Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2010, Article ID 537909, 8 pages
doi:10.1155/2010/537909

Research Article

Block-by-Block Method for Solving Nonlinear Volterra-Fredholm Integral Equation

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Received 21 April 2010; Accepted 27 December 2010

Academic Editor: Christos H. Skiadas

We consider a nonlinear Volterra-Fredholm integral equation (NVFIE) of the second kind. The Volterra kernel is time dependent, and the Fredholm kernel is position dependent. Existence and uniqueness of the solution to this equation, under certain conditions, are discussed. The block-by-block method is introduced to solve such equations numerically. Some numerical examples are given to illustrate our results.

1. Introduction

Different methods are used to solve integral equations which are investigated from many physical applications such as the mixed problems in the theory of elasticity. Popov [1] applied the orthogonal polynomials method to solve the mixed problem in the mechanics of continuous media. Badr [2] applied Toeplitz matrix method to solve a NVFIE. Abdou et al. [3] discussed the solution of Harmmerstein-Volterra integral equation of the second kind. In [4], Haci obtained, numerically, the solution of a system of Harmmerstein integral equations in the space $L_2([a,b])$. The equivalence between Volterra integral equation with degenerate kernel and a linear system of differential equations is mentioned by Cochran [5]. Although there are some works on Hermite-type collocation method for the second-kind VIEs with smooth kernels, not too many studies have dealt with weakly singular kernel. For example, Papatheodorou and Jesanis [6] used the collocation method and obtained the solution of Volterra integrodifferential equation with weakly singular kernels. More information about different analytical and numerical solutions of Volterra equations can be found in Davis [7], Linz [8], Volterra [9], and Wolkenfelt [10].

In this paper, we consider the following NVFIE:

$$\phi(x,t) = f(x,t) + \lambda \int_{\tau=0}^{t} \int_{x=a}^{b} F(t,\tau)K(x,y)\gamma(\tau,y,\phi(y,\tau))dyd\tau, \quad 0 \le t \le T < 1. \tag{1.1}$$

The existence of a unique solution for the above equation, under certain conditions, is granted using fixed point theorem, where $K(x,y)$ is the Fredholm kernel and $F(t,\tau)$ is the Volterra kernel. $f(x,t)$ is called the free term, and the unknown function, $\phi(x,t)$, is called the potential function in the applied mathematics, and it will be determined. Both two functions $\phi(x,t)$ and $f(x,t)$ are assumed in the same space. The parameter λ has many physical meanings. A numerical method is applied to this equation, and it is reduced it to a system of Volterra integral equations of the second kind. Finally, the block-by-block method is used to obtain the numerical solution of this system. Some examples are stated to illustrate the results.

2. Existence and Uniqueness of Solution

To guarantee the existence and uniqueness of solution to (1.1), we write (1.1) in the integral operator form

$$\overline{W}\phi(x,t) = f(x,t) + W\phi(x,t), \tag{2.1}$$

where

$$W\phi(x,t) = \lambda \int_{0}^{t} \int_{a}^{b} F(t,\tau)k(x,y)\gamma(\tau,y,\phi(y,\tau))dyd\tau. \tag{2.2}$$

Also, we assume the following conditions:

(i) $k(x,y) \in L_2([a,b] \times [a,b])$ and satisfies, in general, the condition $\{\int_a^b \int_a^b |k(x,y)|^2 dxdy\}^{1/2} = c^*$ (c^* is a constant).

(ii) $F(t,\tau) \in C([0,T] \times [0,T])$, $0 \le \tau \le t \le T$ and satisfies $0 < F(t,\tau) \le M$, where M is a constant.

(iii) The given function $f(x,t)$ with its partial derivatives with respect to position and time is continuous in the space $L_2([a,b] \times [0,T])$, and its norm is defined as $\|f\| = \max_{t,x} |\int_0^t \{\int_a^b |f(x,\tau)|^2 dx\}^{1/2} d\tau| = H$ (H is a constant).

