Research Article

C-N Difference Schemes for Dissipative Symmetric Regularized Long Wave Equations with Damping Term

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We study the initial-boundary problem of dissipative symmetric regularized long wave equations with damping term. Crank-Nicolson nonlinear-implicit finite difference scheme is designed. Existence and uniqueness of numerical solutions are derived. It is proved that the finite difference scheme is of second-order convergence and unconditionally stable by the discrete energy method. Numerical simulations verify the theoretical analysis.

1. Introduction

A symmetric version of regularized long wave equation (SRLWE),

$$
u_{xxt} - u_t = \rho_x + uu_x,
$$
$$
\rho_t + u_x = 0,
$$

(1.1)

has been proposed to model the propagation of weakly nonlinear ion acoustic and space charge waves [1]. The sech^2 solitary wave solutions are

$$
u(x,t) = \frac{3(v^2-1)}{v}\operatorname{sech}^2\frac{1}{2}\sqrt{\frac{v^2-1}{v^2}}(x-vt),
$$
$$
\rho(x,t) = \frac{3(v^2-1)}{v^2}\operatorname{sech}^2\frac{1}{2}\sqrt{\frac{v^2-1}{v^2}}(x-vt).
$$

(1.2)

The four invariants and some numerical results have been obtained in [1], where v is the velocity, $v^2 > 1$. Obviously, eliminating ρ from (1.1), we get a class of SRLWE

$$u_{tt} - u_{xx} + \frac{1}{2}\left(u^2\right)_{xt} - u_{xxtt} = 0. \tag{1.3}$$

Equation (1.3) is explicitly symmetric in the x and t derivatives and is very similar to the regularized long wave equation that describes shallow water waves and plasma drift waves [2, 3]. The SRLW equation also arises in many other areas of mathematical physics [4–6]. Numerical investigation indicates that interactions of solitary waves are inelastic [7] thus, the solitary wave of the SRLWE is not a solution. Research on the well-posedness for its solution and numerical methods has aroused more and more interest. In [8], Guo studied the existence, uniqueness and regularity of the numerical solutions for the periodic initial value problem of generalized SRLW by the spectral method. In [9], Zheng et al. presented a Fourier pseudospectral method with a restraint operator for the SRLWEs, and proved its stability and obtained the optimum error estimates. There are other methods such as pseudospectral method, finite difference method for the initial-boundary value problem of SRLWEs (see [10–15]).

Because of gravity and resistance of propagation medium and air, the principle of dissipation must be considered when studying the move of nonlinear wave. In applications, the viscous damping effect is inevitable and it plays the same important role as the dispersive effect. Therefore, it is more significant to study the dissipative symmetric regularized long wave equations with the damping term

$$u_{xxt} - u_t + vu_{xx} = \rho_x + uu_x,$$
$$\rho_t + u_x + \gamma\rho = 0, \tag{1.4}$$

where v, γ are positive, $v > 0$ is the dissipative coefficient, $\gamma > 0$ is the damping coefficient. Equation (1.4) are a reasonable model to render essential phenomena of nonlinear ion acoustic wave motion when dissipation is considered (see [16–20]). Existence, uniqueness and well-posedness of global solutions to (1.4) are presented (see [16–20]). But it is difficult to find the analytical solution to (1.4), which makes numerical solution important.

In this paper, we study (1.4) with

$$u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x), \quad x \in [x_L, x_R] \tag{1.5}$$

and the boundary conditions

$$u(x_L, t) = u(x_R, t) = 0, \quad \rho(x_L, t) = \rho(x_R, t) = 0, \quad t \in [0, T]. \tag{1.6}$$

In [21] we proposed a three-level implicit finite difference scheme to (1.4)–(1.6) with second-order convergence. But the three-level implicit finite difference scheme can not start by itself, we need to select other two-level schemes (such as the C-N Scheme) to get u^1, ρ^1. Then, reusing initial vale u^0, ρ^0, we can work out $u^2, \rho^2, u^3, \rho^3, \ldots$. Since the form of SRLW equations is similar ro the Rosenau equation and Rosenau-Burgers equation, the established difference

schemes in [22, 23] for solving Rosenau equation and Rosenau-Burgers equation are helpful to investigate the SRLWEs. We propose the Crank-Nicolson finite difference scheme for (1.4)–(1.6) which can start by itself. We will show that this difference scheme is uniquely solvable, convergent and stable in both theoretical and numerical senses.

2. Finite Difference Scheme and Its Error Estimation

Let h and τ be the uniform step size in the spatial and temporal direction, respectively. Denote $x_j = x_L + jh$ $(j = 0, 1, 2, \ldots, J)$, $t_n = n\tau$ $(n = 0, 1, 2, \ldots, N)$, $N = [T/\tau]$, $u_j^n \approx u(x_j, t_n)$, $\rho_j^n \approx \rho(x_j, t_n)$ and $Z_h^0 = \{u = (u_j) \mid u_0 = u_J = 0, j = 0, 1, 2, \ldots, J\}$. Throughout this paper, we will denote C as a generic constant independent of h and τ that varies in the context. We define the difference operators, inner product, and norms that will be used in this paper as follows:

$$\left(u_j^n\right)_x = \frac{u_{j+1}^n - u_j^n}{h}, \qquad \left(u_j^n\right)_{\overline{x}} = \frac{u_j^n - u_{j-1}^n}{h}, \qquad \left(u_j^n\right)_{\hat{x}} = \frac{u_{j+1}^n - u_{j-1}^n}{2h}, \qquad \left(u_j^n\right)_t = \frac{u_j^{n+1} - u_j^n}{\tau},$$

