

Research Article

An Optimal Error Estimates of H^1 -Galerkin Expanded Mixed Finite Element Methods for Nonlinear Viscoelasticity-Type Equation

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We investigate a H^1 -Galerkin mixed finite element method for nonlinear viscoelasticity equations based on H^1 -Galerkin method and expanded mixed element method. The existence and uniqueness of solutions to the numerical scheme are proved. A priori error estimation is derived for the unknown function, the gradient function, and the flux.

1. Introduction

Consider the following nonlinear viscoelasticity-type equation:

$$\begin{aligned}u_{tt} - \nabla \cdot (a(x, u) \nabla u_t + b(x, u) \nabla u) &= f(x, t), \quad (x, t) \in \Omega \times J, \\u(x, t) &= 0, \quad (x, t) \in \partial\Omega \times J, \\u(x, 0) &= u_0(x), \quad x \in \Omega, \\u_t(x, 0) &= u_1(x), \quad x \in \Omega,\end{aligned}\tag{1.1}$$

where Ω is a convex polygonal domain in R^2 with the Lipschitz continuous boundary $\partial\Omega$, $J = (0, T]$ is the time interval with $0 < T < \infty$, and $u_0(x)$ and $u_1(x)$ are, respectively, the initial data functions defined on Ω . The deformation of viscoelastic solid under the external loads is usually considered by means of this viscoelastic model [1–4], and the problem has a unique

sufficiently smooth solution with the regularity condition provided that the given data $u_0(x)$, $u_1(x)$, $a(u)$, $b(u)$, and f are sufficiently smooth [5].

For problem (1.1), by adopting finite element method, Lin et al. [6] established the convergence of the finite element approximations to solutions of Sobolev and viscoelasticity type of equations via Ritz-Volterra projection and an optimal-order error estimates in L_p ($2 \leq p < \infty$). Latter, Lin and Zhang [7] presented a direct analysis for global superconvergence for this problem without using the Ritz projection or its modified forms. Jin et al. [8] and Shi et al. [9] employed the Wilson nonconforming finite element and a Crouzeix-Raviart type nonconforming finite element on the anisotropic meshes to solve viscoelasticity-type equations, and the global superconvergence estimations were obtained by means of post-processing technique. Since the estimation of flux ∇u by the unknown scalar u is usually indirect, thus the quantity of calculation of the finite element method is relatively large.

As an efficient strategy, mixed finite element methods received much attention in solving partial differential equation in recent decades [10–16]. Compared with finite element methods, mixed finite element methods can obtain the unknown scalar u and its flux ∇u directly, and; hence, it can decrease smoothness of solution space. However, the LBB assumption is needed in the approximating subspaces and; hence, confines the choice of finite element spaces.

On the base of the mixed finite element methods, Pani [17] proposed a new mixed finite element method, called the H^1 -Galerkin mixed finite element procedure, to solve a mixed system in unknown scalar and its flux. Compared with the standard mixed finite methods, the new mixed finite element method does not require the LBB condition, and a better order of convergence for the flux in L^2 norm can be obtained if an extra regularity on the solution holds. Recently, H^1 -Galerkin mixed finite element methods were applied to differential equations [18–22]. However, the assumption needed for this method is not suitable for the nonlinear equations and equations with a small tensor. To overcome this, Chen and Wang [23] proposed H^1 -Galerkin expanded mixed finite element methods which combines the H^1 -Galerkin formulation and the expanded mixed finite element methods [24] to deal with a nonlinear parabolic equation in porous medium flow. This method can compute the scalar unknown, its gradient, and its flux directly. Hence, it is suitable to the case where the coefficient of the differential equation is a small tensor and cannot be inverted. Motivated by this, we establish an H^1 -Galerkin expanded mixed finite element method for the viscoelasticity-type equations.

The remainder of this paper is organized as follows. In Section 2, we first establish the equivalence between viscoelasticity-type equations and their weak formulation by using the H^1 -Galerkin expanded mixed finite element methods and then discuss the existence and uniqueness of the formulation. In Section 3, we show that the H^1 -Galerkin expanded mixed finite element method has the same convergence rate as that of the classical mixed finite element methods without requiring the LBB consistency condition.

Throughout this paper, we use H to denote the space $H(\text{div}, \Omega) = \{\mathbf{v} \in (L^2(\Omega))^d : \nabla \cdot \mathbf{v} \in L^2(\Omega)\}$ with norm $\|\mathbf{v}\|_{H(\text{div}; \Omega)} = (\|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2)^{1/2}$ and $H_0^1(\Omega) = \{w \in H^1(\Omega) : w = 0 \text{ on } \partial\Omega\}$. For theoretical analysis, we also need the following assumptions on the functions involved in problem (1.1).

- Assumption 1.1.* (1) There exist constants a_1 and a_2 such that $0 < a_1 \leq a(x, u), b(x, u) \leq a_2$.
 (2) The functions $a(x, u), b(x, u), a_u(x, u)$, and $b_u(x, u)$ are Lipschitz continuous with respect to u , and there exists $C_1 > 0$ such that $|\partial a / \partial u| + |\partial b / \partial u| + |\partial^2 a / \partial u^2| + |\partial^2 b / \partial u^2| \leq C_1$.

2. H^1 -Galerkin Expanded Mixed Finite Element Discrete Scheme

2.1. Weak Formulation

To define the H^1 -Galerkin expanded mixed finite element procedure, we introduce vector

$$\mathbf{p} = a(x, u)\nabla u_t + b(x, u)\nabla u, \quad \boldsymbol{\sigma} = \nabla u, \quad (2.1)$$

and split (1.1) into a first-order system as follows:

$$\begin{aligned} u_{tt} - \nabla \cdot \mathbf{p} &= f, \\ \boldsymbol{\sigma} &= \nabla u, \\ \mathbf{p} &= a(u)\boldsymbol{\sigma}_t + b(u)\boldsymbol{\sigma}, \\ \boldsymbol{\sigma}(x, 0) &= \nabla u_0(x), \\ \boldsymbol{\sigma}_t(x, 0) &= \nabla u_1(x), \\ \mathbf{p}(x, 0) &= a(u_0)\nabla u_1(x) + b(u_0)\nabla u_0(x). \end{aligned} \quad (2.2)$$

Then by Green's formula we can further define the following weak formulation of problem (2.2): find $(u, \boldsymbol{\sigma}, \mathbf{p}) \in H_0^1(\Omega) \times H(\operatorname{div}, \Omega) \times H(\operatorname{div}, \Omega)$ such that

$$\begin{aligned} (\boldsymbol{\sigma}_{tt}, \mathbf{q}) + (\nabla \cdot \mathbf{p}, \nabla \cdot \mathbf{q}) &= -(f, \nabla \cdot \mathbf{q}), \quad \forall \mathbf{q} \in H(\operatorname{div}, \Omega), \\ (\boldsymbol{\sigma}, \nabla v) &= (\nabla u, \nabla v), \quad \forall v \in H_0^1(\Omega), \\ (\mathbf{p}, \mathbf{w}) &= (a(u)\boldsymbol{\sigma}_t, \mathbf{w}) + (b(u)\boldsymbol{\sigma}, \mathbf{w}), \quad \forall \mathbf{w} \in H(\operatorname{div}, \Omega), \\ \boldsymbol{\sigma}(x, 0) &= \nabla u_0(x), \\ \boldsymbol{\sigma}_t(x, 0) &= \nabla u_1(x), \\ \mathbf{p}(x, 0) &= a(u_0)\nabla u_1(x) + b(u_0)\nabla u_0(x). \end{aligned} \quad (2.3)$$

In order to establish the equivalence between problem (2.2) and the weak form (2.3), we need the following technical lemmas.

