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Research Article

Weingarten and Linear Weingarten Type Tubular Surfaces in E^3

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We study tubular surfaces in Euclidean 3-space satisfying some equations in terms of the Gaussian curvature, the mean curvature, the second Gaussian curvature, and the second mean curvature. This paper is a completion of Weingarten and linear Weingarten tubular surfaces in Euclidean 3-space.

1. Introduction

Let f and g be smooth functions on a surface M in Euclidean 3-space E^3. The Jacobi function $\Phi(f, g)$ formed with f, g is defined by

$$\Phi(f, g) = \det\begin{pmatrix} f_s & f_t \\ g_s & g_t \end{pmatrix}, \tag{1.1}$$

where $f_s = \partial f/\partial s$ and $f_t = \partial f/\partial t$. In particular, a surface satisfying the Jacobi equation $\Phi(K, H) = 0$ with respect to the Gaussian curvature K and the mean curvature H on a surface M is called a Weingarten surface or a W-surface. Also, if a surface satisfies a linear equation with respect to K and H, that is, $aK + bH = c$, $(a, b, c) \neq (0, 0, 0)$, $a, b, c \in IR$, then it is said to be a linear Weingarten surface or a LW-surface [1].

When the constant $b = 0$, a linear Weingarten surface M reduces to a surface with constant Gaussian curvature. When the constant $a = 0$, a linear Weingarten surface M reduces to a surface with constant mean curvature. In such a sense, the linear Weingarten surfaces can be regarded as a natural generalization of surfaces with constant Gaussian curvature or with constant mean curvature [1].

If the second fundamental form II of a surface M in E^3 is nondegenerate, then it is regarded as a new pseudo-Riemannian metric. Therefore, the Gaussian curvature K_{II} is the second Gaussian curvature on M [1].

For a pair (X, Y), $X \neq Y$, of the curvatures K, H, K_{II} and H_{II} of M in E^3, if M satisfies $\Phi(X, Y) = 0$ by $aX + bY = c$, then it said to be a (X, Y)-Weingarten surface or (X, Y)-linear Weingarten surface, respectively [1].

Several geometers have studied W-surfaces and LW-surfaces and obtained many interesting results [1–9]. For the study of these surfaces, Kühnel and Stamou investigated ruled (X, Y)-Weingarten surfaces in Euclidean 3-space E^3 [7, 9]. Also, Baikoussis and Koufogiorgos studied helicoidal (H, K_{II})-Weingarten surfaces [10]. Dillen, and sodsiri, and Kühnel, gave a classification of ruled (X, Y)-Weingarten surfaces in Minkowski 3-space E_1^3, where $(X, Y) \in \{K, H, K_{II}\}$ [2–4]. Koufogiorgos, Hasanis, and Koutroufiotis investigated closed ovaloid (X, Y)-linear Weingarten surfaces in E^3 [11, 12]. Yoon, Blair and Koufogiorgos classified ruled (X, Y)-linear Weingarten surfaces in E^3 [8, 13, 14]. Ro and Yoon studied tubes in Euclidean 3-space which are (K, H), (K, K_{II}), (H, K_{II})-Weingarten, and linear Weingarten tubes, satisfying some equations in terms of the Gaussian curvature, the mean curvature, and the second Gaussian curvature [1].

Following the Jacobi equation and the linear equation with respect to the Gaussian curvature K, the mean curvature H, the second Gaussian curvature K_{II}, and the second mean curvature H_{II}, an interesting geometric question is raised: classify all surfaces in Euclidean 3-space satisfying the conditions

$$\begin{aligned} \Phi(X, Y) &= 0, \\ aX + bY &= c, \end{aligned} \tag{1.2}$$

where $X, Y \in \{K, H, K_{II}, H_{II}\}$, $X \neq Y$ and $(a, b, c) \neq (0, 0, 0)$.

In this paper, we would like to contribute the solution of the above question by studying this question for tubes or tubular surfaces in Euclidean 3-space E^3.

2. Preliminaries

We denote a surface M in E^3 by

$$M(s, t) = (m_1(s, t), m_2(s, t), m_3(s, t)). \tag{2.1}$$

Let U be the standard unit normal vector field on a surface M defined by

$$U = \frac{M_s \wedge M_t}{\|M_s \wedge M_t\|}, \tag{2.2}$$

where $M_s = \partial M(s, t)/\partial s$. Then, the first fundamental form I and the second fundamental form II of a surface M are defined by, respectively,

$$\begin{aligned} \mathrm{I} &= E ds^2 + 2F ds\, dt + G dt^2, \\ \mathrm{II} &= e ds^2 + 2f ds\, dt + g dt^2, \end{aligned} \tag{2.3}$$

where

$$
\begin{gathered}
E=\langle M_s,M_s\rangle,\quad F=\langle M_s,M_t\rangle,\quad G=\langle M_t,M_t\rangle,\\
e=-\langle M_s,U_s\rangle=\langle M_{ss},U\rangle,\quad f=-\langle M_s,U_t\rangle=\langle M_{st},U\rangle,\quad g=-\langle M_t,U_t\rangle=\langle M_{tt},U\rangle,
\end{gathered}
\tag{2.4}
$$