$$u_j^{n+1/2} = \frac{u_j^{n+1} + u_j^n}{2}, \qquad \langle u^n, v^n \rangle = h \sum_{j=1}^{J-1} u_j^n v_j^n, \qquad \|u^n\|^2 = \langle u^n, u^n \rangle, \qquad \|u^n\|_\infty = \max_{1 \le j \le J-1} \left|u_j^n\right|.$$

$$(2.1)$$

Then, the Crank-Nicolson finite difference scheme for the solution of (1.4)–(1.6) is as follows:

$$\left(u_j^n\right)_t - \left(u_j^n\right)_{x\overline{x}t} + \left(\rho_j^{n+1/2}\right)_{\hat{x}} - v\left(u_j^{n+1/2}\right)_{x\overline{x}} + \frac{1}{3}\left(u_{j+1}^{n+1/2} + u_j^{n+1/2} + u_{j-1}^{n+1/2}\right)\left(u_j^{n+1/2}\right)_{\hat{x}} = 0,$$

$$(2.2)$$

$$\left(\rho_j^n\right)_t + \left(u_j^{n+1/2}\right)_{\hat{x}} + \gamma\rho_j^{n+1/2} = 0, \tag{2.3}$$

$$u_j^0 = u_0(x_j), \quad \rho_j^0 = \rho_0(x_j), \quad 1 \le j \le J - 1, \tag{2.4}$$

$$u_0^n = u_J^n = 0, \quad \rho_0^n = \rho_J^n = 0, \quad 1 \le n \le N. \tag{2.5}$$

Lemma 2.1. *It follows summation by parts [12, 23] that for any two discrete functions $u, v \in Z_h^0$,*

$$\langle u_x, v \rangle = -\langle u, v_{\overline{x}} \rangle, \qquad \langle u_{x\overline{x}}, v \rangle = -\langle u_x, v_x \rangle. \tag{2.6}$$

Lemma 2.2 (Discrete Sobolev's inequality [12, 23]). *There exist two constants C_1 and C_2 such that*

$$\|u^n\|_\infty \le C_1\|u^n\| + C_2\|u_x^n\|. \tag{2.7}$$

Lemma 2.3 (Discrete Gronwall's inequality [12, 23]). *Suppose $w(k)$, $\rho(k)$ are nonnegative function and $\rho(k)$ is nondecreasing. If $C > 0$ and $w(k) \le \rho(k) + C\tau \sum_{l=0}^{k-1} w(l)$, then*

$$w(k) \le \rho(k)e^{C\tau k}. \tag{2.8}$$

Theorem 2.4. *If $u_0 \in H^1$, $\rho_0 \in L_2$, then the solution of (2.2)–(2.5) satisfies*

$$\|u^n\| \le C, \quad \|u_x^n\| \le C, \quad \|\rho^n\| \le C, \quad \|u^n\|_\infty \le C, \quad (n = 1, 2, \dots, N). \tag{2.9}$$

Proof. Taking an inner product of (2.2) with $2u^{n+1/2}$ (i.e., $u^{n+1} + u^n$) and considering the boundary condition (2.5) and Lemma 2.1, we obtain

$$\frac{1}{\tau}\left(\left\|u^{n+1}\right\|^2 - \|u^n\|^2\right) + \frac{1}{\tau}\left(\left\|u_x^{n+1}\right\|^2 - \|u_x^n\|^2\right) + \left\langle \rho_{\widehat{x}}^{n+1/2}, 2u^{n+1/2}\right\rangle$$
$$- v\left\langle u_{x\overline{x}}^{n+1/2}, 2u^{n+1/2}\right\rangle + \left\langle P^{n+1/2}, 2u^{n+1/2}\right\rangle = 0, \tag{2.10}$$

where $P_j^{n+1/2} = (1/3)(u_{j+1}^{n+1/2} + u_j^{n+1/2} + u_{j-1}^{n+1/2}) \cdot (u_j^{n+1/2})_{\widehat{x}}$.

Since

$$\left\langle \rho_{\widehat{x}}^{n+1/2}, 2u^{n+1/2}\right\rangle = -2\left\langle u_{\widehat{x}}^{n+1/2}, \rho^{n+1/2}\right\rangle,$$
$$\left\langle u_{x\overline{x}}^{n+1/2}, 2u^{n+1/2}\right\rangle = -2\left\|u_x^{n+1/2}\right\|^2, \tag{2.11}$$

$$
\begin{aligned}
\left\langle P^{n+1/2}, 2u^{n+1/2}\right\rangle &= 2h\sum_{j=1}^{J-1} P_j^{n+1/2} u_j^{n+1/2} \\
&= \frac{2}{3}h\sum_{j=1}^{J-1}\left(u_{j+1}^{n+1/2} + u_j^{n+1/2} + u_{j-1}^{n+1/2}\right) \cdot \left(u_j^{n+1/2}\right)_{\widehat{x}} u_j^{n+1/2} \\
&= \frac{1}{3}\sum_{j=1}^{J-1}\left(u_{j+1}^{n+1/2} + u_j^{n+1/2} + u_{j-1}^{n+1/2}\right) \cdot \left(u_{j+1}^{n+1/2} - u_{j-1}^{n+1/2}\right) u_j^{n+1/2} \\
&= \frac{1}{3}\sum_{j=1}^{J-1}\left(u_{j+1}^{n+1/2} + u_j^{n+1/2}\right) u_{j+1}^{n+1/2} u_j^{n+1/2} \\
&\quad - \frac{1}{3}\sum_{j=1}^{J-1}\left(u_j^{n+1/2} + u_{j-1}^{n+1/2}\right) u_{j-1}^{n+1/2} u_j^{n+1/2} = 0,
\end{aligned} \tag{2.12}
$$