Lemma 2.1 (see [25]). *Let Ω be a bounded domain with a Lipschitz continuous boundary $\partial\Omega$. Then, for any $\mathbf{p} \in H(\operatorname{div}, \Omega)$, there exists $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ and divergence free $\boldsymbol{\psi} \in H(\operatorname{div}, \Omega)$ such that $\nabla \cdot \boldsymbol{\psi} = 0$ and $\mathbf{p} = \nabla \phi + \boldsymbol{\psi}$.*

Lemma 2.2 (see [26]). *Let Ω be a bounded domain with a Lipschitz continuous boundary $\partial\Omega$. Then, for any $g \in L^2(\Omega)$, there exists $\mathbf{p} \in (H^1(\Omega))^d \subset H(\operatorname{div}, \Omega)$ such that $\nabla \cdot \mathbf{p} = g$.*

Now we are in a position to state our main result in this subsection.

Theorem 2.3. *Under the conditions of Lemmas 2.1 and 2.2, $(u, \boldsymbol{\sigma}, \mathbf{p}) \in H_0^1(\Omega) \times H(\operatorname{div}, \Omega) \times H(\operatorname{div}, \Omega)$ is a solution to the system (2.2) if and only if it is a solution to the weak form (2.3).*

Proof. It is easy to check that any solution to the system (2.2) is a solution to the weak form (2.3). Hence, to prove the assertion, we only need to show that any solution to the weak form (2.3) is a solution to the system (2.2).

First, taking $\mathbf{w} = \mathbf{p} - a(u)\sigma_t - b(u)\sigma$ in the third equation of (2.3) leads to

$$(\mathbf{p} - a(u)\sigma_t - b(u)\sigma, \mathbf{p} - a(u)\sigma_t - b(u)\sigma) = 0, \quad (2.4)$$

which implies

$$\mathbf{p} = a(u)\sigma_t + b(u)\sigma. \quad (2.5)$$

By Lemma 2.1, there exist $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ and divergence free $\psi \in H(\text{div}, \Omega)$ such that $\nabla \cdot \psi = 0$ and $\sigma = \nabla\phi + \psi$. Choosing $\sigma = \nabla\phi + \psi$ in the second equation of (2.3) yields

$$(\nabla\phi + \psi, \nabla v) = (\nabla u, \nabla v), \quad \forall v \in H_0^1(\Omega). \quad (2.6)$$

By the divergence theorem [1], one has

$$(\psi, \nabla v) = -(\nabla \cdot \psi, v) = 0, \quad \forall v \in H_0^1(\Omega). \quad (2.7)$$

Substituting (2.7) into (2.6) yields

$$(\nabla\phi, \nabla v) = (\nabla u, \nabla v), \quad \forall v \in H_0^1(\Omega), \quad (2.8)$$

which means that

$$\nabla\phi = \nabla u, \quad \sigma = \nabla u + \psi. \quad (2.9)$$

Inserting (2.5) and (2.9) into the first equation of (2.2) and applying the divergence theorem to the first term, for any $\mathbf{q} \in H(\text{div}, \Omega)$, one has

$$(u_{tt}, \nabla \cdot \mathbf{q}) - (\psi_{tt}, \mathbf{q}) - (\nabla \cdot (a(u)(\nabla u_t + \psi_t)), \nabla \cdot \mathbf{q}) + (\nabla \cdot (b(u)(\nabla u + \psi)), \nabla \cdot \mathbf{q}) = (f, \nabla \cdot \mathbf{q}). \quad (2.10)$$

Instituting $\mathbf{q} = \psi_t$ into (2.10) and using $\nabla \cdot \psi_t = 0$ lead to

$$0 = (\psi_{tt}, \psi_t) = \frac{1}{2} \frac{d}{dt} (\psi_t, \psi_t). \quad (2.11)$$

Integrating from 0 to t with respect to time results in

$$(\psi_t(x, t), \psi_t(x, t)) = (\psi_t(x, 0), \psi_t(x, 0)). \quad (2.12)$$

Differentiating (2.9) with respect to t , one obtains

$$\sigma_t = \nabla u_t + \varphi_t. \quad (2.13)$$

By the fifth equation in (2.3), we deduce that

$$\varphi_t(x, 0) = 0, \quad (2.14)$$

which implies

$$\varphi_t(x, t) = 0. \quad (2.15)$$

Integrating the equation $\varphi_t(x, t) = 0$ with respect to t from 0 to t gives

$$\varphi(x, t) = \varphi(x, 0). \quad (2.16)$$

By (2.9) and the fourth equation in (2.2), we deduce

$$\varphi(x, t) = 0, \quad (2.17)$$

which leads to

$$\sigma = \nabla u. \quad (2.18)$$

Therefore, (2.10) can equivalently be transformed into the following equation:

$$(u_{tt}, \nabla \cdot \mathbf{q}) - (\nabla \cdot (a(u)\nabla u_t + b(u)\nabla u), \nabla \cdot \mathbf{q}) = (f, \nabla \cdot \mathbf{q}), \quad \forall \mathbf{q} \in H(\text{div}, \Omega). \quad (2.19)$$

For $f, u_{tt} \in L^2(\Omega)$, by Lemma 2.2, there exists $\mathbf{F} \in H(\text{div}, \Omega)$ such that $\nabla \cdot \mathbf{F} = u_{tt} - f$. Thus, (2.19) reduces to

$$(\nabla \cdot \mathbf{p}, \nabla \cdot \mathbf{q}) = (\nabla \cdot \mathbf{F}, \nabla \cdot \mathbf{q}), \quad \forall \mathbf{q} \in H(\text{div}, \Omega). \quad (2.20)$$

Recalling Lemma 2.1, one concludes that

$$\nabla \cdot \mathbf{F} = \nabla \cdot \mathbf{p}, \quad (2.21)$$

that is,

$$u_{tt} - \nabla \cdot \mathbf{p} = f. \quad (2.22)$$

Combining this with (2.5) and (2.18) results in the desired assertion, and this completes the proof. \square