(iv) The known function $\gamma(\tau,y,\phi(y,\tau))$ satisfies the following conditions:

(a) $\|\gamma(\tau,y,\phi(y,\tau))\|_{L_2([a,b]\times[0,T])} \le B$,

(b) $|\gamma(\tau,y,\phi(y,\tau)) - \gamma(\tau,y,\varphi(y,\tau))| \le N|\phi(y,\tau) - \varphi(y,\tau)|$.

Under these conditions, using the Cauchy-Schwarz and Minkowski inequalities in conjunction with Schauders fixed point theorem, we easily can prove the following theorem.

Theorem 2.1. *If the conditions (i)–(iv) are satisfied, then* (1.1) *has a unique solution in the space* $L_2([a,b] \times C[0,T])$.

3. The Numerical Solution

In this section, we present a numerical method to solve (1.1). This method consists of two phases.

In phase one, we rewrite this equation as a system of linear Volterra integral equations. To do so, we choose a sufficiently small step size h_x, and we assume $a = x_0 < x_1 < x_2 < \cdots < x_{N_x} = b$ is a partition of the interval $[a, b]$ with $h_x = b - a/N_x$. Then, at each point x_n, (1.1) becomes

$$\phi_n(t) = f_n(t) + \lambda \int_0^t \int_a^b F(t,\tau) K(x_n, y) \gamma(\tau, y, \phi(y,\tau)) dy d\tau, \quad 0 \le t \le T, \tag{3.1}$$

where

$$\phi_n(t) = \phi(x_n, t), \qquad f_n(t) = f(x_n, t), \quad n = 0, 1, 2, \ldots, N_x. \tag{3.2}$$

Replacing the integral $\int_a^b K(x_n, y)\gamma(\tau, y, \phi(y,\tau))dy$ by a numerical integration rule of the form $\sum_{j=0}^{n} w_j k_{nj} \gamma_j(\tau, y_j, \phi_j(\tau))$, where $k_{nj} = k(x_n, x_j)$, then an approximate to (1.1) can be found as a system of Volterra integral equations:

$$\phi_n(t) = f_n(t) + \lambda \sum_{j=0}^{n} w_j k_{n,j} \int_0^t F(t,\tau) \gamma_j(\tau, y_j, \phi_j(\tau)) d\tau, \quad n = 0, 1, 2, \ldots, N_x. \tag{3.3}$$

In phase two, we use the block-by-block method to solve such system of integral equations of Volterra type. The following subsection explains how this method works.

3.1. Block-by-Block Method, See [8]

Assume that we need to solve an equation of the form

$$u(t) = v(t) + \lambda \int_0^t H(t, s, u(s)) ds. \tag{3.4}$$

The idea behind the block-by-block method is to divide the interval $[0, t]$ into a mesh $0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots < t_{N_t} = t < T$, and then we try to evaluate the value of the unknown function $u(t)$ at these points except at $t = 0$, where we have that $u(0) = v(0)$. Using any known rule, say Simpson's rule, we have

$$u(t_2) = v(t_2) + \lambda \frac{h_t}{3} \{ H(t_2, t_0, u(t_0)) + 4H(t_2, t_1, u(t_1)) + H(t_2, t_2, u(t_2)) \}. \tag{3.5}$$

To obtain a value for $u(t_1)$, we introduce the point $t_{1/2} = h_t/2$, $h_t = t/N_t$, and then we use Simpson's rule again to obtain

$$u(t_1) = v(t_1) + \lambda \frac{h_t}{3} \{ H(t_1, t_0, u(t_0)) + 4H(t_1, t_{1/2}, u(t_{1/2})) + H(t_1, t_1, u(t_1)) \}. \tag{3.6}$$

