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Research Article

On Multiple Generalized w-Genocchi Polynomials and Their Applications

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We define the multiple generalized w-Genocchi polynomials. By using fermionic p-adic invariant integrals, we derive some identities on these generalized w-Genocchi polynomials, for example, fermionic p-adic integral representation, Witt's type formula, explicit formula, multiplication formula, and recurrence formula for these w-Genocchi polynomials.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of integers, the ring of p-adic integers, the field of p-adic rational numbers, the complex number field, and. the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$.

The q-basic natural numbers are defined by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1} \tag{1.1}$$

for $n \in \mathbb{N}$, and the binomial coefficient is defined as

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\cdots(n-k+1)}{k!}. \tag{1.2}$$

The binomial formulas are well known that

$$(1-b)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i b^i, \qquad \frac{1}{(1-b)^n} = \sum_{i=0}^{n} \binom{n+i-1}{i} b^i \tag{1.3}$$

(see, [1, 2]). When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes that $|q-1|_p < 1$. We use the notation

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}, \tag{1.4}$$

see [1–13] for all $x \in \mathbb{Z}_p$. Note that $\lim_{q\to 1}[x]_q = x$ for $x \in \mathbb{Z}_p$ in presented p-adic case.

Let $UD(\mathbb{Z}_p)$ be denoted by the set of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, an invariant p-adic q-integral on $\mathbb{Z}_p$ is defined as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{n\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x. \tag{1.5}$$

Thus, we have the following integral relation:

$$\lim_{q\to 1} q I_{-q}(f_1) + I_{-q}(f) = (1+q) f(0), \tag{1.6}$$

where $f_1(x) = f(x+1)$, and the fermionic p-adic invariant integral relation:

$$I_{-1}(f) = \lim_{q\to 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x), \tag{1.7}$$

$$I_{-1}(f_1) + I(f) = 2f(0). \tag{1.8}$$

Now, we recall that the definitions of w-Euler polynomials and w-Genocchi polynomials are defined as

$$\frac{2e^{xt}}{we^t+1} = \sum_{n=0}^{\infty} E_{n,w}(x) \frac{t^n}{n!},$$
$$\frac{2te^{xt}}{we^t+1} = \sum_{n=0}^{\infty} G_{n,w}(x) \frac{t^n}{n!}, \quad t \in \mathbb{R}, \ w \in \mathbb{C}, \tag{1.9}$$

with $|1-w|_p < 1$, respectively. In the special case $x = 0$, $E_{n,w}(0) = E_{n,w}$, and $G_{n,w}(0) = G_{n,w}$ are called w-Euler numbers and w-Genocchi numbers (see [2, 9]).

In [13], Bayard and Simsek have studied multiple generalized Bernoulli polynomials as follows:

$$\prod_{j=1}^{r}\left(\frac{a_j t+\log(w^{a_j})}{(we^t)^{a_j}-1}\right)e^t=\sum_{n=0}^{\infty}B_{n,w}^{(r)}(x;a_1,\ldots a_r)\frac{t^n}{n!},\qquad \left|t+\log(|w|)\right|<\min\left\{\frac{\pi}{a_1}\cdots\frac{\pi}{a_p}\right\}, \tag{1.10}$$

where $a_1,\ldots,a_r$ are strictly positive real numbers.

The purpose of this paper is to define another construction of multiple generalized w-Genocchi polynomials and numbers, which are different from multiple generalized Bernoulli polynomials and numbers in [13]. By using fermionic p-adic invariant integrals, we derive some identities on these generalized w-Genocchi polynomials, for example, fermionic p-adic integral representation, Witt's type formula, explicit formulas, multiplication formula, and recurrence formula for these w-Genocchi polynomials.