2.2. Numerical Scheme

Let T_h be a quasi-uniform family of subdivision of domain Ω ; that is, $\Omega = \cup_{K \in T_h} K$ with $h = \max \{\text{diam}(K) : K \in T_h\}$, and let V_h be the finite-dimensional subspaces of $H_0^1(\Omega)$ defined by

$$V_h = \left\{ v_h \in H_0^1(\Omega); v_h|_K \in P_m(K) \right\}, \quad (2.23)$$

where $P_m(K)$ denotes the space of polynomials of degree at most m on K . Moreover, we denote the vector space in mixed finite element spaces with index k by H_h . It is well known that both H_h and V_h satisfy the inverse property and the following approximation properties [26, 27]:

$$\begin{aligned} \inf_{v_h \in V_h} \|v - v_h\| + h\|v - v_h\|_1 &\leq Ch^{m+1}\|v\|_{m+1}, \quad v \in H^{m+1}(\Omega), \\ \inf_{\mathbf{q}_h \in W_h} \|\mathbf{q} - \mathbf{q}_h\| &\leq Ch^{k+1}\|\mathbf{q}\|_{k+1}, \quad \mathbf{q}_h \in (H^{k+1}(\Omega))^d. \end{aligned} \quad (2.24)$$

Let $\Pi_h : H \rightarrow H_h$ denote the Raviart-Thomas interpolation operator [28] which satisfies

$$(\nabla \cdot (\mathbf{q} - \Pi_h \mathbf{q}), \nabla \cdot \mathbf{q}_h) = 0, \quad \forall \mathbf{q}_h \in H_h, \quad (2.25)$$

and the following estimates [26, 28, 29]

$$\|\mathbf{q} - \Pi_h \mathbf{q}\| \leq Ch^{k+1}\|\mathbf{q}\|_{k+1}, \quad (2.26)$$

$$\|\nabla \cdot (\mathbf{q} - \Pi_h \mathbf{q})\| \leq Ch^k\|\mathbf{q}\|_{k+1}. \quad (2.27)$$

With the above notations, the semidiscrete H^1 -Galerkin expanded mixed finite element method for system (2.3) is reduced to find a triple $(u_h, \sigma_h, \mathbf{p}_h) \in V_h \times H_h \times H_h$ such that

$$\begin{aligned} (\sigma_{htt}, \mathbf{q}_h) + (\nabla \cdot \mathbf{p}_h, \nabla \cdot \mathbf{q}_h) &= -(f, \nabla \cdot \mathbf{q}_h), \quad \forall \mathbf{q}_h \in H_h, \\ (\sigma_h, \nabla v_h) &= (\nabla u_h, \nabla v_h), \quad \forall v_h \in V_h, \\ (\mathbf{p}_h, \mathbf{w}_h) &= (a(u_h)\sigma_{ht}, \mathbf{w}_h) + (b(u_h)\sigma_h, \mathbf{w}_h), \quad \forall \mathbf{w}_h \in H_h, \\ \mathbf{p}_h(x, 0) &= \Pi_h \mathbf{p}(x, 0), \\ \sigma_h(x, 0) &= \Pi_h \nabla u_0(x), \\ \sigma_{ht}(x, 0) &= \Pi_h \nabla u_1(x). \end{aligned} \quad (2.28)$$

For the H^1 -Galerkin expanded mixed finite element scheme (2.28), we claim that there exists a unique solution.

In fact, set $V_h = \text{span} \{\varphi_i\}_{i=1}^N$ and $H_h = \text{span} \{\psi_j\}_{j=1}^M$. Then $\sigma_h, \mathbf{p}_h \in H_h$ and $u_h \in V_h$, and; hence,

$$\sigma_h = \sum_{j=1}^M p_j(t) \varphi_j(x), \quad \mathbf{p}_h = \sum_{j=1}^M \lambda_j(t) \varphi_j(x), \quad u_h = \sum_{i=1}^N u_i(t) \varphi_i(x). \quad (2.29)$$

Taking $\mathbf{q}_h = \varphi_j$, $\mathbf{w}_h = \varphi_j$, $j = 1, 2, \dots, M$, $\mathbf{v}_h = \varphi_i$, $i = 1, 2, \dots, N$ in (2.28) leads to

$$\begin{aligned} AP_{tt} + B\Lambda &= F, \\ DU &= CP, \\ A\Lambda &= M(U)P_t + N(U)P, \end{aligned} \quad (2.30)$$

where

$$\begin{aligned} A &= (\varphi_i(x), \varphi_j(x))_{M \times M'}, & P &= (p_1, p_2, \dots, p_M)^T, \\ B &= (\nabla \cdot \varphi_i(x), \nabla \cdot \varphi_j(x))_{M \times M'}, & \Lambda &= (\lambda_1, \lambda_2, \dots, \lambda_M)^T, \\ D &= (\nabla \varphi_i(x), \nabla \varphi_j(x))_{N \times N'}, & U &= (u_1, u_2, \dots, u_N)^T, \\ C &= (\varphi_i(x), \nabla \varphi_j(x))_{N \times M'}, & M(U) &= (a(U) \varphi_i(x), \varphi_j(x))_{M \times M'}, \\ N(U) &= (b(U) \varphi_i(x), \varphi_j(x))_{M \times M'}, & F &= -(f, \nabla \cdot \varphi_j(x))_{M \times 1'} \end{aligned} \quad (2.31)$$

and $P(0)$, $P_t(0)$ are given.

Note that matrix A in (2.31) is positive definite. Thus, by the third equation in (2.30), one has

$$\Lambda = A^{-1}(MP_t + N)P. \quad (2.32)$$

Inserting the above equality into the first equation of (2.30) yields

$$AP_{tt} + BA^{-1}MP_t + BA^{-1}NP = F. \quad (2.33)$$

By the standard arguments on the initial-value problem of a system of ordinary differential equations, we can obtain existence and uniqueness of P . The existence and uniqueness of U and Λ follow from the existence and uniqueness of P .

3. Error Analysis

This section is devoted to the error estimates for the H^1 -Galerkin expanded mixed finite element method.

