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Research Article

Solution to the Linear Fractional Differential Equation Using Adomian Decomposition Method

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We obtain the analytical general solution of the linear fractional differential equations with constant coefficients by Adomian decomposition method under nonhomogeneous initial value condition, which is in the sense of the Caputo fractional derivative.

1. Introduction

Fractional differential equations are hot topics both in mathematics and physics. Recently, the fractional differential equations have been the subject of intensive research. There are several methods to obtain the solution, such as the Laplace transform method, power series method, and Green function method. Many remarkable results for the fractional differential equations can be found in the literature [1–11]. In particular, the Adomian decomposition method has attracted the attention of many mathematicians [12–15].

For a better understanding of the fractional derivatives and for a physical understanding of the fractional equations, the readers can refer to the recent publications in [16, 17]. Ebaid [18] suggested a modification of the Adomian method, and a few iterations lead to exact solution. Das [19] compared the variational iteration method with the Adomian method for fractional equations and found that the variational iteration method is much more effective. For other methods of the fractional differential equations, especially the homotopy perturbation method, variational iteration method and differential transform method were presented in [20, 21].

Consider the following n-term fractional differential equation with constant coefficients:

$$a_n\left[{}^{C}D^{\beta_n}\right]y(t) + a_{n-1}\left[{}^{C}D^{\beta_{n-1}}\right]y(t) + \cdots + a_1\left[{}^{C}D^{\beta_1}\right]y(t) + a_0\left[{}^{C}D^{\beta_0}\right]y(t) = f(t), \tag{1.1}$$

where $n+1 > \beta \geq n > \beta_{n-1} > \cdots > \beta_1 > \beta_0$ and a_i $(i = 0, 1, \ldots, n)$ is a real constant. In [12], the authors obtain the particular solution of (1.1) of the homogeneous initial value problem of the form

$$y^{i}(0) = 0, \quad i = 0, 1, \ldots, n. \tag{1.2}$$

However, it seems also more meaningful and more complicated for solving general solution of (1.1) under nonhomogeneous initial value condition. Therefore, in this paper, we will remove the restriction of the homogeneous initial value, consider the nonhomogeneous initial value problems of the form

$$y^{j_i}(0) = c_{ij_i}, \quad i = 0, 1, \ldots, n,\ j_i = 1, 2, \ldots, l_i,\ l_i - 1 \leqslant \beta_i < l_i, \tag{1.3}$$

and obtain the analytical general solution of (1.1), which generalizes the result in [12].

We organize the paper as follows. In Section 2, we give some basic definitions and properties. In Section 3, we obtain the analytical general solution of the linear fractional differential equations by Adomian decomposition method. Some explicit examples are given in Section 4.

2. Basic Definitions and Notations

Definition 2.1 (see [1]). The Riemann-Liouville integral of order p is defined by

$${}_aD_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau)\, d\tau, \quad p > 0. \tag{2.1}$$

From Definition 2.1 , we clearly see that

$${}_aD_t^{-\alpha}\left((t-a)^{\nu}\right) = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu+\alpha)}(t-a)^{\nu+\alpha}, \tag{2.2}$$

$${}_aD_t^{\alpha}\left((t-a)^{\nu}\right) = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\alpha)}(t-a)^{\nu-\alpha}, \tag{2.3}$$

where $\alpha > 0$ and ν is a real number.

Definition 2.2 (see [1]). For $f(t) \in C^m (m \in N)$, the Caputo fractional derivative of $f(t)$ is defined by

$$ {}_a^C D_t^p f(t) = \begin{cases} D^{m-p} f^{(m)}(t), & m-1 < p < m, \\ \dfrac{d^{(m)}}{dt^m} f(t), & p = m, \\ {}_a D_t^p f(t), & p \leqslant 0. \end{cases} \tag{2.4} $$

Therefore,

$$ {}_a^c D_t^{-\alpha}\left((t-a)^{\nu}\right) = {}_a D_t^{-\alpha}\left((t-a)^{\nu}\right) = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu+\alpha)}(t-a)^{\nu+\alpha}. \tag{2.5} $$

Lemma 2.3 (see [1]). *If $f(t)$ is continuous, then*

$$ {}_a^C D_t^{-n} f^{(n)}(t) = {}_a D_t^{-n} f^{(n)}(t) = f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)(t-a)^j}{\Gamma(j+1)}, \quad n \in Z. \tag{2.6} $$

Lemma 2.4 (see [1]). *If $f(t)$ is continuous, then*

$$ {}_a^C D_t^{-p}\left({}_a^C D_t^{-q}\right) f(t) = {}_a D_t^{-p}\left({}_a D_t^{-q}\right) f(t) = {}_a D_t^{-p-q} f(t) = {}_a^C D_t^{-p-q} f(t) = {}_a^C D_t^{-q}\left({}_a^C D_t^{-p}\right) f(t). \tag{2.7} $$

Lemma 2.5. *If $f(t)$ is continuous, then*

$$ {}_a^C D_t^{-p}\left({}_a^C D_t^{q}\right) f(t) = {}_a D_t^{q-p} f(t) - \sum_{j=0}^{m-1} f^{(j)}(a) \frac{(t-a)^{j-q+p}}{\Gamma(j-q+p+1)}, \tag{2.8} $$

where $m-1 < q < m$ and $q \leqslant p$.