2. Multiple Generalized w-Genocchi Polynomials and Numbers

Let $r\in\mathbb{N}$ and $a_1,\ldots,a_r$ be strictly positive real numbers. The multiple generalized w-Genocchi polynomials $G_{n,w}^{(r)}(x;a_1,\ldots,a_r)$ are defined as

$$\prod_{j=1}^{r}\frac{(2t)^r}{(we^t)^{a_j}+1}e^{xt}=\sum_{n=0}^{\infty}G_{n,w}^{(r)}(x;a_1,\ldots a_r)\frac{t^n}{n!},\quad \text{for } t\in\mathbb{R}, w\in\mathbb{C}, \tag{2.1}$$

where $|\log w+t|\le\min_{1\le j\le r}\{\pi/a_j\}$. The values of $G_{n,w}^{(r)}(x;a_1,\ldots,a_r)$ at $x=0$ are called the multiple generalized w-Genocchi numbers: when $r=1$, $w=1$, and $a_j=0$ $(j=1,\ldots r)$, the polynomials or numbers are called the ordinary Genocchi polynomials or numbers.

It is known that

$$t\int_{\mathbb{Z}_p}w^z e^{t(z+x)}d\mu_{-1}(z)=\frac{2t}{we^t+1}=\sum_{n=0}^{\infty}G_{n,w}(x)\frac{t^n}{n!},$$

$$t^r\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}w^{z_1+z_2+\cdots+z_r}e^{t(z_1+\cdots+z_r+x)}d\mu_{-1}(z_1)\cdots d\mu_{-1}(z_r)=\left(\frac{2t}{we^t+1}\right)^r e^{xt}=\sum_{n=0}^{\infty}G_{n,w}^{(r)}(x)\frac{t^n}{n!}. \tag{2.2}$$

In fact, let us take $t\in\mathbb{R}$, $w\in\mathbb{C}$, and we apply the above difference integral formula (1.8) for $f(z)=w^{az}e^{taz}$, then we obtain

$$\frac{2}{(we^t)^a+1}e^{tx}=\int_{\mathbb{Z}_p}w^{az}e^{t(az+x)}d\mu_{-1}(z). \tag{2.3}$$

By (2.3), we easily see that

$$\prod_{j=1}^{r}\frac{(2t)^r}{(we^t)^{a_j}+1}e^{xt}=t^r\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}w^{a_1z_1+\cdots+a_rz_r}e^{t(a_1z_1+\cdots+a_rz_r+x)}d\mu_{-1}(z_1)\cdots d\mu_{-1}(z_r)$$
$$=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}w^{a_1z_1+\cdots+a_rz_r}(a_1z_1+\cdots+a_rz_r+x)^n \times d\mu_{-1}(z_1)\cdots d\mu_{-1}(z_r)\frac{t^{n+r}}{n!}, \tag{2.4}$$

$$G_{0,w}^{(r)}(x;a_1,\ldots,a_r)=\cdots=G_{r-1,w}^{(r)}(x;a_1,\ldots,a_r))=0. \tag{2.5}$$

By (2.4) and (2.5), we obtain the following fermionic p-adic integral representation formula for these numbers.

Theorem 2.1 (p-adic integral representation). *Let $r\in\mathbb{N}$ and $a_1,\ldots,a_r$ be strictly positive real numbers. Then one has a fermionic p-adic invariant integral representation for the multiple generalized w-Genocchi polynomials $G_{n,w}^{(r)}(x;a_1,\ldots,a_r)$ as follows:*

$$\frac{G_{n+r,w}^{(r)}(x;a_1,\ldots,a_r)}{r!\binom{n+r}{r}}=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}w^{a_1z_1+\cdots+a_rz_r}(a_1z_1+\cdots+a_rz_r+x)^n d\mu_{-1}(z_1)\cdots d\mu_{-1}(z_r) \tag{2.6}$$

for $n\geq r$ and

$$G_{0,w}^{(r)}(x;a_1,\ldots,a_r)=\cdots=G_{r-1,w}^{(r)}(x;a_1\ldots,a_r))=0. \tag{2.7}$$

We remark that if we set $r=1$ and $a_1=1$, then we have the following equation:

$$\frac{G_{n+r,w}^{(1)}(x;1)}{1!\binom{n+1}{1}}=\frac{G_{n+1,w}^{(r)}(x)}{n+1}=E_{n,w}(x). \tag{2.8}$$