For error analysis in the following, we need to introduce a projection operator. Let $R_h : H_0^1(\Omega) \rightarrow V_h$ be the Ritz projection defined by

$$(\nabla(u - R_h u), \nabla v_h) = 0, \quad \forall v_h \in V_h. \quad (3.1)$$

Then the following approximation holds [27]:

$$\|u - R_h u\| + h\|\nabla(u - R_h u)\| \leq Ch^{m+1}\|u\|_{m+1}. \quad (3.2)$$

Let

$$\begin{aligned} \mathbf{p} - \mathbf{p}_h &= (\mathbf{p} - \Pi_h \mathbf{p}) + (\Pi_h \mathbf{p} - \mathbf{p}_h) = \eta + \zeta, \\ \boldsymbol{\sigma} - \boldsymbol{\sigma}_h &= (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}) + (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) = \theta + \xi, \\ u - u_h &= (u - R_h u) + (R_h u - u_h) = \alpha + \beta. \end{aligned} \quad (3.3)$$

Utilizing (2.3), (2.28), and auxiliary projections (3.1), (2.25), we can obtain the following error equations:

$$(\xi_{tt}, \mathbf{q}_h) + (\nabla \cdot \zeta, \nabla \cdot \mathbf{q}_h) = -(\theta_t, \mathbf{q}_h), \quad \forall \mathbf{q}_h \in H_h, \quad (3.4)$$

$$(\xi, \nabla v_h) = (\nabla \beta, \nabla v_h) - (\theta, \nabla v_h), \quad \forall v_h \in V_h, \quad (3.5)$$

$$\begin{aligned} (\zeta, \mathbf{w}_h) - (a(u_h)\xi_t, \mathbf{w}_h) - (b(u_h)\xi, \mathbf{w}_h) &= (\boldsymbol{\sigma}_t(a(u) - a(u_h)), \mathbf{w}_h) + (\boldsymbol{\sigma}(b(u) - b(u_h)), \mathbf{w}_h) \\ &\quad + (a(u_h)\theta_t, \mathbf{w}_h) + (b(u_h)\theta, \mathbf{w}_h) - (\eta, \mathbf{w}_h), \quad \forall \mathbf{w}_h \in H_h. \end{aligned} \quad (3.6)$$

Theorem 3.1. *Let $(u, \boldsymbol{\sigma}, \mathbf{p})$ and $(u_h, \boldsymbol{\sigma}_h, \mathbf{p}_h)$ be the solutions to (2.3) and (2.28), respectively. Then the following error estimates hold:*

$$\begin{aligned} (a) \quad \|u - u_h\|_1 &\leq Ch^{\min(k+1, m)}, \\ (b) \quad \|\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\| &\leq Ch^{\min(k, m+1)}, \\ (c) \quad \|u - u_h\| + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| + \|\mathbf{p} - \mathbf{p}_h\| &\leq Ch^{\min(k+1, m+1)}, \end{aligned} \quad (3.7)$$

where $k \geq 1$ and $m \geq 1$ for $d = 2, 3$, and the positive constant C depends on $\|u_t\|_{L^\infty(H^{m+1})}$, $\|u\|_{L^\infty(H^{m+1})}$, $\|\mathbf{p}_t\|_{L^\infty(H^{k+1})}$, $\|\mathbf{p}\|_{L^\infty(H^{k+1})}$, $\|\boldsymbol{\sigma}_t\|_{L^\infty(H^{k+1})}$, $\|\boldsymbol{\sigma}_{tt}\|_{L^\infty(H^{k+1})}$, $\|\boldsymbol{\sigma}\|_{L^\infty(H^{k+1})}$.

Proof. Since estimates of θ , η , and α can be obtained by (3.2) and (2.26), it suffices to estimate ξ , ζ , and β .

Instituting $\mathbf{w}_h = \xi_{tt}$ into (3.6) and $\mathbf{q}_h = \zeta$ in (3.4) gives

$$\begin{aligned} (\nabla \cdot \zeta, \nabla \cdot \zeta) + (a(u_h)\xi_t, \xi_{tt}) + (b(u_h)\xi, \xi_{tt}) &= -(\boldsymbol{\sigma}_t(a(u) - a(u_h)), \xi_{tt}) - (\boldsymbol{\sigma}(b(u) - b(u_h)), \xi_{tt}) \\ &\quad - (a(u_h)\theta_t, \xi_{tt}) - (b(u_h)\theta, \xi_{tt}) + (\eta, \xi_{tt}) - (\theta_t, \zeta). \end{aligned} \quad (3.8)$$

It is easy to check that

$$\begin{aligned}
(a(u_h)\xi_{tt}, \xi_t) &= \frac{1}{2} \frac{d}{dt} (a(u_h)\xi_t, \xi_t) - \frac{1}{2} (a_u(u_h)u_{ht}\xi_t, \xi_t), \\
(b(u_h)\xi, \xi_{tt}) &= \frac{d}{dt} (b(u_h)\xi, \xi_t) - (b_u(u_h)u_{ht}\xi, \xi_t) - (b(u_h)\xi_t, \xi_t), \\
(\eta, \xi_{tt}) &= \frac{d}{dt} (\eta, \xi_t) - (\eta_t, \xi_t), \\
(\sigma_t(a(u) - a(u_h)), \xi_{tt}) &= \frac{d}{dt} (\sigma_t(a(u) - a(u_h)), \xi_t) - (\sigma_{tt}(a(u) - a(u_h)), \xi_t) \\
&\quad - (\sigma_t(a_u(u)u_t - a_u(u_h)u_{ht}), \xi_t), \\
(\sigma(b(u) - b(u_h)), \xi_{tt}) &= \frac{d}{dt} (\sigma(b(u) - b(u_h)), \xi_t) - (\sigma_t(b(u) - b(u_h)), \xi_t) \\
&\quad - (\sigma(b_u(u)u_t - b_u(u_h)u_{ht}), \xi_t), \\
(a(u_h)\theta_t, \xi_{tt}) &= \frac{d}{dt} (a(u_h)\theta_t, \xi_t) - (a_u(u_h)u_{ht}\theta_t, \xi_t) - (a(u_h)\theta_{tt}, \xi_t), \\
(b(u_h)\theta, \xi_{tt}) &= \frac{d}{dt} (b(u_h)\theta, \xi_t) - (b_u(u_h)u_{ht}\theta, \xi_t) - (b(u_h)\theta_t, \xi_t).
\end{aligned} \tag{3.9}$$

Thus, (3.8) can be written as

$$\begin{aligned}
&(\nabla \cdot \zeta, \nabla \cdot \zeta) + \frac{1}{2} \frac{d}{dt} (a(u_h)\xi_t, \xi_t) + \frac{d}{dt} (b(u_h)\xi, \xi_t) \\
&= \frac{1}{2} (a_u(u_h)u_{ht}\xi_t, \xi_t) + (b_u(u_h)u_{ht}\xi, \xi_t) + (b(u_h)\xi_t, \xi_t) - (\theta_{tt}, \zeta) \\
&\quad + \frac{d}{dt} (\eta, \xi_t) - (\eta_t, \xi_t) - \frac{d}{dt} (\sigma_t(a(u) - a(u_h)), \xi_t) \\
&\quad + (\sigma_{tt}(a(u) - a(u_h)), \xi_t) + (\sigma_t(a_u(u)u_t - a_u(u_h)u_{ht}), \xi_t) \\
&\quad - \frac{d}{dt} (\sigma(b(u) - b(u_h)), \xi_t) + (\sigma_t(b(u) - b(u_h)), \xi_t) \\
&\quad + (\sigma(b_u(u)u_t - b_u(u_h)u_{ht}), \xi_t) - \frac{d}{dt} (a(u_h)\theta_t, \xi_t) + (a_u(u_h)u_{ht}\theta_t, \xi_t) \\
&\quad + (a(u_h)\theta_{tt}, \xi_t) - \frac{d}{dt} (b(u_h)\theta, \xi_t) + (b_u(u_h)u_{ht}\theta, \xi_t) + (b(u_h)\theta_t, \xi_t).
\end{aligned} \tag{3.10}$$