Proof. From Definition 2.2 and Lemmas 2.3–2.4, we get

$$ \begin{aligned} {}_a^C D_t^{-p}\left({}_a^C D_t^{q}\right) f(t) &= {}_a D_t^{-p}\left({}_a D_t^{q-m}\right) f^{(m)}(t) = {}_a D_t^{-p+q}\left[{}_a D_t^{-q}\left({}_a D_t^{q-m}\right) f^{(m)}(t)\right] \\ &= {}_a D_t^{-p+q}\left[{}_a D_t^{-m}(t) f^{(m)}(t)\right] = {}_a D_t^{-p+q}\left[f(t) - \sum_{j=0}^{m-1} \frac{f^{(j)}(a)(t-a)^j}{\Gamma(j+1)}\right] \\ &= {}_a D_t^{q-p} f(t) - \sum_{j=0}^{m-1} \frac{f^{(j)}(a)(t-a)^{j-q+p}}{\Gamma(j-q+p+1)}. \end{aligned} \tag{2.9} $$

It is easy to see that

$$ {}_a^C D_t^{-p}\left({}_a^C D_t^p\right) f(t) = f(t) - \sum_{j=0}^{m-1} f^{(j)}(a)\frac{(t-a)^j}{\Gamma(j+1)}. \tag{2.10} $$

□

Proposition 2.6 (see [12]). *One has*

$$ \sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} a_{k_1,k_2\cdots k_{n-1}k_n} = \sum_{m=0}^{\infty}\sum_{\substack{k_1,\ldots,k_{n-1},k_n\geqslant 0\\ k_1+\cdots+k_{n-1}+k_n=m}} a_{k_1,k_2\cdots k_{n-1}k_n}. \tag{2.11} $$

Proposition 2.7 (see [12]). *More over, one has*

$$ \sum_{m=0}^{\infty}\sum_{\substack{k_1,\ldots,k_{n-1},k_n\geqslant 0\\ k_1+\cdots+k_{n-1}+k_n=m}} a_{k_1,k_2\cdots k_{n-1}k_n} = \sum_{s=0}^{\infty}\sum_{\substack{k_1,\ldots,k_{n-1}\geqslant 0\\ k_1+\cdots+k_{n-1}=s}}\sum_{k_n=0}^{\infty} a_{k_1,k_2\cdots k_{n-1}k_n}. \tag{2.12} $$

3. The Analytical Solution of the Linear Constant Coefficient Fractional Differential Equation

For simplicity, if $a = 0$, then we denote ${}_a^C D_t^p$ or ${}_a^C D_t^{-p}$ by ${}^C D^p$ or ${}^C D^{-p}$.

In this section, we use Adomian decomposition method to discuss the general form of the linear fractional differential equations with constant coefficients, and apply and some basic transformation and integration to obtain the solution of the equations.

Let us consider the following n-term linear fractional differential equations with constant coefficients:

$$ \begin{aligned} &a_n\left[{}^C D^{\beta_n}\right]y(t) + a_{n-1}\left[{}^C D^{\beta_{n-1}}\right]y(t) + \cdots + a_1\left[{}^C D^{\beta_1}\right]y(t) + a_0\left[{}^C D^{\beta_0}\right]y(t) = f(t),\\ &y^{(j_i)}(0) = c_{ij_i}, \quad i = 0,1,\ldots,n,\ j_i = 1,2,\ldots,l_i,\ l_i - 1 \leqslant \beta_i < l_i, \end{aligned} \tag{3.1} $$

where $n+1 > \beta_n \geqslant n > \beta_{n-1}\cdots > \beta_1 > \beta_0$, a_i and c_{ij_i} are real constants, ${}^C D^p = {}_0^C D_t^p$ denotes Caputo fractional derivative of order α.

Applying ${}^C D^{-\beta_n}$ to both sides of (1.1) and utilizing Lemma 2.5, we get

$$ \begin{aligned} &y(t) + \frac{a_{n-1}}{a_n}\,{}^C D^{\beta_{n-1}-\beta_n} y(t) + \cdots + \frac{a_0}{a_n}\,{}^C D^{\beta_0-\beta_n} y(t)\\ &\quad = \frac{1}{a_n}\,{}^C D^{-\beta_n} f(t) + \frac{1}{a_n}\sum_{i=0}^{n} a_i \sum_{j_i=0}^{l_i-1} y^{(j_i)}(0)\frac{t^{\beta_n-\beta_i+j_i}}{\Gamma(1+\beta_n-\beta_i+j_i)}. \end{aligned} \tag{3.2} $$