we obtain

$$\frac{1}{\tau}\left(\left\|u^{n+1}\right\|^2 - \|u^n\|^2\right) + \frac{1}{\tau}\left(\left\|u_x^{n+1}\right\|^2 - \|u_x^n\|^2\right) - 2\left\langle u_{\widehat{x}}^{n+1/2}, \rho^{n+1/2}\right\rangle + 2v\left\|u_x^{n+1/2}\right\|^2 = 0. \tag{2.13}$$

Taking an inner product of (2.3) with $2\rho^{n+1/2}$ (i.e., $\rho^{n+1} + \rho^n$) and considering the boundary condition (2.5) and Lemma 2.1, we obtain

$$\frac{1}{\tau}\left(\left\|\rho^{n+1}\right\|^2 - \|\rho^n\|^2\right) + \left\langle u_{\widehat{x}}^{n+1/2}, 2\rho^{n+1/2}\right\rangle + 2\gamma\left\|\rho^{n+1/2}\right\|^2 = 0. \tag{2.14}$$

Adding (2.13) to (2.14), we have

$$\left(\left\|u^{n+1}\right\|^2 - \|u^n\|^2\right) + \left(\left\|u_x^{n+1}\right\|^2 - \|u_x^n\|^2\right) + \left(\left\|\rho^{n+1}\right\|^2 - \|\rho^n\|^2\right)$$
$$= -2\tau\left(v\left\|u_x^{n+1/2}\right\|^2 + \gamma\left\|\rho^{n+1/2}\right\|^2\right) \le 0, \tag{2.15}$$

which implies

$$\|u^n\|^2 + \|u_x^n\|^2 + \|\rho^n\|^2 \le \left\|u^{n-1}\right\|^2 + \left\|u_x^{n-1}\right\|^2 + \left\|\rho^{n-1}\right\|^2 \le \cdots \le \left\|u^0\right\|^2 + \left\|u_x^0\right\|^2 + \left\|\rho^0\right\|^2 = C. \tag{2.16}$$

Then, it holds

$$\|u^n\| \le C, \qquad \|u_x^n\| \le C, \qquad \|\rho^n\| \le C. \tag{2.17}$$

By Lemma 2.2, we obtain $\|u^n\|_\infty \le C$. $\qquad\qquad\square$

Theorem 2.5. *Assume that $u^0 \in H^2$, $\rho^0 \in H^1$, the solution of difference scheme (2.2)–(2.5) satisfies:*

$$\|\rho_x^n\| \le C, \quad \|u_{xx}^n\| \le C, \quad \|u_x^n\| \le C, \quad \|\rho^n\|_\infty \le C, \quad (n = 1, 2, \ldots, N). \tag{2.18}$$

Proof. Differentiating backward (2.2)–(2.5) with respect to x, we obtain

$$\left(u_j^n\right)_{xt} - \left(u_j^n\right)_{xx\overline{x}t} + \left(\rho_j^{n+1/2}\right)_{x\widehat{x}} - v\left(u_j^{n+1/2}\right)_{xx\overline{x}} + \left(P_j^{n+1/2}\right)_x = 0, \tag{2.19}$$

$$\left(\rho_j^n\right)_{xt} + \left(u_j^{n+1/2}\right)_{x\widehat{x}} + \gamma\left(\rho_j^{n+1/2}\right)_x = 0, \tag{2.20}$$

$$\left(u_j^0\right)_x = u_{0,x}(x_j), \quad \left(\rho_j^0\right)_x = \rho_{0,x}(x_j), \quad 1 \le j \le J - 1, \tag{2.21}$$

$$(u_0^n)_x = \left(u_J^n\right)_x = 0, \quad (\rho_0^n)_x = \left(\rho_J^n\right)_x = 0, \quad 1 \le n \le N, \tag{2.22}$$

where $(P_j^{n+1/2})_x = (1/3)[(u_{j+1}^{n+1/2} + u_j^{n+1/2} + u_{j-1}^{n+1/2})(u_j^{n+1/2})_{\widehat{x}}]_x$.

Computing the inner product of (2.19) with $2u_x^{n+1/2}$ (i.e., $u_x^{n+1} + u_x^n$) and considering (2.22) and Lemma 2.1, we obtain

$$\frac{1}{\tau}\left(\left\|u_x^{n+1}\right\|^2 - \|u_x^n\|^2\right) + \frac{1}{\tau}\left(\left\|u_{xx}^{n+1}\right\|^2 - \|u_{xx}^n\|^2\right)$$

$$+ \left\langle \rho_{x\hat{x}}^{n+1/2}, 2u_x^{n+1/2}\right\rangle - \upsilon\left\langle u_{xx\overline{x}}^{n+1/2}, 2u_x^{n+1/2}\right\rangle + \left\langle P_x^{n+1/2}, 2u_x^{n+1/2}\right\rangle = 0. \tag{2.23}$$

It follows from Theorem 2.4 that

$$\left|u_{j+1}^{n+1/2} + u_j^{n+1/2} + u_{j-1}^{n+1/2}\right| \leq C, \quad (j = 0, 1, 2, \ldots, J). \tag{2.24}$$