[14]. On the other hand, the Gaussian curvature K and the mean curvature H are

$$
\begin{aligned}
K&=\frac{eg-f^2}{EG-F^2},\\
H&=\frac{Eg-2Ff+Ge}{2(EG-F^2)},
\end{aligned}
\tag{2.5}
$$

respectively. From Brioschi's formula in a Euclidean 3-space, we are able to compute K_{II} and H_{II} of a surface by replacing the components of the first fundamental form E, F, and G by the components of the second fundamental form e, f, and g, respectively [14]. Consequently, the second Gaussian curvature K_{II} of a surface is defined by

$$
K_{\mathrm{II}}=\frac{1}{\left(|eg|-f^2\right)^2}\left\{\begin{vmatrix}-\frac{1}{2}e_{tt}+f_{st}-\frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s-\frac{1}{2}e_t\\ f_t-\frac{1}{2}g_s & e & f\\ \frac{1}{2}g_t & f & g\end{vmatrix}-\begin{vmatrix}0 & \frac{1}{2}e_t & \frac{1}{2}g_s\\ \frac{1}{2}e_t & e & f\\ \frac{1}{2}g_s & f & g\end{vmatrix}\right\},
\tag{2.6}
$$

and the second mean curvature H_{II} of a surface is defined by

$$
H_{\mathrm{II}}=H-\frac{1}{2\sqrt{|\det \mathrm{II}|}}\sum_{i,j}\frac{\partial}{\partial u^i}\left(\sqrt{|\det \mathrm{II}|}L^{ij}\frac{\partial}{\partial u^j}\left(\ln\sqrt{|K|}\right)\right),
\tag{2.7}
$$

where u^i and u^j stand for "s" and "$\theta = t$", respectively, and $L^{ij} = (L_{ij})^{-1}$, where L_{ij} are the coefficients of the second fundamental form [3, 4].

Remark 2.1. It is well known that a minimal surface has a vanishing second Gaussian curvature, but that a surface with the vanishing second Gaussian curvature need not to be minimal [14].

3. Weingarten Tubular Surfaces

Definition 3.1. Let $\alpha : [a,b] \to \mathrm{E}^3$ be a unit-speed curve. A tubular surface of radius $\lambda > 0$ about α is the surface with parametrization

$$
M(s,\theta)=\alpha(s)+\lambda[N(s)\cos\theta+B(s)\sin\theta],
\tag{3.1}
$$

$a \leq s \leq b$, where $N(s)$, $B(s)$ are the principal normal and the binormal vectors of α, respectively [1].

The curvature and the torsion of the curve α are denoted by κ, τ. Then, Frenet formula of $\alpha(s)$ is defined by

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{3.2}$$

[1]. Furthermore, we have the natural frame $\{M_S, M_\theta\}$ given by

$$\begin{gathered} M_s = (1 - \lambda\kappa\cos\theta)T - \lambda\tau\sin\theta N + \lambda\tau\cos\theta B, \\ M_\theta = -\lambda\sin\theta N + \lambda\cos\theta B. \end{gathered} \tag{3.3}$$

The components of the first fundamental form are

$$E = \lambda^2\tau^2 + \sigma^2, \qquad F = \lambda^2\tau, \qquad G = \lambda^2, \tag{3.4}$$

where $\sigma = 1 - \lambda\kappa\cos\theta$.

On the other hand, the unit normal vector field U is obtained by

$$U = \frac{M_s \wedge M_\theta}{\|M_s \wedge M_\theta\|} = -\varepsilon\cos\theta N - \varepsilon\sin\theta B. \tag{3.5}$$

As $\lambda > 0$, ε is the sign of σ such that if $\sigma < 0$, then $\varepsilon = -1$ and if $\sigma > 0$, then $\varepsilon = 1$. From this, the components of the second fundamental form of M are given by

$$e = \varepsilon\lambda\tau^2 - \varepsilon\kappa\cos\theta\sigma, \qquad f = \varepsilon\lambda\tau, \qquad g = \varepsilon\lambda. \tag{3.6}$$

If the second fundamental form is nondegenerate, $eg - f^2 \neq 0$, that is, κ, σ and $\cos\theta$ are nowhere vanishing. In this case, we can define formally the second Gaussian curvature K_{II} and the second mean curvature H_{II} on M. On the other hand, the Gauss curvature K, the mean curvature H, the second Gaussian curvature K_{II} and the second mean curvature H_{II} are obtained by using (2.5), (2.6) and (2.7) as follows:

$$
\begin{aligned}
K &= -\frac{\kappa\cos\theta}{\lambda\sigma}, \\
H &= \frac{\varepsilon(1-2\lambda\kappa\cos\theta)}{2\lambda\sigma}, \\
K_{\mathrm{II}} &= -\frac{\varepsilon\kappa\left(\cos^2\theta - 6\kappa\lambda\cos^3\theta + 4\kappa^2\lambda^2\cos^4\theta + 1\right)}{4\cos\theta\sigma}, \\
H_{\mathrm{II}} &= \frac{1}{-8\varepsilon\lambda\kappa^3\cos^3\theta\sigma^3}\left(\sum_{i=0}^{6} g_i\cos^i\theta\right),
\end{aligned}
\tag{3.7}
$$

and where the coefficients g_i are

$$
\begin{aligned}
g_0 &= 3\lambda^2\kappa^2\tau^2, \\
g_1 &= 2\lambda\kappa(\kappa_s\tau - \kappa\tau_s)\sin\theta - \left(1+6\lambda^2\tau^2\right)\kappa^3, \\
g_2 &= 2\lambda^2\kappa^2(\kappa\tau_s - 4\kappa_s\tau)\sin\theta + \lambda\left(3(\kappa_s)^2 + 3\kappa^4 - 2\kappa\kappa_{ss} - \kappa^2\tau^2\right), \\
g_3 &= 2\lambda^2\kappa\left(2\kappa^2\tau^2 - \kappa^3 + \kappa\kappa_{ss} - 3(\kappa_s)^2\right) - \kappa^3, \\
g_4 &= 16\lambda\kappa^4, \\
g_5 &= -20\lambda^2\kappa^5, \\
g_6 &= 8\lambda^3\kappa^6.
\end{aligned}
\tag{3.8}
$$

Differentiating K, K_{II}, H, and H_{II} with respect to s and θ, after straightforward calculations, we get,

$$
K_s = -\frac{\kappa_s\cos\theta}{\lambda\sigma^2}, \qquad K_\theta = \frac{\kappa\sin\theta}{\lambda\sigma^2}, \tag{3.9}
$$

$$
H_s = -\frac{\varepsilon\kappa_s\cos\theta}{2\sigma^2}, \qquad H_\theta = \frac{\varepsilon\kappa\sin\theta}{2\sigma^2}, \tag{3.10}
$$

$$
(K_{\mathrm{II}})_s = \frac{\varepsilon\kappa_s\left(8\lambda^3\kappa^3\cos^5\theta - 18\lambda^2\kappa^2\cos^4\theta + 12\lambda\kappa\cos^3\theta - \cos^2\theta - 1\right)}{4\cos\theta\sigma^2}, \tag{3.11}
$$

$$
(K_{\mathrm{II}})_\theta = -\frac{\varepsilon\kappa\sin\theta\left(8\lambda^3\kappa^3\cos^5\theta - 18\lambda^2\kappa^2\cos^4\theta + 12\lambda\kappa\cos^3\theta + \sin^2\theta - 2\lambda\kappa\cos\theta\right)}{4\cos^2\theta\sigma^2}, \tag{3.12}
$$

$$
(H_{\mathrm{II}})_s = \frac{1}{8\varepsilon\kappa^4\cos^3\theta\sigma^4}\left(\sum_{i=0}^{6} f_i\cos^i\theta\right), \tag{3.13}
$$

and where f_i are

$$
\begin{aligned}
f_0 &= 3\kappa^2\tau(\kappa_s\tau - 2\kappa\tau_s),\\
f_1 &= 2\kappa(2\kappa_s(\kappa_s\tau - \kappa\tau_s) - \kappa\kappa_{ss}\tau)\sin\theta + (3\kappa\tau_s - 2\kappa_s\tau)6\lambda\kappa^3\tau,\\
f_2 &= 2\lambda\kappa^2(9\kappa_s(\kappa\tau_s - \kappa_s\tau) + 2\kappa\kappa_{ss}\tau)\sin\theta + 6\lambda^2\kappa^4\tau(3\kappa_s\tau - 2\kappa\tau_s) + \kappa_s\left(9(\kappa_s)^2 - 10\kappa\kappa_{ss}\right)\\
&\quad + \kappa^2\tau(2\kappa\tau_s - \kappa_s\tau),\\
f_3 &= 2\lambda^2\kappa^3(\kappa_s(16\kappa_s\tau - 7\kappa\tau_s) - 4\kappa\tau\kappa_{ss})\sin\theta\\
&\quad + 2\lambda\kappa\left(15\kappa\kappa_s\kappa_{ss} - \left(15(\kappa_s)^2 + \kappa^4\right)\kappa_s + \kappa^2\tau(2\tau\kappa_s - 5\kappa\tau_s)\right),\\
f_4 &= 2\lambda^2\kappa^2\left(5\kappa_s\left(3(\kappa_s)^2 - 2\kappa\kappa_{ss}\right) + 2\kappa^2\tau(2\kappa\tau_s - 3\tau\kappa_s) + \kappa^4\kappa_s\right) - 2\kappa^4\kappa_s,\\
f_5 &= 6\lambda\kappa^5\kappa_s,\\
f_6 &= -4\lambda^2\kappa^6\kappa_s,
\end{aligned}
\tag{3.14}
$$