By the Adomian decomposition method, we obtain the recursive relationship as follows:

$$
\begin{aligned}
y_0(t) &= \frac{1}{a_n}\,{}^{C}D^{-\beta_n} f(t) + \frac{1}{a_n}\sum_{i=0}^{n} a_i \sum_{j_i=0}^{l_i-1} y^{(j_i)}(0)\frac{t^{\beta_n-\beta_i+j_i}}{\Gamma(1+\beta_n-\beta_i+j_i)},\\
y_1(t) &= -\left(\frac{a_{n-1}}{a_n}\,{}^{C}D^{\beta_{n-1}-\beta_n} + \cdots + \frac{a_0}{a_n}\,{}^{C}D^{\beta_0-\beta_n}\right) y_0(t),\\
y_2(t) &= (-1)^2\left(\frac{a_{n-1}}{a_n}\,{}^{C}D^{\beta_{n-1}-\beta_n} + \cdots + \frac{a_0}{a_n}\,{}^{C}D^{\beta_0-\beta_n}\right)^2 y_0(t),\\
&\vdots\\
y_s(t) &= (-1)^s\left(\frac{a_{n-1}}{a_n}\,{}^{C}D^{\beta_{n-1}-\beta_n} + \cdots + \frac{a_0}{a_n}\,{}^{C}D^{\beta_0-\beta_n}\right)^s y_0(t),\\
&\vdots
\end{aligned}
\tag{3.3}
$$

By Adomian decomposition method, adding all terms of the recursion, we obtain the solution of (1.1) as

$$
\begin{aligned}
y(t) &= \sum_{s=0}^{\infty} y_s(t)\\
&= \sum_{s=0}^{\infty}(-1)^s\left(\frac{a_{n-1}}{a_n}\,{}^{C}D^{\beta_{n-1}-\beta_n} + \cdots + \frac{a_0}{a_n}\,{}^{C}D^{\beta_0-\beta_n}\right)^s y_0(t)\\
&= \frac{1}{a_n}\sum_{s=0}^{\infty}(-1)^s\left(\frac{a_{n-1}}{a_n}D^{\beta_{n-1}-\beta_n} + \cdots + \frac{a_0}{a_n}D^{\beta_0-\beta_n}\right)^s D^{-\beta_n} f(t)\\
&\quad + \frac{1}{a_n}\sum_{s=0}^{\infty}(-1)^s\left(\frac{a_{n-1}}{a_n}D^{\beta_{n-1}-\beta_n} + \cdots + \frac{a_0}{a_n}D^{\beta_0-\beta_n}\right)^s \sum_{i=0}^{n} a_i \sum_{j_i=0}^{l_i-1} y^{(j_i)}(0)\frac{t^{\beta_n-\beta_i+j_i}}{\Gamma(1+\beta_n-\beta_i+j_i)}.
\end{aligned}
\tag{3.4}
$$

Let

$$
\begin{aligned}
I_1 &= \frac{1}{a_n}\sum_{s=0}^{\infty}(-1)^s\left(\frac{a_{n-1}}{a_n}D^{\beta_{n-1}-\beta_n} + \cdots + \frac{a_0}{a_n}D^{\beta_0-\beta_n}\right)^s D^{-\beta_n} f(t)\\
I_2 &= \frac{1}{a_n}\sum_{s=0}^{\infty}(-1)^s\left(\frac{a_{n-1}}{a_n}D^{\beta_{n-1}-\beta_n} + \cdots + \frac{a_0}{a_n}D^{\beta_0-\beta_n}\right)^s \sum_{i=0}^{n} a_i \sum_{j_i=0}^{l_i-1} y^{(j_i)}(0)\frac{t^{\beta_n-\beta_i+j_i}}{\Gamma(1+\beta_n-\beta_i+j_i)}.
\end{aligned}
\tag{3.5}
$$

Then,

$$
y(t) = I_1 + I_2. \tag{3.6}
$$

Next, we estimate I_1 and I_2, respectively.

For I_1, by [12] we obtain

$$\begin{aligned}
&\frac{1}{a_n}\sum_{s=0}^{\infty}(-1)^s\left(\frac{a_{n-1}}{a_n}D^{\beta_{n-1}-\beta_n}+\cdots+\frac{a_0}{a_n}D^{\beta_0-\beta_n}\right)^s D^{-\beta_n}f(t),\\
&\quad=\int_0^t\frac{1}{a_n}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{\substack{k_0,k_1,\ldots,k_{n-2}\geqslant 0\\ k_0+k_1+\cdots+k_{n-2}=m}}(m;k_0,k_1,\ldots,k_{n-2})\\
&\quad\times\prod_{p=0}^{n-2}\left(\frac{a_p}{a_n}\right)^{k_p}(t-\tau)^{(\beta_n-\beta_{n-1})m+\beta_n+\sum_{j=0}^{n-2}(\beta_{n-1}-\beta_j)k_j-1}\\
&\quad\times E^{(m)}_{\beta_n-\beta_{n-1},\beta_n+\sum_{j=0}^{n-2}(\beta_{n-1}-\beta_j)k_j}\left(-\frac{a_{n-1}}{a_n}D^{\beta_n-\beta_{n-1}}\right)f(\tau)d\tau.
\end{aligned}\tag{3.7}$$