Integrating this system from 0 to t yields

$$\begin{aligned}
& \int_0^t \|\nabla \cdot \zeta\|^2 d\tau + \frac{1}{2}(a(u_h)\xi_t, \xi_t) + (b(u_h)\xi, \xi_t) \\
&= (\eta, \xi_t) - (\sigma_t(a(u) - a(u_h)), \xi_t) - (\sigma(b(u) - b(u_h)), \xi_t) - (a(u_h)\theta_t, \xi_t) \\
&\quad - (b(u_h)\theta, \xi_t) + \frac{1}{2} \int_0^t (a_u(u_h)u_{ht}\xi_t, \xi_t) d\tau + \int_0^t (b_u(u_h)u_{ht}\xi, \xi_t) d\tau + \int_0^t (b(u_h)\xi_t, \xi_t) d\tau \\
&\quad - \int_0^t (\theta_{ht}, \zeta) d\tau - \int_0^t (\eta_t, \xi_t) d\tau + \int_0^t (\sigma_{tt}(a(u) - a(u_h)), \xi_t) d\tau \\
&\quad + \int_0^t (\sigma_t(a_u(u)u_t - a_u(u_h)u_{ht}), \xi_t) d\tau + \int_0^t (\sigma_t(b(u) - b(u_h)), \xi_t) d\tau \\
&\quad + \int_0^t (\sigma(b_u(u)u_t - b_u(u_h)u_{ht}), \xi_t) d\tau + \int_0^t (a_u(u_h)u_{ht}\theta_t, \xi_t) d\tau \\
&\quad + \int_0^t (a(u_h)\theta_{ht}, \xi_t) d\tau + \int_0^t (b_u(u_h)u_{ht}\theta, \xi_t) d\tau + \int_0^t (b(u_h)\theta_t, \xi_t) d\tau.
\end{aligned} \tag{3.11}$$

In what follows, we, respectively, analyze the terms on the right-hand side of (3.11). By the Cauchy-Schwartz inequality, we can bound the sixth term on the right-hand side of (3.11) as follows:

$$\begin{aligned}
\left| \int_0^t \frac{1}{2} (a_u(u_h)u_{ht}\xi_t, \xi_t) d\tau \right| &= \frac{1}{2} \left| \int_0^t (a_u(u_h)u_t\xi_t, \xi_t) d\tau + \int_0^t \frac{1}{2} (a_u(u_h)(u_{ht} - u_t)\xi_t, \xi_t) d\tau \right| \\
&\leq C \int_0^t \|\xi_t\|^2 d\tau + C \|\xi_t\|_{L^\infty(0,t;L^\infty)} \int_0^t (\|\alpha_t\|^2 + \|\beta_t\|^2 + \|\xi_t\|^2) d\tau.
\end{aligned} \tag{3.12}$$

For the seventh term on the right-hand side of (3.11), one has

$$\begin{aligned}
\left| \int_0^t (b_u(u_h)u_{ht}\xi, \xi_t) d\tau \right| &= \left| \int_0^t (b_u(u_h)(u_{ht} - u_t)\xi, \xi_t) + (b_u(u_h)u_t\xi, \xi_t) d\tau \right| \\
&\leq C \int_0^t (\|\xi_t\|^2 + \|\xi\|^2) d\tau + C \|\xi_t\|_{L^\infty(0,t;L^\infty)} \int_0^t (\|\alpha_t\|^2 + \|\beta_t\|^2 + \|\xi\|^2) d\tau.
\end{aligned} \tag{3.13}$$

For the term $\int_0^t (\sigma_t(a_u(u)u_t - a_u(u_h)u_{ht}), \xi_t) d\tau$ on the right side of (3.11), we have

$$\begin{aligned} \left| \int_0^t (\sigma_t(a_u(u)u_t - a_u(u_h)u_{ht}), \xi_t) d\tau \right| &= \left| \int_0^t (\sigma_t(a_u(u) - a_u(u_h))u_t + a_u(u_h)(u_t - u_{ht}), \xi_t) d\tau \right| \\ &\leq C \int_0^t (\|\alpha\|^2 + \|\beta\|^2 + \|\alpha_t\|^2 + \|\beta_t\|^2 + \|\xi_t\|^2) d\tau. \end{aligned} \quad (3.14)$$

Similarly,

$$\begin{aligned} \left| \int_0^t (\sigma (b_u(u)u_t - b_u(u_h)u_{ht}), \xi_t) d\tau \right| &\leq \int_0^t |(\sigma (b_u(u) - b_u(u_h))u_t + b_u(u_h)(u_t - u_{ht}), \xi_t)| d\tau \\ &\leq C \int_0^t (\|\alpha\|^2 + \|\beta\|^2 + \|\alpha_t\|^2 + \|\beta_t\|^2 + \|\xi_t\|^2) d\tau, \\ \left| \int_0^t (a_u(u_h)u_{ht}\theta_t, \xi_t) d\tau \right| &= \left| \int_0^t (a_u(u_h)(u_{ht} - u_t)\theta_t, \xi_t) d\tau + \int_0^t (a_u(u_h)u_t\theta_t, \xi_t) d\tau \right| \\ &\leq C \|\xi_t\|_{L^\infty(0,t;L^\infty)} \int_0^t (\|\alpha_t\|^2 + \|\beta_t\|^2 + \|\theta_t\|^2) d\tau \\ &\quad + C \int_0^t (\|\theta_t\|^2 + \|\xi_t\|^2) d\tau, \\ \left| \int_0^t (b_u(u_h)u_{ht}\theta, \xi_t) d\tau \right| &\leq \int_0^t |(b_u(u_h)(u_{ht} - u_t)\theta, \xi_t)| d\tau + \int_0^t |(b_u(u_h)u_t\theta, \xi_t)| d\tau \\ &\leq C \|\xi_t\|_{L^\infty(0,t;L^\infty)} \int_0^t (\|\alpha_t\|^2 + \|\beta_t\|^2 + \|\theta\|^2) d\tau \\ &\quad + C \int_0^t (\|\theta\|^2 + \|\xi_t\|^2) d\tau. \end{aligned} \quad (3.15)$$