The generalized w-Genocchi polynomials are given by

$$\frac{2t}{(we^t)^a+1}e^{xt}=\sum_{n=0}^{\infty}G_{n,w}(x;a)\frac{t^n}{n!},$$
$$\int_{\mathbb{Z}_p}w^{az}e^{t(az+x)}d\mu_{-1}(z)=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}w^{az}(az+x)^n d\mu_{-1}(z)t^n. \tag{2.9}$$

By comparing the coefficients on both sides in (2.9), we obtain the following identity on the generalized w-Genocchi polynomials

$$\frac{G_{n,w}(x;a)}{n!} = \int_{\mathbb{Z}_p} w^{az}(az+x)^n d\mu_{-1}(z). \tag{2.10}$$

Similarly, from (2.4), we can obtain the following Witt's type formula for the multiple generalized w-Genocchi polynomials.

Theorem 2.2 (Witt's type formula). *Let $r \in \mathbb{N}$ and $a_1, \ldots, a_r$ be strictly positive real numbers. Then one has*

$$\frac{G_{n,w}^{(r)}(x;a_1,\ldots,a_r)}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{a_1z_1+\cdots+a_rz_r}(a_1z_1+\cdots+a_rz_r+x)^n d\mu_{-1}(z_1)\cdots d\mu_{-1}(z_r). \tag{2.11}$$

From (2.4), we can directly calculate the following:

$$\begin{aligned} G_{n,w}^{(r)}&(x;a_1,\ldots,a_r) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{a_1z_1+\cdots+a_rz_r}(a_1z_1+\cdots+a_rz_r+x)^n \times \; d\mu_{-1}(z_1)\cdots d\mu_{-1}(z_r)n! \\ &= \sum_{i=0}^{n}\binom{n}{i}x^{n-i}\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{a_1z_1+\cdots+a_rz_r} \times (a_1z_1+\cdots+a_rz_r)^i d\mu_{-1}(z_1)\cdots d\mu_{-1}(z_r)n! \\ &= \sum_{i=0}^{n}\binom{n}{i}^2 (n-i)!x^{n-i}G_{i,w}^{(r)}(a_1,\ldots,a_r). \end{aligned} \tag{2.12}$$

From (2.12), we get the following explicit formula.

Theorem 2.3 (explicit formula). *Let $r \in \mathbb{N}$ and $a_1, \ldots, a_r$ be strictly positive real numbers. Then one has*

$$G_{n,w}^{(r)}(x;a_1,\ldots,a_r) = \sum_{i=0}^{n}\binom{n}{i}^2 (n-i)!x^{n-i}G_{i,w}^{(r)}(a_1,\ldots,a_r). \tag{2.13}$$

Next we discuss the multiplication formula for the multiple generalized w-Genocchi polynomials as follows:

$$\begin{aligned}
&G_{n,w}^{(r)}(x;a_1,\dots,a_r)\\
&\quad=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} w^{a_1z_1+\cdots+a_rz_r}(a_1z_1+\cdots+a_rz_r+x)^n\,d\mu_{-1}(z_1)\cdots d\mu_{-1}(z_r)n!\\
&\quad=\lim_{N\to\infty}\sum_{z_1,\dots,z_r=0}^{mp^N-1} w^{a_1z_1+\cdots+a_rz_r}(a_1z_1+\cdots+a_rz_r+x)^n(-1)^{z_1+\cdots+z_r}\\
&\quad=m^n\sum_{t_1,\dots,t_r=0}^{m-1} w^{a_1t_1+\cdots+a_rt_r}(-1)^{t_1+\cdots+t_r}\lim_{N\to\infty}\sum_{y_1,\dots,y_r=0}^{p^N-1}(-1)^{m(y_1+\cdots+y_r)}\\
&\qquad\times(w^m)^{a_1y_1+\cdots+a_ry_r}\left(\frac{x+a_1t_1+\cdots+a_rt_r}{m}+a_1y_1+\cdots+a_ry_r\right)^n n!\\
&\quad=m^n\sum_{t_1,\dots,t_r=0}^{m-1} w^{a_1t_1+\cdots+a_rt_r}(-1)^{t_1+\cdots+t_r}n!\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(w^m)^{a_1y_1+\cdots+a_ry_r}\\
&\qquad\times\left(\frac{x+a_1t_1+\cdots+a_rt_r}{m}+a_1y_1+\cdots+a_ry_r\right)^n d\mu_{-1}(y_1)\cdots d\mu_{-1}(y_r)\\
&\quad=m^n\sum_{t_1,\dots,t_r=0}^{m-1} w^{a_1t_1+\cdots+a_rt_r}(-1)^{t_1+\cdots+t_r}\times G_{n,w^n}^{(r)}\left(\frac{x+a_1t_1+\cdots+a_rt_r}{m};a_1,\dots,a_r\right).
\end{aligned}\tag{2.14}$$