For I_2, by the initial conditions (2.10) we get

$$\begin{aligned}
&\frac{1}{a_n}\sum_{s=0}^{\infty}(-1)^s\left(\frac{a_{n-1}}{a_n}D^{\beta_{n-1}-\beta_n}+\cdots+\frac{a_0}{a_n}D^{\beta_0-\beta_n}\right)^s\sum_{i=0}^{n}a_i\sum_{j_i=0}^{l_i-1}y^{(j_i)}(0)\frac{t^{\beta_n-\beta_i+j_i}}{\Gamma(1+\beta_n-\beta_i+j_i)}\\
&\quad=\frac{1}{a_n}\sum_{s=0}^{\infty}(-1)^s\sum_{\substack{k_0,k_1,\ldots,k_{n-1}\geqslant 0\\ k_0+k_1+\cdots+k_{n-1}=s}}\frac{s!}{k_0!k_1!\cdots k_{n-1}!}\left(\frac{a_{n-1}}{a_n}\right)^{k_{n-1}}\left(\frac{a_{n-2}}{a_n}\right)^{k_{n-2}}\cdots\left(\frac{a_0}{a_n}\right)^{k_0}\\
&\quad\times D^{k_{n-1}(\beta_{n-1}-\beta_n)+k_{n-2}(\beta_{n-2}-\beta_n)+\cdots+k_0(\beta_0-\beta_n)}\sum_{i=0}^{n}a_i\sum_{j_i=0}^{l_i-1}c_{ij_i}\frac{t^{\beta_n-\beta_i+j_i}}{\Gamma(1+\beta_n-\beta_i+j_i)}.
\end{aligned}\tag{3.8}$$

Using formulas (2.2) and (2.3), the above expression can be written as

$$\begin{aligned}
&\frac{1}{a_n}\sum_{s=0}^{\infty}(-1)^s\sum_{\substack{k_0,k_1,\ldots,k_{n-1}\geqslant 0\\ k_0+k_1+\cdots+k_{n-1}=s}}\frac{s!}{k_0!k_1!\cdots k_{n-1}!}\left(\frac{a_{n-1}}{a_n}\right)^{k_{n-1}}\left(\frac{a_{n-2}}{a_n}\right)^{k_{n-2}}\cdots\left(\frac{a_0}{a_n}\right)^{k_0}\\
&\times\sum_{i=0}^{n}a_i\sum_{j_i=0}^{l_i-1}c_{ij_i}\frac{\Gamma(\mathscr{A})t^{\beta_n-\beta_i+j_i+k_{n-1}(\beta_n-\beta_{n-1})+k_{n-2}(\beta_n-\beta_{n-2})+\cdots+k_0(\beta_n-\beta_0)}}{\Gamma(\mathscr{A})\cdot\Gamma\left(\mathscr{A}+k_{n-1}(\beta_n-\beta_{n-1})+k_{n-2}(\beta_n-\beta_{n-2})+\cdots+k_0(\beta_n-\beta_0)\right)}\\
&=\frac{1}{a_n}\sum_{s=0}^{\infty}(-1)^s\sum_{\substack{k_0,k_1,\ldots,k_{n-2}\geqslant 0\\ k_0+k_1+\cdots+k_{n-1}=s}}\frac{s!}{k_0!k_1!\cdots k_{n-1}!}\left(\frac{a_{n-1}}{a_n}\right)^{k_{n-1}}\left(\frac{a_{n-2}}{a_n}\right)^{k_{n-2}}\cdots\left(\frac{a_0}{a_n}\right)^{k_0}\\
&\times\sum_{i=0}^{n}a_i\sum_{j_i=0}^{l_i-1}c_{ij_i}\frac{t^{\beta_n-\beta_i+j_i+k_{n-1}(\beta_n-\beta_{n-1})+k_{n-2}(\beta_n-\beta_{n-2})+\cdots+k_0(\beta_n-\beta_0)}}{\Gamma\left(\mathscr{A}+k_{n-1}(\beta_n-\beta_{n-1})+k_{n-2}(\beta_n-\beta_{n-2})+\cdots+k_0(\beta_n-\beta_0)\right)}
\end{aligned}$$

$$= \sum_{i=0}^{n} a_i \sum_{j_i=0}^{l_i-1} c_{ij_i} D^{(\beta_i - j_i - 1)} \left[\frac{1}{a_n} \sum_{s=0}^{\infty} (-1)^s \sum_{\substack{k_0,k_1,\ldots,k_{n-1} \geqslant 0 \\ k_0+k_1+\cdots+k_{n-1}=s}} \frac{s!}{k_0!k_1!\cdots k_{n-1}!} \right.$$

$$\times \left(\frac{a_{n-1}}{a_n}\right)^{k_{n-1}} \left(\frac{a_{n-2}}{a_n}\right)^{k_{n-2}} \cdots \left(\frac{a_0}{a_n}\right)^{k_0}$$

$$\left. \times \frac{t^{\beta_n - 1 + k_{n-1}(\beta_n - \beta_{n-1}) + k_{n-2}(\beta_n - \beta_{n-2}) + \cdots + k_0(\beta_n - \beta_0)}}{\Gamma\left(\beta_n + k_{n-1}(\beta_n - \beta_{n-1}) + k_{n-2}(\beta_n - \beta_{n-2}) + \cdots + k_0(\beta_n - \beta_0)\right)} \right], \tag{3.9}$$

where $\mathcal{A}$ denotes $1 + \beta_n - \beta_i + j_i$.