Inserting (3.12)–(3.15) into (3.11) and using the Cauchy-Schwartz inequality lead to

$$\begin{aligned} &\int_0^t \|\nabla \cdot \zeta\|^2 d\tau + \frac{1}{2} (a(u_h)\xi_t, \xi_t) + (b(u_h)\xi, \xi_t) \\ &\leq C (\|\eta\|^2 + \|\xi_t\|^2 + \|\alpha\|^2 + \|\beta\|^2 + \|\theta\|^2 + \|\theta_t\|^2) \\ &\quad + C \int_0^t (\|\theta_{tt}\|^2 + \|\theta_t\|^2 + \|\xi_t\|^2 + \|\eta_t\|^2 + \|\alpha\|^2 + \|\beta\|^2 + \|\zeta\|^2) d\tau \\ &\quad + C \|\xi_t\|_{L^\infty(0,t;L^\infty)} \int_0^t (\|\alpha_t\|^2 + \|\beta_t\|^2 + \|\xi_t\|^2) d\tau \end{aligned}$$

$$\begin{aligned}
& + C \|\xi_t\|_{L^\infty(0,t;L^\infty)} \int_0^t (\|\theta_t\|^2 + \|\theta\|^2 + \|\xi\|^2) d\tau \\
& + C \int_0^t (\|\theta_t\|^2 + \|\theta\|^2 + \|\xi\|^2 + \|\xi_t\|^2) d\tau.
\end{aligned} \tag{3.16}$$

Integrating (3.16) from 0 to t , using the fact $(b(u_h)\xi, \xi_t) = (1/2)(d/dt)(b(u_h)\xi, \xi) - (1/2)(b_u(u_h)u_{ht}\xi, \xi_t)$ and the inequality

$$\int_0^t \int_0^\tau |\varphi(s)|^2 ds d\tau \leq C \int_0^t |\varphi(s)|^2 ds, \tag{3.17}$$

yields

$$\begin{aligned}
\|\xi\|^2 & \leq C \|\xi_t\|_{L^\infty(0,t;L^\infty)} \int_0^t (\|\alpha_t\|^2 + \|\beta_t\|^2 + \|\xi_t\|^2) d\tau \\
& + C \|\xi_t\|_{L^\infty(0,t;L^\infty)} \int_0^t (\|\theta_t\|^2 + \|\theta\|^2 + \|\xi\|^2) d\tau \\
& + C \int_0^t (\|\alpha_t\|^2 + \|\beta_t\|^2 + \|\alpha\|^2 + \|\beta\|^2 + \|\theta_t\|^2 \\
& \quad + \|\theta\|^2 + \|\theta_{tt}\|^2 + \|\eta_t\|^2 + \|\xi\|^2 + \|\xi_t\|^2 + \|\zeta\|^2) d\tau.
\end{aligned} \tag{3.18}$$

Thus, to estimate $\|\xi\|$, we need to estimate $\|\beta\|$, $\|\beta_t\|$, $\|\zeta\|$, and $\|\xi_t\|$. Taking $v_h = \beta$ in (3.5) leads to

$$(\nabla\beta, \nabla\beta) = (\xi, \nabla\beta) + (\theta, \nabla\beta). \tag{3.19}$$

By the Cauchy-Schwartz inequality, we obtain

$$\|\nabla\beta\| \leq C(\|\xi\| + \|\theta\|). \tag{3.20}$$

Note that $\beta \in V_h \subset H_0^1(\Omega)$ and $\|\beta\| \leq C\|\nabla\beta\|$. We further have

$$\|\beta\| \leq C(\|\xi\| + \|\theta\|). \tag{3.21}$$

Differentiating (3.5) with respect to t and choosing $v_h = \beta_t$ gives

$$\|\nabla\beta_t\| \leq C(\|\xi_t\| + \|\theta_t\|). \tag{3.22}$$

Similarly, since $\beta \in V_h \subset H_0^1(\Omega)$, one has $\|\beta_t\| \leq \|\nabla\beta_t\| \leq C(\|\xi_t\| + \|\theta_t\|)$.

Taking $\mathbf{w}_h = \zeta$ in (3.6), one has

$$\begin{aligned} (\zeta, \zeta) &= (a(u_h)\xi_t, \zeta) + (b(u_h)\xi, \zeta) + (\sigma_t(a(u) - a(u_h)), \zeta) \\ &\quad + (\sigma(b(u) - b(u_h)), \zeta) + (a(u_h)\theta_t, \zeta) + (b(u_h)\theta, \zeta) - (\eta, \zeta). \end{aligned} \quad (3.23)$$

By the Cauchy-Schwartz inequality, we obtain

$$\|\zeta\| \leq C(\|\xi\| + \|\theta_t\| + \|\theta\| + \|\eta\| + \|\alpha\| + \|\beta\| + \|\xi_t\|). \quad (3.24)$$

To bound $\|\xi_t\|^2$, we differentiate (3.6) with respect to t to obtain

$$\begin{aligned} (\zeta_t, \mathbf{w}_h) &- (a(u_h)\xi_{tt}, \mathbf{w}_h) - (b(u_h)\xi_t, \mathbf{w}_h) \\ &= (a_u(u_h)u_{ht}\xi_t, \mathbf{w}_h) + (b_u(u_h)u_{ht}\xi, \mathbf{w}_h) \\ &\quad + (\sigma_{tt}(a(u) - a(u_h)), \mathbf{w}_h) + (\sigma_t(a_u(u)u_t - a_u(u_h)u_{ht}), \mathbf{w}_h) \\ &\quad + (\sigma_t(b(u) - b(u_h)), \mathbf{w}_h) + (\sigma(b_u(u)u_t - b_u(u_h)u_{ht}), \mathbf{w}_h) \\ &\quad + (a(u_h)\theta_{tt}, \mathbf{w}_h) + (a_u(u_h)u_{ht}\theta_t, \mathbf{w}_h) \\ &\quad + (b(u_h)\theta_t, \mathbf{w}_h) + (b_u(u_h)u_{ht}\theta, \mathbf{w}_h) - (\eta_t, \mathbf{w}_h), \quad \forall \mathbf{w}_h \in H_h. \end{aligned} \quad (3.25)$$

Testing (3.25) with $\mathbf{w}_h = \xi_{tt}$ and (3.4) with $\mathbf{q}_h = \zeta_t$ and combining the resulting equations together lead to

$$\begin{aligned} (\nabla \cdot \zeta, \nabla \cdot \zeta_t) &+ (a(u_h)\xi_{tt}, \xi_{tt}) + (b(u_h)\xi_t, \xi_{tt}) \\ &= -(a_u(u_h)u_{ht}\xi_t, \xi_{tt}) - (b_u(u_h)u_{ht}\xi, \xi_{tt}) \\ &\quad - (\sigma_{tt}(a(u) - a(u_h)), \xi_{tt}) - (\sigma_t(a_u(u)u_t - a_u(u_h)u_{ht}), \xi_{tt}) \\ &\quad - (\sigma_t(b(u) - b(u_h)), \xi_{tt}) - (\sigma(b_u(u)u_t - b_u(u_h)u_{ht}), \xi_{tt}) \\ &\quad - (a(u_h)\theta_{tt}, \xi_{tt}) - (a_u(u_h)u_{ht}\theta_t, \xi_{tt}) - (\theta_t, \zeta_t) \\ &\quad - (b(u_h)\theta_t, \xi_{tt}) - (b_u(u_h)u_{ht}\theta, \xi_{tt}) + (\eta_t, \xi_{tt}). \end{aligned} \quad (3.26)$$