By the Schwarz inequality and Lemma 2.1, we get

$$\left\langle P_x^{n+1/2}, 2u_x^{n+1/2}\right\rangle = -2\left\langle P^{n+1/2}, u_{x\overline{x}}^{n+1/2}\right\rangle$$

$$= -\frac{2}{3}h\sum_{j=1}^{J-1}\left(u_{j+1}^{n+1/2} + u_j^{n+1/2} + u_{j-1}^{n+1/2}\right) \cdot \left(u_j^{n+1/2}\right)_{\hat{x}}\left(u_j^{n+1/2}\right)_{x\overline{x}}$$

$$\leq Ch\sum_{j=1}^{J-1}\left|\left(u_j^{n+1/2}\right)_{\hat{x}}\right| \cdot \left|\left(u_j^{n+1/2}\right)_{x\overline{x}}\right| \leq C\left(\left\|u_x^{n+1/2}\right\|^2 + \left\|u_{xx}^{n+1/2}\right\|^2\right)$$

$$\leq C\left(\left\|u_x^{n+1}\right\|^2 + \|u_x^n\|^2 + \left\|u_{xx}^{n+1}\right\|^2 + \|u_{xx}^n\|^2\right). \tag{2.25}$$

By

$$\left\langle \rho_{x\hat{x}}^{n+1/2}, 2u_x^{n+1/2}\right\rangle = -2\left\langle u_{x\hat{x}}^{n+1/2}, \rho_x^{n+1/2}\right\rangle,$$

$$\left\langle u_{xx\overline{x}}^{n+1/2}, 2u_x^{n+1/2}\right\rangle = -2\left\|u_{xx}^{n+1/2}\right\|^2, \tag{2.26}$$

it follows from (2.23) that

$$\left(\left\|u_x^{n+1}\right\|^2 - \|u_x^n\|^2\right) + \left(\left\|u_{xx}^{n+1}\right\|^2 - \|u_{xx}^n\|^2\right) - 2\tau\left\langle u_{x\hat{x}}^{n+1/2}, \rho_x^{n+1/2}\right\rangle$$

$$\leq -2\upsilon\tau\left\|u_{xx}^{n+1/2}\right\|^2 + C\tau\left(\left\|u_x^{n+1}\right\|^2 + \|u_x^n\|^2 + \left\|u_{xx}^{n+1}\right\|^2 + \|u_{xx}^n\|^2\right). \tag{2.27}$$

Computing the inner product of (2.20) with $2\rho_x^{n+1/2}$ (i.e., $\rho_x^{n+1} + \rho_x^n$) and considering (2.22) and Lemma 2.1, we obtain

$$\left(\left\|\rho_x^{n+1}\right\|^2 - \|\rho_x^n\|^2\right) + 2\tau\left\langle u_{x\hat{x}}^{n+1/2}, \rho_x^{n+1/2}\right\rangle + 2\gamma\tau\left\|\rho_x^{n+1/2}\right\|^2 = 0. \tag{2.28}$$

Adding (2.28) to (2.27),we have

$$
\left(\left\| u_x^{n+1} \right\|^2 - \| u_x^n \|^2 \right) + \left(\left\| u_{xx}^{n+1} \right\|^2 - \| u_{xx}^n \|^2 \right) + \left(\left\| \rho_x^{n+1} \right\|^2 - \| \rho_x^n \|^2 \right)
$$

$$
\leq -2\upsilon\tau \left\| u_{xx}^{n+1/2} \right\|^2 - 2\gamma\tau \left\| \rho_x^{n+1/2} \right\|^2 + C\tau \left(\left\| u_x^{n+1} \right\|^2 + \| u_x^n \|^2 + \left\| u_{xx}^{n+1} \right\|^2 + \| u_{xx}^n \|^2 \right) \tag{2.29}
$$

$$
\leq C\tau \left(\left\| u_x^{n+1} \right\|^2 + \| u_x^n \|^2 + \left\| u_{xx}^{n+1} \right\|^2 + \| u_{xx}^n \|^2 + \left\| \rho_x^{n+1} \right\|^2 + \| \rho_x^n \|^2 \right).
$$

Let $A^n = \| u_x^n \|^2 + \| u_{xx}^n \|^2 + \| \rho_x^n \|^2$, we obtain $A^{n+1} - A^n \leq C\tau(A^{n+1} + A^n)$.
Choosing suitable τ which is small enough to satisfy $1 - C\tau > 0$, we get

$$
A^{n+1} - A^n \leq C\tau A^n. \tag{2.30}
$$

Summing up (2.30) from 0 to $n - 1$, we have

$$
A^n \leq A^0 + C\tau \sum_{l=0}^{n-1} A^l. \tag{2.31}
$$

By Lemma 2.3, we get $A^n \leq C$, which implies $\| \rho_x^n \| \leq C$, $\| u_{xx}^n \| \leq C$. It follows from Theorem 2.4 and Lemma 2.2 that $\| u_x^n \|_\infty \leq C$, $\| \rho^n \|_\infty \leq C$. $\qquad\square$

3. Solvability, Convergence, and Stability

The following Brouwer fixed point theorem will be needed in order to show the existence of solution for (2.2)–(2.5). For the proof, see [24].