Thus, we obtain the following multiplication formula for the multiple generalized w-Genocchi polynomials.

Theorem 2.4 (multiplication formula). *Let $r\in\mathbb{N}$ and $a_1,\dots,a_r$ be strictly positive real numbers. For any $m\in\mathbb{N}$, one has*

$$\begin{aligned}
&G_{n,w}^{(r)}(mx;a_1,\dots,a_r)\\
&\quad=m^n\sum_{t_1,\dots,t_r=0}^{m-1} w^{a_1t_1+\cdots+a_rt_r}(-1)^{t_1+\cdots+t_r}\times G_{n,w^n}^{(r)}\left(\frac{x+a_1t_1+\cdots+a_rt_r}{m};a_1,\dots,a_r\right).
\end{aligned}\tag{2.15}$$

Corollary 2.5. *(1) If one sets $w=a_1=\cdots=a_r=1$ and $r,n\in\mathbb{N}$, then one obtains Raabe type formula for multiple Genocchi polynomials $G_n^{(r)}(x)$ as follows:*

$$G_n^{(r)}(mx)=m^n\sum_{t_1,\dots,t_r=0}^{m-1}G_n^{(r)}\left(x+\sum_{i=1}^{n}\frac{t_i}{m}\right),\tag{2.16}$$

where $(2t/(e^t+1))^r e^{xt}=\sum_{n=0}^{\infty}G_n^{(r)}(x)(t^n/n!)$.

(2) If one sets $w = 1$ and $r, n \in \mathbb{N}$, then one obtains Carlitz's multiplication formula for the multiple generalized Genocchi polynomials $G_n^{(r)}(x; a_1, \ldots, a_r)$ as follows:

$$G_n^{(r)}(mx; a_1, \ldots, a_r) = m^n \sum_{t_1,\ldots,t_r=0}^{m-1} G_n^{(r)}\left(x + \sum_{i=1}^{n} a_i \frac{t_i}{m}; a_1, \ldots, a_r \right), \tag{2.17}$$

where $((2t)^r / (\prod_{j=1}^{r}(e^{a_j t} + 1)))e^{xt} = \sum_{n=0}^{\infty} G_n^{(r)}(mx; a_1, \ldots, a_r)(t^n / n!)$.