$$
(H_{\mathrm{II}})_\theta = \frac{1}{8\varepsilon\lambda\kappa^3\cos^4\theta\sigma^4}\left(\sum_{i=0}^{6} h_i\cos^i\theta\right), \tag{3.15}
$$

and where the coefficients h_i are

$$
\begin{aligned}
h_0 &= -9\lambda\kappa^2\tau^2\sin\theta,\\
h_1 &= 2\kappa^3\left(1 + 15\lambda^2\tau^2\right)\sin\theta + 4\lambda\kappa(\kappa\tau_s - \kappa_s\tau),\\
h_2 &= \lambda\left(2\kappa\kappa_{ss} - 8\kappa^4 + \kappa^2\tau^2\left(1 - 30\lambda^2\kappa^2\right) - 3(\kappa_s)^2\right)\sin\theta + 6\lambda^2\kappa^2(3\kappa_s\tau - 2\kappa\tau_s),\\
h_3 &= 4\lambda^2\kappa\left(2\kappa^4 - \kappa^2\tau^2 - 2\kappa\kappa_{ss} + 3(\kappa_s)^2\right)\sin\theta + 2\lambda\kappa\left(\kappa_s\tau - \kappa\tau_s + 4\lambda^2\kappa^2(\kappa\tau_s - 4\kappa_s\tau)\right),\\
h_4 &= 2\lambda\kappa^3\left(3\lambda^2\left(2\kappa\tau^2 - \kappa^3 + \kappa_{ss}\right) + \kappa\right)\sin\theta + 2\lambda^2\kappa^2\left(4(\kappa\tau_s - \tau\kappa_s) - 9\lambda(\kappa_s)^3\right),\\
h_5 &= 6\lambda^2\kappa^3\left(\lambda(4\kappa_s\tau - \kappa\tau_s) - \kappa^2\sin\theta\right),\\
h_6 &= 4\lambda^3\kappa^6\sin\theta.
\end{aligned}
\tag{3.16}
$$

Now, we consider a tubular surface M in E^3 satisfying the Jacobi equation $\Phi(K, H_{\mathrm{II}}) = 0$. By using (3.9), (3.13), and (3.15), we obtain $\Phi(K, H_{\mathrm{II}})$ in the following form:

$$
K_s(H_{\mathrm{II}})_\theta - K_\theta(H_{\mathrm{II}})_s = \frac{-\varepsilon}{4\lambda^2\kappa^3\sigma^5\cos^3\theta}\sum_{i=0}^{4} u_i\cos^i\theta, \tag{3.17}
$$

with respect to the Gaussian curvature K and the second mean curvature H_{II}, where

$$
\begin{aligned}
u_0 &= -3\lambda\tau\kappa^2(\kappa_s\tau + \kappa\tau_s)\sin\theta,\\
u_1 &= \kappa^3\Big(\Big(6\lambda^2\tau^2 + 1\Big)\kappa_s + 6\lambda^2\kappa\tau\tau_s\Big)\sin\theta - \lambda\kappa^2\kappa_{ss}\tau + \lambda\kappa^3\tau_{ss},\\
u_2 &= \lambda\Big(\kappa^2\kappa_{sss} - 4\kappa\kappa_s\kappa_{ss} - 3\kappa^4\kappa_s + 3(\kappa_s)^3 + \kappa^3\tau\tau_s\Big)\sin\theta + \lambda^2\kappa^3(3\kappa_s\tau_s + 4\kappa_{ss}\tau - \kappa\tau_{ss}),\\
u_3 &= \lambda\kappa\Big\{\Big(7\lambda\kappa\kappa_s\kappa_{ss} - \lambda\kappa^2\kappa_{sss} - 6\lambda(\kappa_s)^3 + 2\lambda\kappa^4\kappa_s - 4\lambda\kappa^3\tau\tau_s\Big)\sin\theta\\
&\qquad + \Big(\kappa\kappa_{ss}\tau + \kappa\kappa_s\tau_s - (\kappa_s)^2\tau - \kappa^2\tau_{ss}\Big)\Big\},\\
u_4 &= -\lambda^2\kappa^2\Big\{4\kappa\kappa_s\tau_s - 4\tau(\kappa_s)^2 - \kappa^2\tau_{ss} + \kappa\kappa_{ss}\tau\Big\}.
\end{aligned}
\tag{3.18}
$$

Then, by $\Phi(K, H_{\mathrm{II}}) = 0$, (3.17) becomes

$$
\sum_{i=0}^{4} u_i \cos^i\theta = 0. \tag{3.19}
$$

Hence, we have the following theorem.

Theorem 3.2. *Let M be a tubular surface defined by* (3.1) *with nondegenerate second fundamental form. M is a (K, H_{II})-Weingarten surface if and only if M is a tubular surface around a circle or a helix.*

Proof. Let us assume that M is a (K, H_{II})-Weingarten surface, then the Jacobi equation (3.19) is satisfied. Since polynomial in (3.19) is equal to zero for every θ, all its coefficients must be zero. Therefore, the solutions of $u_0 = u_1 = u_2 = u_3 = u_4 = 0$ are $\kappa_s = 0$, $\tau = 0$ and $\kappa_s = 0$, $\tau_s = 0$ that is, M is a tubular surface around a circle or a helix, respectively.