Lemma 3.1 (Brouwer fixed point theorem). *Let H be a finite dimensional inner product space, suppose that $g : H \to H$ is continuous and there exists an $\alpha > 0$ such that $< g(x), x > 0$ for all $x \in H$ with $\|x\| = \alpha$. Then there exists $x^* \in H$ such that $g(x^*) = 0$ and $\|x^*\| \leq \alpha$.*

Let $Z_\Delta = \{v = (v_1, v_2) = (v_{1,j}, v_{2,j}) \mid v_{1,0} = v_{1,J} = v_{2,0} = v_{2,J} = 0, \ j = 0, 1, 2, \ldots, J\}$, equipped with the inner product $\langle v, v' \rangle = \langle (v_1, v_2), (v_1', v_2') \rangle = \langle v_1, v_1' \rangle + \langle v_2, v_2' \rangle$ and the norm $\|v\|^2 = \|v_1\|^2 + \|v_2\|^2$.

Theorem 3.2. *There exists $(u^n, \rho^n) \in Z_\Delta$ which satisfies the difference scheme (2.2)–(2.5).*

Proof. In order to prove the theorem by the mathematical induction, we assume that (u^0, ρ^0), $(u^1, \rho^1), \ldots, (u^n, \rho^n) \in Z_\Delta$ satisfying (2.2)–(2.5). Next prove there exists (u^{n+1}, ρ^{n+1}) which satisfies (2.2)–(2.5).
Let $g = (g_1, g_2)$ be a operator on Z_Δ defined by

$$
g_1(v) = 2v_1 - 2u^n - 2v_{1x\overline{x}} + 2u_{x\overline{x}}^n + \tau v_{2\hat{x}} - \upsilon\tau v_{1x\overline{x}} + \tau W,
$$

$$
g_2(v) = 2v_2 - 2\rho^n + \tau v_{1\hat{x}} + \gamma\tau v_2, \quad \forall v = (v_1, v_2) \in Z_\Delta, \tag{3.1}
$$

where $W_j = (1/3)(v_{1,j+1} + v_{1,j} + v_{1,j-1})(v_{1,j})_{\hat{x}}$. Computing the inner product of (3.1) with $v = (v_1, v_2)$, similarly to (2.11) and (2.12), we obtain

$$\langle v_{2\hat{x}}, v_1 \rangle = -\langle v_{1\hat{x}}, v_2 \rangle, \qquad \langle W, v_1 \rangle = 0. \tag{3.2}$$

By (2.5) and the Schwarz inequality, we obtain

$$
\begin{aligned}
\langle g(v), v \rangle &= \langle g_1(v), v_1 \rangle + \langle g_2(v), v_2 \rangle \\
&= 2\|v_1\|^2 - 2\langle u^n, v_1 \rangle + 2\|v_{1x}\|^2 - 2\langle u^n_x, v_{1x} \rangle + \upsilon\tau\|v_{1x}\|^2 + 2\|v_2\|^2 \\
&\quad - 2\langle \rho^n, v_2 \rangle + \gamma\tau\|v_2\|^2 \\
&\geq 2\|v_1\|^2 - \left(\|u^n\|^2 + \|v_1\|^2 \right) + 2\|v_{1x}\|^2 - \left(\|u^n_x\|^2 + \|v_{1x}\|^2 \right) + \upsilon\tau\|v_{1x}\|^2 \\
&\quad + 2\|v_2\|^2 - \left(\|\rho^n\|^2 + \|v_2\|^2 \right) + \gamma\tau\|v_2\|^2 \\
&= \|v_1\|^2 + \|v_2\|^2 + \|v_{1x}\|^2 - \|u^n\|^2 - \|u^n_x\|^2 - \|\rho^n\|^2 + \upsilon\tau\|v_{1x}\|^2 + \gamma\tau\|v_2\|^2 \\
&\geq \|v\|^2 - \left(\|u^n\|^2 + \|u^n_x\|^2 + \|\rho^n\|^2 \right).
\end{aligned}
\tag{3.3}
$$

Hence it is obvious that $< g(v), v > 0$ for all $v \in Z_\Delta$ with $\|v\|^2 = (\|u^n\|^2 + \|u^n_x\|^2 + \|\rho^n\|^2) + 1$. It follows from Lemma 3.1 that there exists $v^* \in Z_\Delta$ such that $g(v^*) = 0$. If we take $u^{n+1} = 2v_1 - u^n$, $\rho^{n+1} = 2v_2^* - \rho^n$, then (u^{n+1}, ρ^{n+1}) satisfies (2.2)–(2.5). This completes the proof. □

Next we show that the difference scheme (2.2)–(2.5) is convergent and stable.

Let $v(x, t)$ and $\o(x, t)$ be the solution of problem (1.4)–(1.6), that is, $v_j^n = u(x_j, t_n)$, $\o_j^n = \rho(x_j, t_n)$, then the truncation of the difference scheme (2.2)–(2.5) is

$$r_j^n = \left(v_j^n \right)_t - \left(v_j^n \right)_{x\overline{x}t} + \left(\o_j^{n+1/2} \right)_{\hat{x}} - \upsilon\left(v_j^{n+1/2} \right)_{x\overline{x}} + \frac{1}{3}\left(v_{j+1}^{n+1/2} + v_j^{n+1/2} + v_{j-1}^{n+1/2} \right)\left(v_j^{n+1/2} \right)_{\hat{x}}, \tag{3.4}$$

$$s_j^n = \left(\o_j^n \right)_t + \left(v_j^{n+1/2} \right)_{\hat{x}} + \gamma\o_j^{n+1/2}. \tag{3.5}$$

Making use of Taylor expansion, we know that $|r_j^n| + |s_j^n| = O(\tau^2 + h^2)$ hold if $h, \tau \to 0$.