Note that

$$\begin{aligned} (b(u_h)\xi_t, \xi_{tt}) &= \frac{1}{2} \frac{d}{dt} (b(u_h)\xi_t, \xi_t) - \frac{1}{2} (b_u(u_h)u_{ht}\xi, \xi_t), \\ (\nabla \cdot \zeta, \nabla \cdot \zeta_t) &= \frac{1}{2} \frac{d}{dt} (\nabla \cdot \zeta, \nabla \cdot \zeta), \\ (\theta_t, \zeta_t) &= \frac{d}{dt} (\theta_t, \zeta) - (\theta_{tt}, \zeta). \end{aligned} \quad (3.27)$$

Thus, (3.26) can be rewritten as

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (\nabla \cdot \zeta, \nabla \cdot \zeta) + (a(u_h) \xi_{tt}, \xi_{tt}) + \frac{1}{2} \frac{d}{dt} (b(u_h) \xi_t, \xi_t) \\
&= -(a_u(u_h) u_{ht} \xi_t, \xi_{tt}) - \frac{1}{2} (b_u(u_h) u_{ht} \xi, \xi_{tt}) \\
&\quad - (\sigma_{tt}(a(u) - a(u_h)), \xi_{tt}) - (\sigma_t(a_u(u) u_t - a_u(u_h) u_{ht}), \xi_{tt}) \\
&\quad - (\sigma_t(b(u) - b(u_h)), \xi_{tt}) - (\sigma(b_u(u) u_t - b_u(u_h) u_{ht}), \xi_{tt}) \\
&\quad - (a(u_h) \theta_{tt}, \xi_{tt}) - (a_u(u_h) u_{ht} \theta_t, \xi_{tt}) - \frac{d}{dt} (\theta_t, \zeta) + (\theta_{tt}, \zeta) \\
&\quad - (b(u_h) \theta_t, \xi_{tt}) - (b_u(u_h) u_{ht} \theta, \xi_{tt}) + (\eta_t, \xi_{tt}).
\end{aligned} \tag{3.28}$$

Integrating (3.28) from 0 to t yields

$$\begin{aligned}
& (\nabla \cdot \zeta, \nabla \cdot \zeta) + \int_0^t (a(u_h) \xi_{tt}, \xi_{tt}) + (b(u_h) \xi_t, \xi_t) \\
&= - \int_0^t (a_u(u_h) u_{ht} \xi_t, \xi_{tt}) d\tau - \frac{1}{2} \int_0^t (b_u(u_h) u_{ht} \xi, \xi_{tt}) d\tau \\
&\quad - \int_0^t (\sigma_{tt}(a(u) - a(u_h)), \xi_{tt}) d\tau - \int_0^t (\sigma_t(a_u(u) u_t - a_u(u_h) u_{ht}), \xi_{tt}) d\tau \\
&\quad - \int_0^t (\sigma_t(b(u) - b(u_h)), \xi_{tt}) d\tau - \int_0^t (\sigma(b_u(u) u_t - b_u(u_h) u_{ht}), \xi_{tt}) d\tau \\
&\quad - \int_0^t (a(u_h) \theta_{tt}, \xi_{tt}) d\tau - \int_0^t (a_u(u_h) u_{ht} \theta_t, \xi_{tt}) d\tau - (\theta_t, \zeta) + \int_0^t (\theta_{tt}, \zeta) d\tau \\
&\quad - \int_0^t (b(u_h) \theta_t, \xi_{tt}) d\tau - \int_0^t (b_u(u_h) u_{ht} \theta, \xi_{tt}) d\tau + \int_0^t (\eta_t, \xi_{tt}) d\tau.
\end{aligned} \tag{3.29}$$

For the first term on the right-hand side of (3.29), by the Cauchy-Schwarz inequality and Young's inequality, for sufficiently small constant $\varepsilon > 0$, it holds that

$$\begin{aligned}
& \left| - \int_0^t (a_u(u_h) u_{ht} \xi_t, \xi_{tt}) d\tau \right| \leq \left| \int_0^t (a_u(u_h) (u_{ht} - u_t) \xi_t, \xi_{tt}) d\tau \right| + \left| \int_0^t (a_u(u_h) u_t \xi_t, \xi_{tt}) d\tau \right| \\
&\leq C \|\xi_t\|_{L^\infty(0,t;L^\infty)} \int_0^t (\|\alpha_t\|^2 + \|\beta_t\|^2) d\tau \\
&\quad + C \int_0^t \|\xi_t\|^2 d\tau + \varepsilon (1 + \|\xi_t\|_{L^\infty(0,t;L^\infty)}) \int_0^t \|\xi_{tt}\|^2 d\tau.
\end{aligned} \tag{3.30}$$

Similarly, we can bound (3.29) as follows:

$$\begin{aligned}
\|\nabla \cdot \zeta\|^2 + \|\xi_t\|^2 + \int_0^t \|\xi_{tt}\|^2 d\tau &\leq C \|\xi_t\|_{L^\infty(0,t;L^\infty)} \int_0^t (\|\alpha_t\|^2 + \|\beta_t\|^2) d\tau \\
&\quad + \varepsilon (1 + \|\xi_t\|_{L^\infty(0,t;L^\infty)}) \int_0^t \|\xi_{tt}\|^2 d\tau \\
&\quad + C \|\xi\|_{L^\infty(0,t;L^\infty)} \int_0^t (\|\alpha_t\|^2 + \|\beta_t\|^2) d\tau \\
&\quad + \varepsilon (1 + \|\xi\|_{L^\infty(0,t;L^\infty)}) \int_0^t \|\xi_{tt}\|^2 d\tau \\
&\quad + C \int_0^t (\|\xi\|^2 + \|\zeta\|^2 + \|\theta_{tt}\|^2 + \|\theta_t\|^2 + \|\theta\|^2 \\
&\quad \quad + \|\alpha\|^2 + \|\alpha_t\|^2 + \|\beta\|^2 + \|\beta_t\|^2 + \|\eta_t\|^2 + \|\xi_t\|^2) d\tau \\
&\quad + \|\theta_t\| \|\zeta\|.
\end{aligned} \tag{3.31}$$

In the following error analysis, we make an induction hypothesis:

$$\left(\|\xi_t\|_{L^\infty(0,t;L^\infty)} + \|\xi\|_{L^\infty(0,t;L^\infty)} \right) \leq 1. \tag{3.32}$$