Conversely, suppose that M is a tubular surface around a circle or a helix, then it is easily to see that $\Phi(K, H_{\mathrm{II}}) = 0$ is satisfied for the cases both $\kappa_s = 0$, $\tau = 0$ and $\kappa_s = 0$, $\tau_s = 0$. Thus M is a (K, H_{II})-Weingarten surface.

We suppose that a tubular surface M with nondegenerate second fundamental form in E^3 is (H, H_{II})-Weingarten surface. From (3.10), (3.13), and (3.15), $\Phi(H, H_{\mathrm{II}})$ is

$$
H_s(H_{\mathrm{II}})_\theta - H_\theta(H_{\mathrm{II}})_s = \frac{1}{8\lambda\kappa^3\sigma^5\cos^3\theta}\sum_{i=0}^{4} v_i\cos^i\theta, \tag{3.20}
$$

with respect to the variable $\cos\theta$, where

$$
\begin{aligned}
v_0 &= 3\lambda\tau\kappa^2(\kappa\tau_s + \kappa_s\tau)\sin\theta,\\
v_1 &= -\kappa^3\Big(\kappa_s + 6\lambda^2\tau(\kappa_s\tau + \kappa\tau_s)\Big)\sin\theta + \lambda\kappa^2(\kappa_{ss}\tau - \kappa\tau_{ss}),
\end{aligned}
$$

$$v_2 = \lambda\left(3\kappa^4\kappa_s - 3(\kappa_s)^3 + 4\kappa\kappa_s\kappa_{ss} - \kappa^3\tau\tau_s - \kappa^2\kappa_{sss}\right)\sin\theta$$
$$+ \lambda^2\kappa^3(\kappa\tau_{ss} - 3\kappa_s\tau_s - 4\kappa_{ss}\tau),$$
$$v_3 = \lambda^2\kappa\left(6(\kappa_s)^3 + \kappa^2\kappa_{sss} - 7\kappa\kappa_s\kappa_{ss} - 2\kappa^4\kappa_s + 4\kappa^3\tau\tau_s\right)\sin\theta$$
$$+ \lambda\kappa\left(\kappa^2\tau_{ss} + (\kappa_s)^2\tau - \kappa\kappa_{ss}\tau - \kappa\kappa_s\tau_s\right),$$
$$v_4 = -\lambda^2\kappa^2\left(\kappa^2\tau_{ss} - 4\kappa\kappa_{ss}\tau - 4\kappa\kappa_s\tau_s + 4(\kappa_s)^2\tau\right). \quad (3.21)$$

Then, by $\Phi(H, H_{\mathrm{II}}) = 0$, (3.22) becomes in following form:

$$\sum_{i=0}^{4} v_i \cos^i\theta = 0. \quad (3.22)$$

Thus, we state the following theorem. □

Theorem 3.3. *Let M be a tubular surface defined by* (3.1) *with nondegenerate second fundamental form. M is a (H, H_{II})-Weingarten surface if and only if M is a tubular surface around a circle or a helix.*

Proof. Considering $\Phi(H, H_{\mathrm{II}}) = 0$ and by using (3.13), one can obtaine the solutions $\kappa_s = 0$, $\tau = 0$, and $\kappa_s = 0$, $\tau_s = 0$ of the equations $v_0 = v_1 = v_2 = v_3 = v_4 = 0$ for all θ. Thus, it is easly proved that M is a (H, H_{II})-Weingarten surface if and only if M is a tubular surface around a circle or a helix. □

We consider a tubular surface M is $(K_{\mathrm{II}}, H_{\mathrm{II}})$-Weingarten surface with nondegenerate second fundamental form in E^3. By using (3.11), (3.12), (3.13), and (3.15), $\Phi(K_{\mathrm{II}}, H_{\mathrm{II}})$ is

$$(K_{\mathrm{II}})_s(H_{\mathrm{II}})_\theta - (K_{\mathrm{II}})_\theta(H_{\mathrm{II}})_s = \frac{-1}{16\lambda\kappa^3\sigma^5\cos^5\theta}\sum_{i=0}^{9} w_i\cos^i\theta, \quad (3.23)$$

where

$$w_0 = 3\lambda\tau\kappa^2(\kappa\tau_s - 2\kappa_s\tau)\sin\theta,$$
$$w_1 = \kappa^3\left(\kappa_s + 18\lambda^2\tau(\kappa_s\tau - 2\kappa\tau_s)\right)\sin\theta + \lambda\kappa\left(4\kappa_s(\kappa\tau_s - \kappa_s\tau) + \kappa\kappa_{ss}\tau - \kappa^2\tau_{ss}\right),$$
$$w_2 = \left\{\left(6\kappa\kappa_{ss} - 18\lambda^2\kappa^4\tau^2 - 3\kappa^4 - 6(\kappa_s)^2 - 2\kappa^2\tau^2\right)\lambda\kappa_s + 4\left(3\lambda^2\kappa^2 - 1\right)\lambda\kappa^3\tau\tau_s - \lambda\kappa^2\kappa_{sss}\right\}\sin\theta$$
$$+ 3\lambda^2\kappa^2\left(\kappa_s(6\kappa_s\tau - 5\kappa\tau_s) - 2\kappa\kappa_{ss}\tau + \kappa^2\tau_{ss}\right),$$