Lemma 3.3. *Suppose that $u_0 \in H^1$, $\rho_0 \in L_2$, the solution of (1.4)–(1.6) satisfies $\|u\|_{L_2} \leq C$, $\|u_x\|_{L_2} \leq C$, $\|\rho\|_{L_2} \leq C$ and $\|u\|_{L_\infty} \leq C$.*

Proof. See Lemma 1.1 in [21].   □

Theorem 3.4. *Suppose $u_0 \in H^1$, $\rho_0 \in L_2$, then the solution u^n and ρ^n to the difference scheme (2.2)–(2.5) converges to the solution of problem (1.4)–(1.6), and the rate of convergence is $O(\tau^2 + h^2)$.*

Proof. Subtracting (2.2) from (3.4) and subtracting (2.3) from (3.5), letting $e_j^n = v_j^n - u_j^n$, $\eta_j^n = \phi_j^n - \rho_j^n$, we have

$$r_j^n = \left(e_j^n\right)_t - \left(e_j^n\right)_{x\overline{x}t} + \left(\eta_j^{n+1/2}\right)_{\widehat{x}} - v\left(e_j^{n+1/2}\right)_{x\overline{x}} + Q_j^{n+1/2}, \tag{3.6}$$

$$s_j^n = \left(\eta_j^n\right)_t + \left(e_j^{n+1/2}\right)_{\widehat{x}} + \gamma\eta_j^{n+1/2}, \tag{3.7}$$

where

$$Q_j^{n+1/2} = \frac{1}{3}\left(v_{j+1}^{n+1/2} + v_j^{n+1/2} + v_{j-1}^{n+1/2}\right) \cdot \left(v_j^{n+1/2}\right)_{\widehat{x}} - \frac{1}{3}\left(u_{j+1}^{n+1/2} + u_j^{n+1/2} + u_{j-1}^{n+1/2}\right) \cdot \left(u_j^{n+1/2}\right)_{\widehat{x}}. \tag{3.8}$$

Computing the inner product of (3.6) with $2e^{n+1/2}$ we get

$$\left\langle r^n, 2e^{n+1/2}\right\rangle = \frac{1}{\tau}\left(\left\|e^{n+1}\right\|^2 - \|e^n\|^2\right) + \frac{1}{\tau}\left(\left\|e_x^{n+1}\right\|^2 - \|e_x^n\|^2\right) + 2v\left\|e_x^{n+1/2}\right\|^2$$

$$+ \left\langle \eta_{\widehat{x}}^{n+1/2}, 2e^{n+1/2}\right\rangle + \left\langle Q^{n+1/2}, 2e^{n+1/2}\right\rangle. \tag{3.9}$$

Similarly to (2.11), we have

$$\left\langle \eta_{\widehat{x}}^{n+1/2}, 2e^{n+1/2}\right\rangle = -2\left\langle e_{\widehat{x}}^{n+1/2}, \eta^{n+1/2}\right\rangle. \tag{3.10}$$

Then (3.9) can be changed to

$$\left(\left\|e^{n+1}\right\|^2 - \|e^n\|^2\right) + \left(\left\|e_x^{n+1}\right\|^2 - \|e_x^n\|^2\right) - 2\tau\left\langle e_{\widehat{x}}^{n+1/2}, \eta^{n+1/2}\right\rangle$$

$$= -2v\tau\left\|e_x^{n+1/2}\right\|^2 + \tau\left\langle r^n, 2e^{n+1/2}\right\rangle + \tau\left\langle -Q^{n+1/2}, 2e^{n+1/2}\right\rangle. \tag{3.11}$$

It follow from Lemma 3.3, Theorems 2.4 and 2.5 that

$$\left|v_{j+1}^{n+1/2} + v_j^{n+1/2} + v_{j-1}^{n+1/2}\right| \leq C, \quad \left|\left(u_j^{n+1/2}\right)\right| \leq C, \quad (j = 0, 1, 2, \ldots, J). \tag{3.12}$$

Since

$$\left\langle -Q^{n+1/2}, 2e^{n+1/2} \right\rangle = -\frac{2}{3}h\sum_{j=1}^{J-1}\left(v_{j+1}^{n+1/2} + v_j^{n+1/2} + v_{j-1}^{n+1/2}\right)\left(v_j^{n+1/2}\right)_{\widehat{x}}e_j^{n+1/2}$$

$$+\frac{2}{3}h\sum_{j=1}^{J-1}\left(u_{j+1}^{n+1/2} + u_j^{n+1/2} + u_{j-1}^{n+1/2}\right)\left(u_j^{n+1/2}\right)_{\widehat{x}}e_j^{n+1/2}$$

$$= -\frac{2}{3}h\sum_{j=1}^{J-1}\left(v_{j+1}^{n+1/2} + v_j^{n+1/2} + v_{j-1}^{n+1/2}\right)\left(e_j^{n+1/2}\right)_{\widehat{x}}e_j^{n+1/2} \tag{3.13}$$

$$-\frac{2}{3}h\sum_{j=1}^{J-1}\left(e_{j+1}^{n+1/2} + e_j^{n+1/2} + e_{j-1}^{n+1/2}\right)\left(u_j^{n+1/2}\right)_{\widehat{x}}e_j^{n+1/2},$$