Replacing the value $u(t_{1/2})$ by a quadratic interpolation using the values u_0, u_1, and u_2, we have

$$u(t_{1/2}) = \frac{3}{8}u(t_0) + \frac{3}{4}u(t_1) - \frac{1}{8}u(t_2) \tag{3.7}$$

so that we can compute $u(t_1)$ by

$$\begin{aligned} u(t_1) &= v(t_1) \\ &\quad + \lambda\frac{h_t}{3}\left\{H(t_1, t_0, u(t_0)) + 4H\left(t_1, t_{1/2}, \left[\frac{3}{8}u(t_0) + \frac{3}{4}u(t_1) - \frac{1}{8}u(t_2)\right]\right) + H(t_1, t_1, u(t_1))\right\}. \end{aligned} \tag{3.8}$$

Equations (3.5) and (3.8) are a pair of simultaneous equations for $u(t_1)$ and $u(t_2)$. For sufficiently small h_t, $u(t_1)$ and $u(t_2)$ can be found uniquely using any procedure such as Netwon's method.

In general, for $m = 0, 1, 2, \ldots, N_t - 1$, the approximate solution of (3.4) is evaluated using the following two equations:

$$\begin{aligned} u(t_{2m+1}) &= v(t_{2m+1}) + \lambda h_t \sum_{s=0}^{2m} \kappa_s H(t_{2m+1}, t_s, u(t_s)) \\ &\quad + \frac{h_t}{6}\left\{H(t_{2m+1}, t_{2m}, u(t_{2m})) \right. \\ &\qquad + 4H\left(t_{2m+1}, t_{2m+1/2}, \left[\frac{3}{8}u(t_{2m}) + \frac{3}{4}u(t_{2m+1}) - \frac{1}{8}u(t_{2m+2})\right]\right) \\ &\qquad \left. + H(t_{2m+1}, t_{2m+1}, u(t_{2m+1}))\right\}, \\ u(t_{2m+2}) &= v(t_{2m+2}) + \lambda h_t \sum_{s=0}^{2n} \kappa_s H(t_{2m+2}, t_s, u(t_s)) \\ &\quad + \frac{h_t}{3}\{H(t_{2m+2}, t_{2m}, u(t_{2m})) \\ &\qquad + 4H(t_{2m+2}, t_{2m+1}, u(t_{2m+2})) + H(t_{2m+2}, t_{2m+2}, u(t_{2m+2}))\}, \end{aligned} \tag{3.9}$$

where

$$\kappa_s = \frac{1}{3}\{1, 4, 2, 2, \ldots, 2, 4, 1\}, \quad s = 0, 1, 2, \ldots, 2m, \tag{3.10}$$

$$t_{2m+1/2} = t_{2m} + \frac{h_t}{2}. \tag{3.11}$$

At each subinterval $[t_{2m+1}, t_{2m+2}]$, we solve these two equations simultaneously for the unknowns $u(t_{2m+1})$ and $u(t_{2m+2})$, so that we obtain a block of unknowns at a time.

3.2. Application of the Block-by-Block Method on the NVFIE

Combining the previous results, we obtain a new scheme defined by the following two equations:

$$\begin{aligned}
\phi_n(t_{2m+1}) &= f_n(t_{2m+1}) + \lambda \sum_{j=0}^{n} w_j \cdot h_t \sum_{s=0}^{2m} \kappa_s F(t_{2m+1}, t_s) k(x_n, y_j) \gamma_j\left(t_s, \phi_j(t_s)\right) \\
&\quad + \frac{h_t}{6} \Bigg\{ F(t_{2m+1}, t_{2m}) \gamma_n\left(t_{2m}, \phi_n(t_{2m})\right) + 4F(t_{2m+1}, t_{2m+1/2}) \\
&\qquad \times \left[\frac{3}{8} \gamma_n\left(t_{2m}, \phi_n(t_{2m})\right) + \frac{3}{4} \gamma_j\left(t_{2m+1}, \phi_n(t_{2m+1})\right) - \frac{1}{8} \gamma_n\left(t_{2m+2}, \phi_n(t_{2m+2})\right) \right] \\
&\qquad + F(t_{2m+1}, t_{2m+1}) \gamma_n\left(t_{2m+1}, \phi_n(t_{2m+1})\right) \Bigg\}, \\
\phi_n(t_{2m+2}) &= f_n(t_{2m+2}) + \lambda \sum_{j=0}^{n} w_j h_t \sum_{s=0}^{2m} \kappa_s F(t_{2n+2}, t_s) k(x_n, y_j) \gamma_j\left(t_s, \phi_j(t_s)\right) \\
&\quad + \frac{h_t}{3} \Big\{ F(t_{2m+2}, t_{2m}) \gamma_n\left(t_{2m}, \phi_n(t_{2m})\right) + 4F(t_{2m+2}, t_{2m+1}) \gamma_n\left(t_{2m+2}, \phi_n(t_{2m+2})\right) \\
&\qquad + F(t_{2m+2}, t_{2m+2}) \gamma_n\left(t_{2m+2}, \phi_n(t_{2m+2})\right) \Big\},
\end{aligned} \tag{3.12}$$

where $g_\alpha(t_\beta) = g(x_\alpha, t_\beta)$.

These two equations are two nonlinear equations in the two unknown $\phi(\cdot, t_{2m+1})$ and $\phi(\cdot, t_{2m+2})$, and so they can be determined uniquely using any procedure such as Netwons method. By repetitions, we get the solution $\phi(\cdot, t_m)$ for each $m = 1, 2, \ldots, 2N_t$.

4. Examples

We use the block-by-block method to solve numerically the following examples when

(i) case 1: $\gamma(t, x, \phi(x, t)) = \phi(x, t)$,

(ii) case 2: $\gamma(t, x, \phi(x, t)) = \phi^2(x, t)$.

Example 4.1. Solve the Volterra-Fredholm integral equation

$$\phi(x, t) = f(x, t) + \lambda \int_0^t \int_0^1 e^{x+y} \cdot \tau^2 \gamma\left(\tau, \phi(y, \tau)\right) dy d\tau, \quad \lambda = 0.1,\ 0 \le t \le T. \tag{4.1}$$

The exact solution of this integral equation is $\phi(x, t) = t^2 e^x$. Tables 1 and 2 list the error at different values of x and t for different value of N_x (number of partitions of the interval $[a, b]$ and $2N_t$ (number of partitions of the interval $[0, T]$; $T = 0.5$).

Table 1

$N_x = 20$, $N_t = 8$, case 1			$N_x = 20$, $N_t = 8$, case 2		
x	t	Error	x	t	Error
0.0	0.03125	$5.6968e-9$	0.0	0.03125	$1.187911448e-13$
0.0	0.0625	$8.2956e-8$	0.0	0.0625	$1.520526653e-11$
0.0	0.09375	$1.153468e-6$	0.0	0.09375	$2.597962336e-10$
0.0	0.125	$2.45626e-6$	0.0	0.125	$1.946274116e-9$
0.0	0.15625	$1.153968e-6$	0.0	0.15625	$9.280558183e-9$
0.0	0.1875	$1.676568e-5$	0.0	0.1875	$3.325391791e-8$
0.0	0.21875	$4.9747e-5$	0.0	0.21875	$9.782961574e-8$
0.0	0.25	$6.225697e-5$	0.0	0.25	$2.491230869e-7$
0.0	0.28125	$1.44479962e-4$	0.0	0.28125	$5.681743627e-7$
0.0	0.3125	$1.6765394e-4$	0.0	0.3125	$1.187911447e-6$
0.0	0.34375	$3.358295e-4$	0.0	0.34375	$2.314903342e-6$
0.0	0.375	$3.708329e-4$	0.0	0.375	$4.256501464e-6$
0.0	0.40625	$6.718015e-4$	0.0	0.40625	$7.453968013e-6$
0.0	0.4375	$7.185560e-3$	0.0	0.4375	$1.252219045e-5$
0.0	0.46875	$1.2112901e-3$	0.0	0.46875	$2.029657908e-5$