$$w_3 = \left\{ \left(\kappa^2 + 38\lambda^2\kappa^2\tau^2 + 4\lambda^2\kappa^4 - 23\lambda^2\kappa\kappa_{ss} + 24\lambda^2(\kappa_s)^2\right)\kappa\kappa_s + 48\lambda^2\kappa^4\tau\tau_s + 3\lambda^2\kappa^3\kappa_{sss} \right\} \sin\theta$$
$$- \lambda\kappa\left\{2\left(\lambda^2\kappa^2 - 1\right)\kappa^2\tau_{ss} + \left(32\lambda^2\kappa^2 - 3\right)(\kappa_s)^2\tau_s - \left(14\lambda^2\kappa^2 - 3\right)\kappa\kappa_s\tau_s \right.$$
$$\left. +2\left(1 - 4\lambda^2\kappa^2\right)\kappa\kappa_{ss}\tau\right\},$$
$$w_4 = -\lambda\left\{ \left(2\lambda^2\kappa^2 - 1\right)\kappa^2\kappa_{ss} + 4\left(1 - 5\lambda^2\kappa^2\right)\kappa\kappa_s\kappa_{ss} + \left(134\lambda^2\kappa^2 - 1\right)\kappa^3\tau\tau_s \right.$$
$$\left. +\left(3\left(10\lambda^2\kappa^2 - 1\right)(\kappa_s)^2 + \left(2\left(57\tau^2 + \kappa^2\right)\lambda^2 + 13\right)\kappa^4\right)\kappa_s \right\} \sin\theta$$
$$+ \lambda^2\kappa^2\left(22\kappa\kappa_{ss}\tau + 17\kappa\kappa_s\tau_s - 14(\kappa_s)^2\tau - 16\kappa^2\tau_{ss}\right),$$
$$w_5 = \lambda^2\kappa\left\{55\kappa\kappa_s\kappa_{ss} + 4\left(33\lambda^2\kappa^2 - 4\right)\kappa^3\tau\tau_s + 2\left(66\lambda^2\tau^2 + 25\right)\kappa^4\kappa_s - \left(13\kappa^2 + 42\right)\kappa_{sss}\right\} \sin\theta$$
$$- \lambda\kappa\left\{ \left(50\lambda^2\kappa^2 - 1\right)\kappa\kappa_s\tau_s + \left(1 - 32\lambda^2\kappa^2\right)(\kappa_s)^2\tau + \left(1 - 32\lambda^2\kappa^2\right)\kappa^2\tau_{ss} \right.$$
$$\left. +\left(74\lambda^2\kappa^2 - 1\right)\kappa\kappa_{ss}\tau\right\},$$
$$w_6 = 2\lambda^3\kappa^2\left(63(\kappa_s)^3 - 24\lambda^2\kappa^4\kappa_s\tau^2 - 41\kappa^4\kappa_s + 33\kappa^3\tau\tau_s - 78\kappa\kappa_s\kappa_{ss} - 24\lambda^2\kappa^5\tau\tau_s + 15\kappa^2\kappa_{sss}\right) \sin\theta$$
$$+ \lambda^2\kappa^2\left(16(\kappa_s)^2\tau - 26\lambda^2\kappa^4\tau_{ss} - 16\kappa\kappa_{ss}\tau + 13\kappa^2\tau_{ss} - 16\kappa\kappa_s\tau_s + 54\lambda^2\kappa^3\kappa_s\tau_s + 80\lambda^2\kappa^3\kappa_{ss}\tau\right),$$
$$w_7 = 2\lambda^4\kappa^3\left(30\kappa^4\kappa_s - 13\kappa^2\kappa_{sss} - 40\kappa^3\tau\tau_s + 79\kappa\kappa_s\kappa_{ss} - 60(\kappa_s)^3\right) \sin\theta +$$
$$- 2\lambda^3\kappa^3\left(33(\kappa_s)^2\tau_s - 33\kappa\kappa_s\tau_s - 4\lambda^2\kappa^4\tau_{ss} + 12\lambda^2\kappa^3\kappa_s\tau_s + 15\kappa^2\tau_{ss} + 16\lambda^2\kappa^3\kappa_{ss}\tau - 33\kappa\kappa_{ss}\tau\right),$$
$$w_8 = -8\lambda^5\kappa^4\left(-6(\kappa_s)^3 + 2\kappa^4\kappa_s - 4\kappa^3\tau\tau_s + 7\kappa\kappa_s\kappa_{ss} - \kappa^2\kappa_{sss}\right) \sin\theta$$
$$+ 2\lambda^4\kappa^4\left(13\kappa^2\tau_{ss} + 4(\kappa_s)^2\tau - 40\kappa(\kappa_{ss}\tau + \kappa_s\tau_s)\right),$$
$$w_9 = 8\lambda^5\kappa^5\left\{4\kappa\kappa_s\tau_s - 4(\kappa_s)^2\tau - \kappa^2\tau_{ss} + 4\kappa\kappa_{ss}\tau\right\}. \tag{3.24}$$