and the Schwarz inequality, we obtain

$$\left\langle -Q^{n+1/2}, 2e^{n+1/2} \right\rangle \leq \frac{2}{3}Ch\sum_{j=1}^{J-1}\left|\left(e_j^{n+1/2}\right)_{\widehat{x}}\right| \cdot \left|e_j^{n+1/2}\right|$$

$$+\frac{2}{3}Ch\sum_{j=1}^{J-1}\left(\left|e_{j+1}^{n+1/2}\right| + \left|e_j^{n+1/2}\right| + \left|e_{j-1}^{n+1/2}\right|\right)\left|e_j^{n+1/2}\right|$$

$$\leq C\left(\left\|e^{n+1/2}\right\|^2 + \left\|e_x^{n+1/2}\right\|^2\right) \leq C\left(\left\|e^{n+1}\right\|^2 + \|e^n\|^2 + \left\|e_x^{n+1}\right\|^2 + \|e_x^n\|^2\right). \tag{3.14}$$

According to

$$\left\langle r^n, 2e^{n+1/2} \right\rangle = \left\langle r^n, e^{n+1} + e^n \right\rangle \leq \|r^n\|^2 + \frac{1}{2}\left[\left\|e^{n+1}\right\|^2 + \|e^n\|^2\right]. \tag{3.15}$$

It follows from (3.14), (3.15), and (3.11) that

$$\left(\left\|e^{n+1}\right\|^2 - \|e^n\|^2\right) + \left(\left\|e_x^{n+1}\right\|^2 - \|e_x^n\|^2\right) - 2\tau\left\langle e_{\widehat{x}}^{n+1/2}, \eta^{n+1/2} \right\rangle$$

$$\leq C\tau\left[\left\|e^{n+1}\right\|^2 + \|e^n\|^2 + \left\|e_x^{n+1}\right\|^2 + \|e_x^n\|^2\right] + \tau\|r^n\|^2. \tag{3.16}$$

Computing the inner product of (3.7) with $2\eta^{n+1/2}$, we obtain

$$\left(\|\eta^{n+1}\|^2 - \|\eta^n\|^2\right) + 2\tau\left\langle e_{\widehat{x}}^{n+1/2}, \eta^{n+1/2} \right\rangle = \tau\langle s^n, 2\eta^{n+1/2}\rangle - 2\gamma\tau\left\|\eta^{n+1/2}\right\|^2$$

$$\leq C\tau\left[\left\|\eta^{n+1}\right\|^2 + \|\eta^n\|^2\right] + \tau\|s^n\|^2. \tag{3.17}$$

Adding (3.17) to (3.16) we have

$$
\left(\left\| e^{n+1} \right\|^2 - \| e^n \|^2 \right) + \left(\left\| e_x^{n+1} \right\|^2 - \| e_x^n \|^2 \right) + \left\| \eta^{n+1} \right\|^2 - \left\| \eta^n \right\|^2
$$

$$
\leq \tau \| r^n \|^2 + \tau \| s^n \|^2 + C\tau \left[\left\| e^{n+1} \right\|^2 + \| e^n \|^2 + \left\| e_x^{n+1} \right\|^2 + \| e_x^n \|^2 + \left\| \eta^{n+1} \right\|^2 + \| \eta^n \|^2 \right].
$$

(3.18)

Let $B^n = \| e^n \|^2 + \| e_x^n \|^2 + \| \eta^n \|^2$, we get

$$
B^{n+1} - B^n \leq \tau \| r^n \|^2 + \tau \| s^n \|^2 + C\tau \left(B^{n+1} + B^n \right).
$$

(3.19)

If τ is sufficiently small which satisfies $1 - C\tau > 0$, then

$$
B^{n+1} - B^n \leq C\tau B^n + C\tau \| r^n \|^2 + C\tau \| s^n \|^2.
$$

(3.20)

Summing up (3.20) from 0 to $n - 1$, we have

$$
B^n \leq B^0 + C\tau \sum_{l=0}^{n-1} \left\| r^l \right\|^2 + C\tau \sum_{l=0}^{n-1} \left\| s^l \right\|^2 + C\tau \sum_{l=0}^{n-1} B^l.
$$

(3.21)

Since

$$
\tau \sum_{l=0}^{n-1} \left\| r^l \right\|^2 \leq n\tau \max_{0 \leq l \leq n-1} \left\| r^l \right\|^2 \leq T \cdot O\left(\tau^2 + h^2 \right)^2,
$$

$$
\tau \sum_{l=0}^{n-1} \left\| s^l \right\|^2 \leq n\tau \max_{0 \leq l \leq n-1} \left\| s^l \right\|^2 \leq T \cdot O\left(\tau^2 + h^2 \right)^2,
$$

(3.22)

and $B^0 = O(\tau^2 + h^2)^2$, we obtain

$$
B^n \leq O\left(\tau^2 + h^2 \right)^2 + C\tau \sum_{l=0}^{n-1} B^l.
$$

(3.23)

From Lemma 2.3, we get

$$
B^n \leq O\left(\tau^2 + h^2 \right)^2,
$$

(3.24)

which implies

$$
\| e^n \| \leq O\left(\tau^2 + h^2 \right), \qquad \| e_x^n \| \leq O\left(\tau^2 + h^2 \right), \qquad \| \eta^n \| \leq O\left(\tau^2 + h^2 \right).
$$

(3.25)

Table 1: The error comparison in the sense of l_∞ at various time step when $v = \gamma = 0.2$.

