 7, pp. 1069–1082, 2000.

[20] W. D. Li, *Analysis and control of hybrid dynamic system [D]*, A Dissertation, Zhejiang University, 2003.