Table 2

$N_x = 20$, $N_t = 12$, case 1			$N_x = 20$, $N_t = 12$, case 2		
x	t	Error	x	t	Error
0.15	0.0208	$9.043e-10$	0.15	0.0208	$7e-15$
0.15	0.0417	$1.3740e-8$	0.15	0.0417	$9.1e-13$
0.15	0.0625	$1.85378e-7$	0.15	0.0625	$1.544e-11$
0.15	0.0833	$4.10254e-7$	0.15	0.0833	$1.165e-10$
0.15	0.1042	$1.89677e-6$	0.15	0.10417	$5.528e-10$
0.15	0.1250	$2.84822e-6$	0.15	0.125	$1.9913e-9$
0.15	0.1458	$8.37159e-6$	0.15	0.1458	$5.8391e-9$
0.15	0.16667	$1.080563e-5$	0.15	0.16667	$1.49276e-9$
0.15	0.1875	$2.493218e-5$	0.15	0.1875	$3.39692e-8$
0.15	0.2083	$2.975994e-5$	0.15	0.2083	$7.12269e-8$
0.15	0.2292	$5.912387e-5$	0.15	0.2292	$1.38578e-7$
0.15	0.25	$6.732566e-5$	0.15	0.250	$2.55370e-7$
0.15	0.2708	$1.2085744e-4$	0.15	0.27083	$4.46663e-7$
0.15	0.2917	$1.3339764e-4$	0.15	0.2917	$7.51666e-7$
0.15	0.3125	$2.225555e-4$	0.15	0.31250	$3.00819388e-4$

Example 4.2. Solve the Volterra-Fredholm integral equation

$$\phi(x,t) = f(x,t) + \lambda \int_0^t \int_0^1 \sin(x+y) \cdot \tau^2 \phi(y,\tau)\, dy\, d\tau, \quad \lambda = 0.2,\ 0 \le t \le 1. \tag{4.2}$$

The exact solution of this integral equation is $\phi(x,t) = t^2 \cdot \sin(x)$. Tables 3 and 4 list the error at different values of x and t for different value of Nx and $2Nt$, $T = 0.5$.

Table 3

$N_x = 20$, $N_t = 15$, case 1			$N_x = 20$, $N_t = 15$, case 2		
x	t	Error	x	t	Error
0.3	0.01666666667	$5.945e-11$	0.3	0.0167	0.0
0.3	0.03333333334	$6.357e-10$	0.3	0.033	$1.0e-13$
0.3	0.05000000001	$1.14903e-8$	0.3	0.05	$3.6e-12$
0.3	0.06666666668	$1.7798e-8$	0.3	0.067	$2e-12$
0.3	0.08333333335	$1.05510e-7$	0.3	0.083	$9.5e-11$
0.3	0.1000000000	$1.10750\ e-7$	0.3	0.1	$6.1-10$
0.3	0.1166666667	$4.09431e-7$	0.3	0.117	$5.94e-10$
0.3	0.1333333334	$3.58103e-7$	0.3	0.133	$7.52e-10$
0.3	0.1500000001	0.000001.053654	0.3	0.15	$1.533e-9$
0.3	0.1666666668	$8.03699e-7$	0.3	0.167	$5.029e-9$
0.3	0.1833333335	$2.114448e-6$	0.3	0.183	$2.06e-9$
0.3	0.2000000002	$1.39937e-6$	0.3	0.2	$2.278e-8$
0.3	0.2166666669	$3.55673e-6$	0.3	0.217	$1.446e-8$
0.3	0.2333333336	$1.94771e-6$	0.3	0.233	$7.929e-8$
0.3	0.2500000003	$5.17671e-6$	0.3	0.25	$7.235e-8$