Using Propositions 2.6 and 2.7, the above solution is equivalent to the following form:

$$\sum_{i=0}^{n} a_i \sum_{j_i=0}^{l_i-1} c_{ij_i} D^{(\beta_i - j_i - 1)} \left[\frac{1}{a_n} \sum_{k_{n-1}=0}^{\infty} \sum_{m=0}^{\infty} \sum_{\substack{k_0,k_1,\ldots,k_{n-2} \geqslant 0 \\ k_0+k_1+\cdots+k_{n-2}=m}} (-1)^{k_{n-1}+m} \frac{(k_{n-1}+m)!}{k_0!k_1!\cdots k_{n-1}!} \right.$$

$$\times \left(\frac{a_{n-1}}{a_n}\right)^{k_{n-1}} \left(\frac{a_{n-2}}{a_n}\right)^{k_{n-2}} \cdots \left(\frac{a_0}{a_n}\right)^{k_0}$$

$$\left. \times \frac{t^{\beta_n - 1 + k_{n-1}(\beta_n - \beta_{n-1}) + k_{n-2}(\beta_n - \beta_{n-2}) + \cdots + k_0(\beta_n - \beta_0)}}{\Gamma\left(\beta_n + k_{n-1}(\beta_n - \beta_{n-1}) + k_{n-2}(\beta_n - \beta_{n-2}) + \cdots + k_0(\beta_n - \beta_0)\right)} \right]$$

$$= \sum_{i=0}^{n} a_i \sum_{j_i=0}^{l_i-1} c_{ij_i} D^{(\beta_i - j_i - 1)} \left[\frac{1}{a_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{k_0,k_1,\ldots,k_{n-2} \geqslant 0 \\ k_0+k_1+\cdots+k_{n-2}=m}} \frac{m!}{k_0!k_1!\cdots k_{n-2}!} \right.$$

$$\times \prod_{r=0}^{n-2} \left(\frac{a_r}{a_n}\right)^{k_r} t^{(\beta_n - \beta_{n-1})m + \sum_{\xi=0}^{n-2} (\beta_{n-1} - \beta_\xi) k_\xi + \beta_n - 1}$$

$$\times \sum_{k_{n-1}=0}^{\infty} (-1)^{k_{n-1}} \left(\frac{a_{n-1}}{a_n}\right)^{k_{n-1}} \frac{(k_{n-1}+m)!}{k_{n-1}!}$$

$$\left. \times \frac{t^{k_{n-l}(\beta_n - \beta_{n-1})}}{\Gamma\left(k_{n-1}(\beta_n - \beta_{n-1}) + (\beta_n - \beta_{n-1})m + \sum_{\delta=0}^{n-2} (\beta_{n-1} - \beta_\delta) k_\delta + \beta_n\right)} \right]$$

$$= \sum_{i=0}^{n} a_i \sum_{j_i=0}^{l_i-1} c_{ij_i} D^{(\beta_i - j_i - 1)} \left[\frac{1}{a_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{k_0,k_1,\ldots,k_{n-2} \geqslant 0 \\ k_0+k_1+\cdots+k_{n-2}=m}} (m; k_0, k_1, \ldots, k_{n-2}) \right.$$
$$\times \prod_{r=0}^{n-2} \left(\frac{a_r}{a_n} \right)^{k_r} t^{(\beta_n - \beta_{n-1})m + \sum_{\xi=0}^{n-2} (\beta_{n-1} - \beta_\xi) k_\xi + \beta_n - 1}$$
$$\left. \times E^{(m)}_{\beta_n - \beta_{n-1}, \sum_{\delta=0}^{n-2} (\beta_{n-1} - \beta_l) k_\delta + \beta_n} \left(-\frac{a_{n-1}}{a_n} t^{\beta_n - \beta_{n-1}} \right) \right]. \tag{3.10}$$

Therefore,

$$\begin{aligned} y(t) &= I_1 + I_2 \\ &= \int_0^t \frac{1}{a_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{k_0,k_1,\ldots,k_{n-2} \geqslant 0 \\ k_0+k_1+\cdots+k_{n-2}=m}} (m; k_0, k_1, \ldots, k_{n-2}) \\ &\quad \times \prod_{p=0}^{n-2} \left(\frac{a_p}{a_n} \right)^{k_p} (t-\tau)^{(\beta_n - \beta_{n-1})m + \beta_n + \sum_{j=0}^{n-2} (\beta_{n-1} - \beta_j) k_j - 1} \\ &\quad \times E^{(m)}_{\beta_n - \beta_{n-1}, \beta_n + \sum_{j=0}^{n-2} (\beta_{n-1} - \beta_j) k_j} \left(-\frac{a_{n-1}}{a_n} D^{\beta_n - \beta_{n-1}} \right) f(\tau) d\tau \\ &\quad + \sum_{i=0}^{n} a_i \sum_{j_i=0}^{l_i-1} c_{ij_i} D^{(\beta_i - j_i - 1)} \left[\frac{1}{a_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{k_0,k_1,\ldots,k_{n-2} \geqslant 0 \\ k_0+k_1+\cdots+k_{n-2}=m}} (m; k_0, k_1, \ldots, k_{n-2}) \right. \\ &\qquad \times \prod_{r=0}^{n-2} \left(\frac{a_r}{a_n} \right)^{k_r} t^{(\beta_n - \beta_{n-1})m + \sum_{\xi=0}^{n-2} (\beta_{n-1} - \beta_\xi) k_\xi + \beta_n - 1} \\ &\qquad \left. \times E^{(m)}_{\beta_n - \beta_{n-1}, \sum_{\delta=0}^{n-2} (\beta_{n-1} - \beta_l) k_\delta + \beta_n} \left(-\frac{a_{n-1}}{a_n} t^{\beta_n - \beta_{n-1}} \right) \right], \end{aligned} \tag{3.11}$$