Finally, we discuss the recurrence formula for the multiple generalized w-Genocchi polynomials as follows. Let $r \in \mathbb{N}$ and $a_1, \ldots, a_r$ be strictly positive real numbers. For any $k = 1, \ldots, r$, we can directly derive the following equation:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \binom{n}{j} G_{j,w}^{(k)}(x \mid a_1, \ldots, a_k) G_{n-j,w}^{(r-k)}(a_{k+1}, \ldots, a_r) \right) \frac{t^n}{n!} \\
&\quad = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} G_{j,w}^{(k)}(x \mid a_1, \ldots, a_k) \frac{t^j}{j!} G_{n-j,w}^{(r-k)}(a_{k+1}, \ldots, a_r) \frac{t^{n-j}}{(n-j)!} \right) \\
&\quad = \sum_{n=0}^{\infty} \left(\sum_{m+l=n,m,l\geq 0} G_{m,w}^{(k)}(x \mid a_1, \ldots, a_k) \frac{t^m}{m!} G_{l,w}^{(r-k)}(a_{k+1}, \ldots, a_r) \frac{t^l}{l!} \right) \\
&\quad = \left(\sum_{m=0}^{\infty} G_{m,w}^{(k)}(x \mid a_1, \ldots, a_k) \frac{t^m}{m!} \right) \left(\sum_{l=0}^{\infty} G_{l,w}^{(r-k)}(a_{k+1}, \ldots, a_r) \frac{t^l}{l!} \right) \\
&\quad = \left(\prod_{j=1}^{k} \frac{(2t)^k}{(we^t)_j^a + 1} e^{xt} \right) \left(\prod_{j=k+1}^{r} \frac{(2t)^{r-k}}{(we^t)_j^a + 1} \right) = \prod_{j=1}^{r} \left(\frac{(2t)^r}{(we^t)_j^a + 1} e^{xt} \right) \\
&\quad = \sum_{n=0}^{\infty} G_n^{(r)}(mx \mid a_1, \ldots, a_r) \frac{t^n}{n!}.
\end{aligned} \tag{2.18}$$

By comparing the coefficients on both sides in (2.18), we obtain the recurrence formula for the multiple generalized w-Genocchi polynomials.

Theorem 2.6 (recurrence formula). *Let $r \in \mathbb{N}$ and $a_1, \ldots, a_r$ be strictly positive real numbers. For any $k = 1, \ldots, r$, one has*

$$G_n^{(r)}(mx \mid a_1, \ldots, a_r) = \sum_{j=0}^{n} \binom{n}{j} G_{j,w}^{(k)}(x \mid a_1, \ldots, a_k) G_{n-j,w}^{(r-k)}(a_{k+1}, \ldots, a_r). \tag{2.19}$$

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References

[1] T. Kim, "q-Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.

[2] T. Kim, L.-C. Jang, and H. K. Pak, "A note on q-Euler and Genocchi numbers," *Proceedings of the Japan Academy, Series A*, vol. 77, no. 8, pp. 139–141, 2001.

[3] L. Carlitz, "The multiplication formulas for the Bernoulli and Euler polynomials," *Mathematics Magazine*, vol. 27, pp. 59–64, 1953.

[4] L. Jang and T. Kim, "q-Genocchi numbers and polynomials associated with fermionic p-adic invariant integrals on $\mathbb{Z}_p$," *Abstract and Applied Analysis*, vol. 2008, Article ID 232187, 8 pages, 2008.

[5] T. Kim, "New approach to q-Euler polynomials of higher order," *Russian Journal of Mathematical Physics*, vol. 17, no. 2, pp. 218–225, 2010.

[6] T. Kim, "New approach to q-Euler, Genocchi numbers and their interpolation functions," *Advanced Studies in Contemporary Mathematics (Kyungshang). Memoirs of the Jangjeon Mathematical Society*, vol. 18, no. 2, pp. 105–112, 2009.

[7] T. Kim, "Barnes-type multiple q-zeta functions and q-Euler polynomials," *Journal of Physics A*, vol. 43, no. 25, Article ID 255201, 11 pages, 2010.

[8] T. Kim, "Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$," *Russian Journal of Mathematical Physics*, vol. 16, no. 4, pp. 484–491, 2009.

[9] T. Kim, "On the q-extension of Euler and Genocchi numbers," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 1458–1465, 2007.

[10] T. Kim, "On p-adic q-l-functions and sums of powers," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 2, pp. 1472–1481, 2007.

[11] T. Kim, "On the symmetries of the q-Bernoulli polynomials," *Abstract and Applied Analysis*, vol. 2008, Article ID 914367, 7 pages, 2008.

[12] T. Kim, "Note on the Euler q-zeta functions," *Journal of Number Theory*, vol. 129, no. 7, pp. 1798–1804, 2009.

[13] A. Bayard and Y. Simsek, "On multiple generalized Bernoulli numbers and polynomials and their Barnes-Lerch zeta functions," preprint.