Since $\Phi(K_{\mathrm{II}}, H_{\mathrm{II}}) = 0$, then (3.23) becomes in following form:

$$\sum_{i=0}^{9} w_i \cos^i\theta = 0. \tag{3.25}$$

Hence, we have the following theorem.

Theorem 3.4. *Let M be a tubular surface defined by* (3.1) *with nondegenerate second fundamental form. M is a $(K_{\mathrm{II}}, H_{\mathrm{II}})$-Weingarten surface if and only if M is a tubular surface around a circle or a helix.*

Proof. It can be easly proved similar to Theorems 3.2 and 3.3. □

Consequently, we can give the following main theorem for the end of this part.

Theorem 3.5. *Let* $(X,Y) \in \{(K,H_{\mathrm{II}}),(H,K_{\mathrm{II}}),(H_{\mathrm{II}},K_{\mathrm{II}})\}$*, and let* M *be a tubular surface defined by* (3.1) *with nondegenerate second fundamental form.* M *is a* (X,Y)*-Weingarten surface if and only if* M *is a tubular surface around a circle or a helix.*

Thus, the study of Weingarten tubular surfaces in 3-dimensional Euclidean space is completed with [1].

4. Linear Weingarten Tubular Surfaces

In last part of this paper, we study on (K,H_{II}), (H,H_{II}), $(H_{\mathrm{II}},K_{\mathrm{II}})$, (K,H,H_{II}), (K,H,K_{II}), $(H,K_{\mathrm{II}},H_{\mathrm{II}})$, $(K,K_{\mathrm{II}},H_{\mathrm{II}})$, and $(K,H,K_{\mathrm{II}},H_{\mathrm{II}})$ linear Weingarten tubular surfaces in $\mathbb{E}^3$. (K,H), (K,K_{II}), and (H,K_{II}) linear Weingarten tubes are studied in [1].

Let a_1, a_2, a_3, a_4, and b be constants. In general, a linear combination of K, H, K_{II} and H_{II} can be constructed as

$$a_1K + a_2H + a_3K_{\mathrm{II}} + a_4H_{\mathrm{II}} = b. \tag{4.1}$$

By the straightforward calculations, we obtained the reduced form of (4.1)

$$8b\kappa^3\varepsilon\sigma^3\cos^3\theta + \sum_{i=0}^{8} p_i\cos^i\theta = 0, \tag{4.2}$$

where the coefficients are

$$\begin{aligned}
p_0 &= 3a_4\lambda\kappa^2\tau^2,\\
p_1 &= a_4\kappa\Big(2\lambda(\kappa_s\tau - \kappa\tau_s)\sin\theta - \kappa^2\Big(6\lambda^2\tau^2 + 1\Big)\Big),\\
p_2 &= a_4\lambda\Big(2\lambda\kappa^2(\kappa\tau_s - 4\kappa_s\tau)\sin\theta + \kappa^2\Big(3\kappa^2 - \tau^2\Big) - 2\kappa\kappa_{ss} + 3{\kappa_s}^2\Big) + 2a_3\lambda\kappa^4,\\
p_3 &= a_4\kappa\Big(2\lambda^2\Big(\kappa\kappa_{ss} - \kappa^4 + 2\kappa^2\tau^2 - 3{\kappa_s}^2\Big) - 5\kappa^2\Big) - 4a_2\kappa^3 - 4a_3\lambda^2\kappa^5,\\
p_4 &= 8a_1\varepsilon\kappa^4 + 16a_2\lambda\kappa^4 + 2a_3\lambda\kappa^4\Big(1 + \lambda^2\kappa^2\Big) + 17a_4\lambda\kappa^4,\\
p_5 &= -16a_1\varepsilon\lambda\kappa^5 - 20a_2\lambda^2\kappa^5 - 16a_3\lambda^2\kappa^5 - 20a_4\lambda^2\kappa^5,\\
p_6 &= 8a_1\varepsilon\lambda^2\kappa^6 + 8a_2\lambda^3\kappa^6 + 34a_3\lambda^3\kappa^6,\\
p_7 &= -28a_3\lambda^4\kappa^7,\\
p_8 &= 8a_3\lambda^5\kappa^8.
\end{aligned} \tag{4.3}$$

Then, p_0, p_1, p_2, p_7, and p_8 are zero for any $b \in IR$. If $a_4 \neq 0$ or $a_3 \neq 0$, from $p_0 = p_1 = p_7 = p_8 = 0$, one has $\kappa = 0$. Hence, we can give the following theorems.