where

$$(m; k_0, k_1, \ldots, k_{n-2}) = \frac{m!}{k_0! k_1! \cdots k_{n-2}!}, \tag{3.12}$$

and $E^{(i)}_{\lambda,\mu}(y)$ is the Mittag-Leffler function

$$E^{(i)}_{\lambda,\mu}(y) = \frac{d^i}{dy^i} E_{\lambda,\mu}(y) = \sum_{j=0}^{\infty} \frac{(i+j)! y^j}{j! \Gamma(\lambda j + \lambda i + \mu)}. \tag{3.13}$$

Substituting the Green function

$$G_n(t) = \frac{1}{a_n}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{\substack{k_0,k_1,\ldots,k_{n-2}\geqslant 0\\ k_0+k_1+\cdots+k_{n-2}=m}}(m;k_0,k_1,\ldots,k_{n-2})$$
$$\times\prod_{p=0}^{n-2}\left(\frac{a_p}{a_n}\right)^{k_p} t^{(\beta_n-\beta_{n-1})m+\beta_n+\sum_{j=0}^{n-2}(\beta_{n-1}-\beta_j)k_j-1} \tag{3.14}$$
$$\times E^{(m)}_{\beta_n-\beta_{n-1},\beta_n+\sum_{j=0}^{n-2}(\beta_{n-1}-\beta_j)k_j}\left(-\frac{a_{n-1}}{a_n}D^{\beta_n-\beta_{n-1}}\right)$$

into the above expression, we know that

$$y(t) = \int_0^t G_n(t-\tau)f(\tau)d\tau + \sum_{i=0}^{n}a_i\sum_{j_i=0}^{l_i-1}c_{ij_i}G_n^{(\beta_i-j_i-1)}(t) \tag{3.15}$$

is the analytical general solution of (1.1).

4. Illustrative Examples

In order to verify our conclusions, we give some examples.

(1). Consider an initial value problem for the relaxation-oscillation equation (see [1])

$$\begin{aligned} {}^{C}D^{\alpha}y(t) + Ay(t) &= f(t), \quad t > 0,\\ y^{(j)}(0) &= b_j, \quad j = 1,2,\ldots,m-1, \quad m-1 < \alpha < m, \end{aligned} \tag{4.1}$$

where b_j are real constants.

Utilizing Lemma 2.5 and applying ${}^{C}D^{-\alpha}$ to both sides of (4.1), we obtain

$$y(t) + AD^{-\alpha}y(t) = D^{-\alpha}f(t) + \sum_{j=0}^{m-1}y^{(j)}(0)\frac{t^j}{\Gamma(j+1)}. \tag{4.2}$$

According to the above procedure of solving the linear fractional differential equations with constant coefficients and using the Adomian decomposition method, let

$$\begin{aligned} y_0(t) &= D^{-\alpha} f(t) + \sum_{j=0}^{m-1} y^{(j)}(0) \frac{t^j}{\Gamma(j+1)}, \\ y_1(t) &= -AD^{-\alpha} y_0(t), \\ y_2(t) &= -AD^{-\alpha} y_1(t) = (-A)^2 D^{-2\alpha} y_0(t), \\ &\vdots \\ y_s(t) &= -AD^{-\alpha} y_{s-1}(t) = (-A)^s D^{-s\alpha} y_0(t), \\ &\vdots \end{aligned} \tag{4.3}$$

Adding all of the above terms, we obtain the solution of the equation by Adomian decomposition method as follows:

$$\begin{aligned} y(t) &= \sum_{s=0}^{\infty} y_s(t) \\ &= \sum_{s=0}^{\infty} (-A)^s D^{-s\alpha} y_0(t) \\ &= \sum_{s=0}^{\infty} (-A)^s D^{(-s-1)\alpha} f(t) + \sum_{s=0}^{\infty} (-A)^s D^{-s\alpha} \sum_{j=0}^{m-1} y^{(j)}(0) \frac{t^j}{\Gamma(j+1)} \\ &= \sum_{s=0}^{\infty} (-A)^s \frac{1}{\Gamma((s+1)\alpha)} \int_0^t (t-\tau)^{(s+1)\alpha-1} f(\tau) d\tau + \sum_{j=0}^{m-1} b_j \sum_{s=0}^{\infty} (-A)^s \frac{t^{j+s\alpha}}{\Gamma(j+s\alpha+1)} \\ &= \int_0^t \sum_{s=0}^{\infty} (t-\tau)^{\alpha-1} \cdot \frac{[-A(t-\tau)^\alpha]^s}{\Gamma(s\alpha+\alpha)} f(\tau) d\tau + \sum_{j=0}^{m-1} b_j D^{(\alpha-j-1)} \left[\sum_{s=0}^{\infty} t^{\alpha-1} \frac{(-At^\alpha)^s}{\Gamma(s\alpha+\alpha)} \right] \\ &= \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-A(t-\tau)^\alpha\right) f(\tau) d\tau + \sum_{j=0}^{m-1} b_j D^{(\alpha-j-1)} \left[t^{\alpha-1} E_{\alpha,\alpha}(-At^\alpha) \right] \\ &= \int_0^t G_2(t-\tau) f(\tau) d\tau + \sum_{j=0}^{m-1} b_j D^{(\alpha-j-1)} G_2(t), \end{aligned} \tag{4.4}$$

where $G_2(t) = t^{\alpha-1} E_{\alpha,\alpha}(-At^\alpha)$.