Table 4

$N_x = 30$, $N_t = 20$, case 1			$N_x = 30$, $N_t = 20$, case 2		
x	t	Error	x	t	Error
0.9667	0.05	$2.4409e-8$	0.9667	0.05	$5.540\ e-9$
0.9667	0.0625	$1.04382e-7$	0.9667	0.0625	$1.6736e-9$
0.9667	0.075	$1.65928e-7$	0.9667	0.075	$4.1744e-8$
0.9667	0.0875	$4.48488e-7$	0.9667	0.0875	$1.928724554e-3$
0.9667	0.1	$6.13893e-7$	0.9667	0.1	$1.928987702e-3$
0.9667	0.1125	$1.30420e-6$	0.9667	0.1125	$3.1009e-7$
0.9667	0.125	$1.65044e-6$	0.9667	0.125	$5.2633e-7$
0.9667	0.1375	$3.02692e-6$	0.9667	0.1375	$8.3017e-7$
0.9667	0.15	$3.65065e-6$	0.9667	0.15	$1.292513e-6$
0.9667	0.1625	$6.06528e-6$	0.9667	0.1625	$3.46926e-3$
0.9667	0.175	$7.08188e-6$	0.9667	0.175	$3.47390471e-3$
0.9667	0.1875	$1.096068e-6$	0.9667	0.1875	$3.82799e-6$
0.9667	0.2	$1.250315e-5$	0.9667	0.2	$5.31250e-6$
0.9667	0.2125	$1.834653e-5$	0.967	0.2125	$4.49134734e-3$
0.9667	0.225	$2.056455e-5$	0.967	0.225	$4.50775985e-3$

5. Conclusion

Two rules of numerical integration methods are used to solve (1.1). One of these rules is of order $O(h_x)$, the rectangle method, which is applied on the integral with respect to the variable x. The other method is of higher-order, $O(h_t^4)$, the block-by-block method, and is applied on the integral with respect to the variable t where we used the fact that $\phi(x,0) = f(x,0)$. More information about the solution could allow us to use higher-order method with the block-by-block method.

References

[1] G. Ya. Popov, *Contact Problems for a Linearly Deformable Function*, Vishcho Shkola, Kiev, Russia, 1982.

[2] A. A. Badr, "Numerical solution of Fredholm-Volterra integral equation in one dimension with time dependent," *Applied Mathematics and Computation*, vol. 167, no. 2, pp. 1156–1161, 2005.

[3] M. A. Abdou, W. G. El-Sayed, and E. I. Deebs, "A solution of a nonlinear integral equation," *Applied Mathematics and Computation*, vol. 160, no. 1, pp. 1–14, 2005.

[4] L. Haci, "Approximate solutions to the system of Hammerstein integral equations," *Fasciculi Mathematici*, no. 18, pp. 112–117, 1988.

[5] J. A. Cochran, *The Analysis of Linear Integral Equations*, McGraw-Hill Series in Modern Applied Mathematics, McGraw-Hill, New York, NY, USA, 1972.

[6] T. S. Papatheodorou and M. E. Jesanis, "Collocation methods for Volterra integro-differential equations with singular kernels," *Journal of Computational and Applied Mathematics*, vol. 6, no. 1, pp. 3–8, 1980.

[7] H. T. Davis, *Introduction to Nonlinear Differential and Integral Equations*, Dover, New York, NY, USA, 1962.

[8] P. Linz, *Analytical and Numerical Methods for Volterra Equations*, vol. 7 of *SIAM Studies in Applied Mathematics*, SIAM, Philadelphia, Pa, USA, 1985.

[9] V. Volterra, *Theory of Functionals and of Integral and Integro-Differential Equations*, Dover, New York, NY, USA, 1959.

[10] P. H. M. Wolkenfelt, *The Numerical Analysis of Reducible Quadrature Methods for Volterra Integral and Integro-Differential Equations*, Akademisch Proefschrift, Mathematisch Centrum, Amsterdam, The Netherlands, 1981.