Utilizing (3.32), (3.24), (3.22), (3.21), and Young's inequality, one can reduce (3.31) to

$$\begin{aligned}
\|\nabla \cdot \zeta\|^2 + \|\xi_t\|^2 &\leq C (\|\xi\|^2 + \|\theta_t\|^2 + \|\theta\|^2 + \|\alpha\|^2 + \|\eta\|^2) \\
&\quad + C \int_0^t (\|\xi\|^2 + \|\theta_{tt}\|^2 + \|\theta_t\|^2 + \|\theta\|^2 + \|\alpha\|^2 + \|\alpha_t\|^2 + \|\eta_t\|^2 + \|\xi_t\|^2) d\tau.
\end{aligned} \tag{3.33}$$

Then by Gronwall's inequality, we obtain

$$\begin{aligned}
\|\nabla \cdot \zeta\|^2 + \|\xi_t\|^2 &\leq C (\|\xi\|^2 + \|\theta_t\|^2 + \|\theta\|^2 + \|\alpha\|^2 + \|\eta\|^2) \\
&\quad + C \int_0^t (\|\xi\|^2 + \|\theta_{tt}\|^2 + \|\theta_t\|^2 + \|\theta\|^2 + \|\alpha\|^2 + \|\alpha_t\|^2 + \|\eta_t\|^2) d\tau.
\end{aligned} \tag{3.34}$$

Furthermore, by (3.24) and (3.34), one has

$$\begin{aligned}
\|\zeta\|^2 &\leq C (\|\xi\|^2 + \|\theta_t\|^2 + \|\theta\|^2 + \|\alpha\|^2 + \|\eta\|^2) \\
&\quad + C \int_0^t (\|\xi\|^2 + \|\theta_{tt}\|^2 + \|\theta_t\|^2 + \|\theta\|^2 + \|\alpha\|^2 + \|\alpha_t\|^2 + \|\eta_t\|^2) d\tau.
\end{aligned} \tag{3.35}$$

Therefore, by the estimates of $\|\beta\|$, $\|\beta_t\|$, $\|\zeta\|$, and $\|\xi_t\|$, it follows that

$$\|\xi\|^2 \leq C \int_0^t \left(\|\xi\|^2 + \|\theta_{tt}\|^2 + \|\theta_t\|^2 + \|\theta\|^2 + \|\alpha\|^2 + \|\alpha_t\|^2 + \|\eta\|^2 + \|\eta_t\|^2 \right) d\tau. \quad (3.36)$$

Applying Gronwall's inequality to the above equation and using the estimates of projection operators give

$$\begin{aligned} \|\xi\|^2 &\leq C \int_0^t \left(\|\theta_{tt}\|^2 + \|\theta_t\|^2 + \|\theta\|^2 + \|\alpha\|^2 + \|\alpha_t\|^2 + \|\eta\|^2 + \|\eta_t\|^2 \right) d\tau \\ &\leq Ch^{\min(2k+2, 2m+2)} \left(\|u_t\|_{L^\infty(H^{m+1})}^2 + \|u\|_{L^\infty(H^{m+1})}^2 + \|\mathbf{p}_t\|_{L^\infty(H^{k+1})}^2 \right. \\ &\quad \left. + \|\mathbf{p}\|_{L^\infty(H^{k+1})}^2 + \|\sigma_t\|_{L^\infty(H^{k+1})}^2 + \|\sigma_{tt}\|_{L^\infty(H^{k+1})}^2 + \|\sigma\|_{L^\infty(H^{k+1})}^2 \right). \end{aligned} \quad (3.37)$$

Inserting the estimate of $\|\xi\|$ into (3.34) yields

$$\|\xi_t\|^2 \leq Ch^{\min(2k+2, 2m+2)}. \quad (3.38)$$

Thus, the estimates of β and ζ follow from the estimate of ξ .

Finally, according to the proof of the induction hypothesis in [23, 30], we can prove that the inductive hypothesis (3.32) holds. In fact, when $t = 0$, then $\xi(0) = 0$, $\xi_t(0) = 0$. Note that $\|\xi\|_{L^\infty(0,t;L^\infty)} + \|\xi_t\|_{L^\infty(0,t;L^\infty)}$ is continuous w.r.t. t . Then, we conclude that there exists $t_1 \in (0, T]$ such that

$$\|\xi\|_{L^\infty(0,t_1;L^\infty)} + \|\xi_t\|_{L^\infty(0,t_1;L^\infty)} \leq 1. \quad (3.39)$$

Set $t^* = \sup t_1$. Thus, $\|\xi\|_{L^\infty(0,t^*;L^\infty)} + \|\xi_t\|_{L^\infty(0,t^*;L^\infty)} \leq 1$. Therefore, we have

$$\|\xi(t^*)\| + \|\xi_t(t^*)\| \leq Ch^{\min(k+1, m+1)}. \quad (3.40)$$

By inverse estimates, we deduce that, for any $0 \leq t \leq t^*$, it holds that

$$\|\xi\|_{L^\infty(0,t;L^\infty)} + \|\xi_t\|_{L^\infty(0,t;L^\infty)} \leq Ch^{\min(k+1, m+1) - d/2}. \quad (3.41)$$

Then we can take $h > 0$ sufficiently small such that

$$\|\xi\|_{L^\infty(0,t^*;L^\infty)} + \|\xi_t\|_{L^\infty(0,t^*;L^\infty)} < 1. \quad (3.42)$$

Again, by the continuity of $\|\xi\|_{L^\infty(0,t;L^\infty)} + \|\xi_t\|_{L^\infty(0,t;L^\infty)}$, we conclude that there exists a positive

constant δ such that

$$\|\xi\|_{L^\infty(0,t^*+\delta;L^\infty)} + \|\xi_t\|_{L^\infty(0,t^*+\delta;L^\infty)} \leq 1, \quad (3.43)$$

which contracts to the definition of t^* . This completes the proof of the induction hypothesis.

Combining (3.21), (3.37), (3.2), (2.26), (2.27) with the estimates of auxiliary projections and utilizing the triangle inequality, we can derive the desired result. \square

Remark 3.2. By Theorem 3.1 and the standard embedding theorem, we can obtain the L^∞ estimate for $d = 1$ and 2 as follows:

$$\|u - u_h\|_{L^\infty(L^\infty)} \leq C_2 |\ln h|^{d-1} h^{\min(k+1, m+1)}. \quad (3.44)$$

4. Conclusion

In this paper, H^1 -Galerkin mixed finite element method combining with expanded mixed element method is discussed for nonlinear viscoelasticity equations. This method solves the scalar unknown, its gradient, and its flux, directly. It is suitable for the case that the coefficient of the differential is a small tensor and does not need to be inverted. Furthermore, the formulation permits the use of standard continuous and piecewise (linear and higher-order) polynomials in contrast to continuously differentiable piecewise polynomials required by the standard H^1 -Galerkin methods and is free of the LBB condition which is required by the mixed finite element methods.

There are also some important issues to be addressed in the area; for example, one can consider numerical implementation and mathematical and numerical analysis of the full discrete procedure. This is an important and challenging topic in the future research.

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