It is easy to see that

$$y(t) = \int_0^t G_2(t-\tau) f(\tau) d\tau + b_0 D^{(\alpha-1)} G_2(t), \quad (0 < \alpha < 1). \tag{4.5}$$

(2). Consider an initial value problem for the nonhomogeneous Bagley-Torvik equation (see [5])

$$\begin{aligned} Ay''(t) + B{}^{C}D^{3/2}y(t) + Cy(t) &= f(t), \quad t > 0, \\ y^{(i)}(0) &= a_i, \ i = 0, 1, \end{aligned} \tag{4.6}$$

where a_i are real constants.

Utilizing Lemma 2.5 and applying ${}^{C}D^{-2}$ to both sides of (4.6), we obtain

$$\begin{aligned} & y(t) + \frac{1}{A}\left[BD^{-1/2}y(t) + CD^{-2}y(t)\right] \\ &= \frac{1}{A}D^{-2}f(t) + \sum_{i=0}^{1} y^{(i)}(0)\frac{t^i}{\Gamma(i+1)} + \frac{B}{A}\sum_{i=0}^{1} y^{(i)}(0)\frac{t^{1/2+i}}{\Gamma(3/2+i)} \\ &= \frac{1}{A}D^{-2}f(t) + \sum_{i=0}^{1} y^{(i)}(0)\left[\frac{t^i}{\Gamma(i+1)} + \frac{B}{A}\cdot\frac{t^{1/2+i}}{\Gamma(3/2+i)}\right] \end{aligned} \tag{4.7}$$

According to the above procedure of solving the linear fractional differential equation with constant coefficients and using the Adomian decomposition method, let

$$\begin{aligned} y_0(t) &= \frac{1}{A}D^{-2}f(t) + \sum_{i=0}^{1} y^{(i)}(0)\left[\frac{t^i}{\Gamma(i+1)} + \frac{B}{A}\cdot\frac{t^{1/2+i}}{\Gamma(3/2+i)}\right], \\ y_1(t) &= -\frac{1}{A}\left[BD^{-1/2}y_0(t) + CD^{-2}y_0(t)\right] = -\frac{C}{A}\left(\frac{B}{C}I + D^{-3/2}\right)D^{-1/2}y_0(t), \\ y_2(t) &= -\frac{C}{A}\left(\frac{B}{C}I + D^{-3/2}\right)D^{-1/2}y_1(t) = \left(-\frac{C}{A}\right)^2\left(\frac{B}{C}I + D^{-3/2}\right)^2 D^{-2/2}y_0(t), \\ &\vdots \\ y_n(t) &= -\frac{C}{A}\left(\frac{B}{C}I + D^{-3/2}\right)D^{-1/2}y_{n-1}(t) = \left(-\frac{C}{A}\right)^n\left(\frac{B}{C}I + D^{-3/2}\right)^n D^{-n/2}y_0(t), \\ &\vdots \end{aligned} \tag{4.8}$$

Adding all of the above terms, we obtain the solution of the equation by Adomian decomposition method as follows:

$$
\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} y_n(t) \\
&= \sum_{n=0}^{\infty} \left(-\frac{C}{A}\right)^n \left(\frac{B}{C} I + D^{-3/2}\right)^n D^{-n/2} y_0(t) \\
&= \sum_{n=0}^{\infty} \left(-\frac{C}{A}\right)^n \left(\frac{B}{C} I + D^{-3/2}\right)^n D^{-n/2} \left[\frac{1}{A} D^{-2} f(t)\right] \\
&\quad + \sum_{n=0}^{\infty} \left(-\frac{C}{A}\right)^n \left(\frac{B}{C} I + D^{-3/2}\right)^n D^{-n/2} \sum_{i=0}^{1} y^{(i)}(0) \left[\frac{t^i}{\Gamma(i+1)} + \frac{B}{A} \cdot \frac{t^{1/2+i}}{\Gamma(3/2+i)}\right] \\
&= \frac{1}{A} \sum_{n=0}^{\infty} \left(-\frac{C}{A}\right)^n \sum_{k=0}^{n} C_n^{n-k} \left(\frac{B}{C} I\right)^{n-k} D^{-3k/2} D^{-n/2} D^{-2} f(t) \\
&\quad + \sum_{i=0}^{1} a_i \sum_{n=0}^{\infty} \left(-\frac{C}{A}\right)^n \sum_{k=0}^{n} C_n^{n-k} \left(\frac{B}{C} I\right)^{n-k} D^{-3k/2} D^{-n/2} \left[\frac{t^i}{\Gamma(i+1)} + \frac{B}{A} \cdot \frac{t^{1/2+i}}{\Gamma(3/2+i)}\right] \\
&= \frac{1}{A} \sum_{n=0}^{\infty} \left(-\frac{C}{A}\right)^n \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \left(\frac{B}{C} I\right)^{n-k} D^{-(3k/2+n/2+2)} f(t) \\
&\quad + \sum_{i=0}^{1} a_i \sum_{n=0}^{\infty} \left(-\frac{C}{A}\right)^n \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \left(\frac{B}{C} I\right)^{n-k} D^{-(3k/2+n/2)} \left[\frac{t^i}{\Gamma(i+1)} + \frac{B}{A} \cdot \frac{t^{1/2+i}}{\Gamma(3/2+i)}\right] \\
&= \frac{1}{A} \int_0^t \sum_{n=0}^{\infty} \left(-\frac{C}{A}\right)^n \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \left(\frac{B}{C} I\right)^{n-k} \frac{1}{\Gamma(3k/2+n/2+2)} (t-\tau)^{3k/2+n/2+2-1} f(\tau) d\tau \\
&\quad + \sum_{i=0}^{1} a_i \sum_{n=0}^{\infty} \left(-\frac{C}{A}\right)^n \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \left(\frac{B}{C} I\right)^{n-k} \left[\frac{t^{i+3k/2+n/2}}{\Gamma(i+3k/2+n/2+1)} \right. \\
&\qquad \left. + \frac{B}{A} \cdot \frac{t^{1/2+i+(3k/2)+(n/2)}}{\Gamma(i+3k/2+n/2+3/2)}\right] \\
&= \frac{1}{A} \int_0^t \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left(-\frac{C}{A}\right)^{k+m} \frac{(k+m)!}{m!k!} \left(\frac{B}{C} I\right)^m \frac{1}{\Gamma(3/2k+(k+m/2)+2)} \\
&\qquad \times (t-\tau)^{3k/2+(k+m)/2+1} f(\tau) d\tau \\
&\quad + \sum_{i=0}^{1} a_i D^{(1-i)} \left[\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left(-\frac{C}{A}\right)^{k+m} \frac{(k+m)!}{m!k!} \left(\frac{B}{C} I\right)^m \frac{t^{3k/2+(k+m)/2+1}}{\Gamma(3/2k+(k+m)/2+2)}\right] \\
&\quad + \sum_{i=0}^{1} a_i D^{(1/2-i)} \left[\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left(-\frac{C}{A}\right)^{k+m} \frac{(k+m)!}{m!k!} \left(\frac{B}{C} I\right)^m \frac{t^{3k/2+(k+m)/2+1}}{\Gamma(3/2k+(k+m)/2+2)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{A}\int_0^t \sum_{k=0}^{\infty} \frac{(-C/A)^k}{k!}(t-\tau)^{2k+1} \sum_{m=0}^{\infty} \frac{(k+m)!}{m!} \cdot \frac{(-(B/A)(t-\tau)^{1/2})^m}{\Gamma(k/2+m/2+3k/2+2)} f(\tau)d\tau \\
&\quad + A\sum_{i=0}^{1} a_i D^{(1-i)} \left[\frac{1}{A} \sum_{k=0}^{\infty} \frac{(-C/A)^k}{k!} \ (t-\tau)^{2k+1} \ \sum_{m=0}^{\infty} \frac{(k+m)!}{m!} \cdot \frac{\left(-(B/A)(t-\tau)^{1/2}\right)^m}{\Gamma(k/2+m/2+3k/2+2)} \right] \\
&\quad + A\sum_{i=0}^{1} a_i D^{(1/2-i)} \left[\frac{1}{A} \sum_{k=0}^{\infty} \frac{(-C/A)^k}{k!} (t-\tau)^{2k+1} \right. \\
&\qquad\qquad \left. \times \sum_{m=0}^{\infty} \frac{(k+m)!}{m!} \cdot \frac{\left(-(B/A)(t-\tau)^{1/2}\right)^m}{\Gamma((k/2)+(m/2)+(3k/2)+2)} \right] \\
&= \frac{1}{A}\int_0^t \sum_{k=0}^{\infty} \frac{(-C/A)^k}{k!}(t-\tau)^{2k+1} E^{(k)}_{1/2,3k/2+2}\left(-\frac{B}{A}(t-\tau)^{1/2}\right) f(\tau)d\tau \\
&\quad + A\sum_{i=0}^{1} a_i D^{(1-i)} \left[\frac{1}{A} \sum_{k=0}^{\infty} \frac{(-C/A)^k}{k!} t^{2k+1} E^{(k)}_{1/2,3k/2+2}\left(-\frac{B}{A}t^{1/2}\right) \right] \\
&\quad + A\sum_{i=0}^{1} a_i D^{(1/2-i)} \left[\frac{1}{A} \sum_{k=0}^{\infty} \frac{(-C/A)^k}{k!} t^{2k+1} E^{(k)}_{1/2,3k/2+2}\left(-\frac{B}{A}t^{1/2}\right) \right] \\
&= \int_0^t G_3(t-\tau) f(\tau)d\tau + A\sum_{i=0}^{1} a_i \left[D^{(1-i)} G_3(t) + D^{(1/2-i)} G_3(t) \right],
\end{aligned}
\tag{4.9}
$$

where $G_3(t) = (1/A)\sum_{k=0}^{\infty}((-C/A)^k/k!)t^{2k+1}E^{(k)}_{1/2,3k/2+2}(-(B/A)t^{1/2})$.

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