Theorem 4.1. *Let* $(X, Y) \in \{(K, H_{III}), (H, H_{II}), (K_{II}, H_{II})\}$*. Then, there are no* (X, Y)*-linear Weingarten tubular surfaces* M *in Euclidean 3-space defined by* (3.1) *with nondegenerate second fundamental form.*

Theorem 4.2. *Let* $(X, Y, Z) \in \{(H, K_{II}, H_{II}), (K, K_{II}, H_{II}), (K, H, H_{II}), (K, H, K_{II})\}$*. Then, there are no* (X, Y, Z)*-linear Weingarten tubular surfaces* M *in Euclidean 3-space defined by* (3.1) *with nondegenerate second fundamental form.*

Theorem 4.3. *Let* M *be a tubular surface defined by* (3.1) *with nondegenerate second fundamental form. Then, there are no* (K, H, K_{II}, H_{II})*-linear Weingarten surface in Euclidean 3-space.*

Consequently, the study of linear Weingarten tubular surfaces in 3-dimensional Euclidean space is completed with [1].

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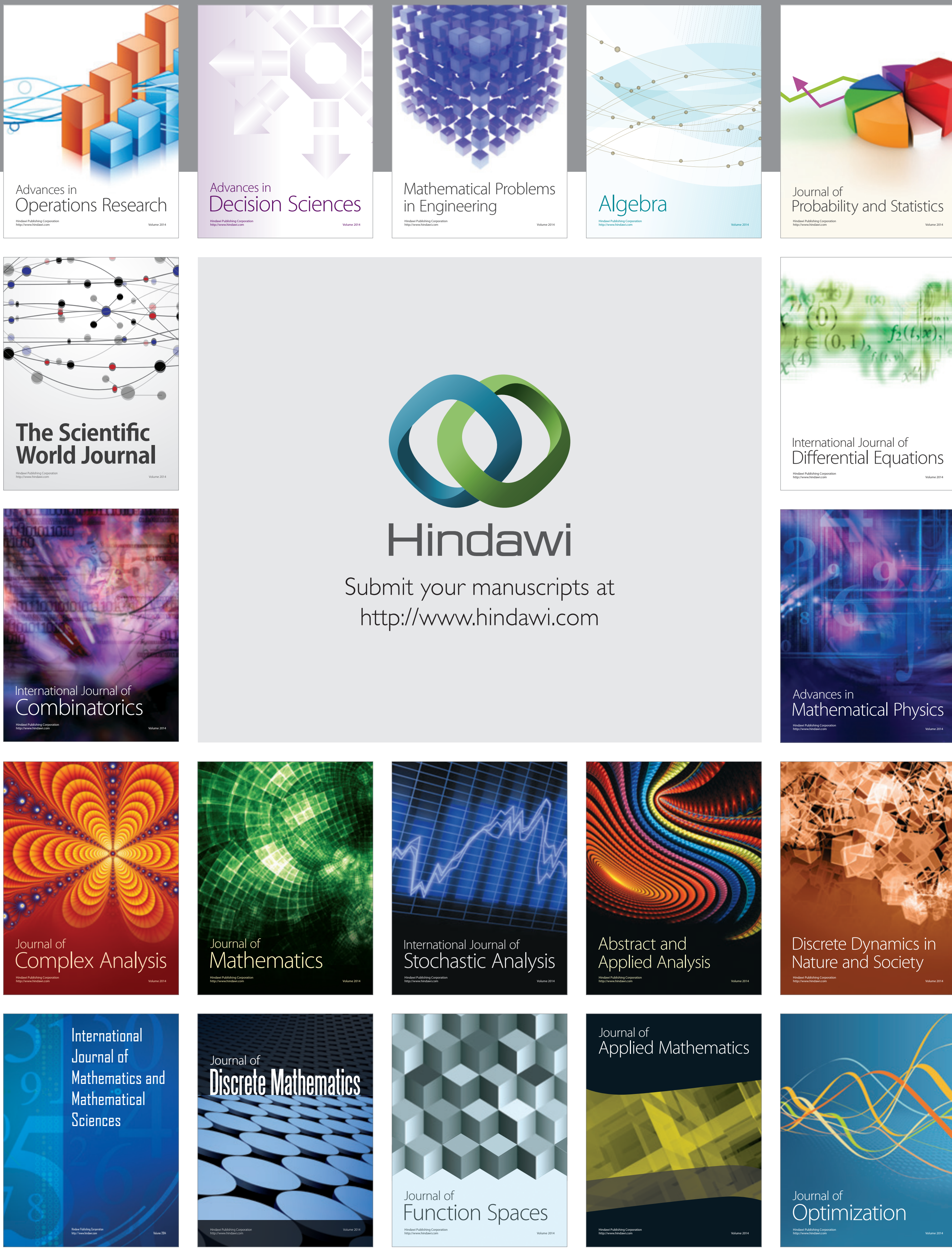
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