		$\tau = h = 0.1$		$\tau = h = 0.05$		$\tau = h = 0.025$	
		Scheme I	Scheme II	Scheme I	Scheme II	Scheme I	Scheme II
u	$t = 1$	2.051828e − 3	3.031228e − 3	5.086997e − 4	7.715042e − 4	1.212062e − 4	1.963616e − 4
	$t = 2$	3.658861e − 3	4.444161e − 3	9.062598e − 4	1.130545e − 3	2.159826e − 4	2.891308e − 4
	$t = 3$	4.659523e − 3	5.248700e − 3	1.154171e − 3	1.332236e − 3	2.749807e − 4	3.363167e − 4
	$t = 4$	5.230463e − 3	5.817171e − 3	1.295681e − 3	1.476326e − 3	3.087002e − 4	3.736609e − 4
	$t = 5$	5.509947e − 3	6.570922e − 3	1.365011e − 3	1.660745e − 3	3.252241e − 4	4.213116e − 4
ρ	$t = 1$	1.672623e − 3	1.981718e − 3	4.146233e − 4	5.009105e − 4	9.882705e − 5	1.274281e − 4
	$t = 2$	2.775247e − 3	3.231087e − 3	6.880969e − 4	8.110353e − 4	1.640128e − 4	2.074091e − 4
	$t = 3$	3.619022e − 3	4.495888e − 3	8.971716e − 4	1.144555e − 3	2.138326e − 4	2.912403e − 4
	$t = 4$	4.150387e − 3	5.169257e − 3	1.028962e − 3	1.314334e − 3	2.452495e − 4	3.329702e − 4
	$t = 5$	4.434692e − 3	5.792717e − 3	1.100089e − 3	1.466326e − 3	2.621296e − 4	3.716609e − 4

Using Lemma 2.2, we get

$$\|e^n\|_\infty \leq O\left(\tau^2 + h^2\right). \tag{3.26}$$

$\square$

Similarly to Theorem 3.4, we can prove the results as follows.

Theorem 3.5. *Under the conditions of Theorem 3.4, the solution u^n and ρ^n of (2.2)–(2.5) is stable in the senses of norm $\|\cdot\|_\infty$ and $\|\cdot\|_{L^2}$, respectively.*

Theorem 3.6. *The solution u^n of (2.2)–(2.5) is unique.*

4. Numerical Simulations

The difference scheme (2.2)–(2.5) is a nonlinear system about u_j^{n+1} that can be easily solved by Newton iterative algorithm. When $t = 0$, the damping does not effect and the dissipative term will not appear. So the initial conditions (1.4)–(1.6) are same as those of (1.1):

$$u_0(x) = \frac{5}{2}\mathrm{sech}^2\frac{\sqrt{5}}{6}x, \quad \rho_0(x) = \frac{5}{3}\mathrm{sech}^2\frac{\sqrt{5}}{6}x, \quad (v = 1.5). \tag{4.1}$$

Let $x_L = -20$, $x_R = 20$, $T = 5.0$. Since we do not know the exact solution of (1.4), an error estimates method in [23] is used: A comparison between the numerical solutions on a coarse mesh and those on a refine mesh is made. We consider the solution on mesh $\tau = h = 1/160$ as the reference solution. We denote the C-N scheme in this paper as Scheme I and the difference scheme in [21] as Scheme II. In Tables 1 and 2 we give the ratios in the sense of l_∞ at various time step when $v = \gamma = 0.2$ and $v = \gamma = 0.5$, respectively.

In Tables 3 and 4 we verify the second convergence of the scheme I using the method in [25] when $v = \gamma = 0.2$ and $v = \gamma = 0.5$, respectively.

When $v = \gamma = 0.2$ and $v = \gamma = 0.5$, a wave figure comparison of u and ρ at various time step with $\tau = h = 0.05$ is as follow: (see Figures 1, 2, 3, and 4).

Table 2: The error comparison in the sense of l_∞ at various time step when $\upsilon = \gamma = 0.5$.

		$\tau = h = 0.1$		$\tau = h = 0.05$		$\tau = h = 0.025$	
		Scheme I	Scheme II	Scheme I	Scheme II	Scheme I	Scheme II
u	$t = 1$	1.795244e − 3	2.268843e − 3	4.445000e − 4	5.685554e − 4	1.059029e − 4	1.454071e − 4
	$t = 2$	2.544783e − 3	3.102494e − 3	6.298484e − 4	7.925500e − 4	1.500033e − 4	2.032788e − 4
	$t = 3$	3.422326e − 3	3.705244e − 3	8.474794e − 4	9.482211e − 4	2.018808e − 4	2.430733e − 4
	$t = 4$	3.779389e − 3	4.207447e − 3	9.355479e − 4	1.079082e − 3	2.223954e − 4	2.799107e − 4
	$t = 5$	4.151289e − 3	4.729322e − 3	1.027876e − 3	1.216704e − 3	2.453155e − 4	3.162708e − 4
ρ	$t = 1$	1.303459e − 3	1.963695e − 3	3.234571e − 4	5.083883e − 4	7.706741e − 5	1.294953e − 4
	$t = 2$	1.868933e − 3	2.928023e − 3	4.633380e − 4	7.506240e − 4	1.104061e − 4	1.907795e − 4
	$t = 3$	2.462890e − 3	3.453731e − 3	6.110918e − 4	8.823517e − 4	1.453867e − 4	2.266268e − 4
	$t = 4$	3.016856e − 3	4.169390e − 3	7.518839e − 4	1.066308e − 3	1.813333e − 4	2.721379e − 4
	$t = 5$	3.446674e − 3	4.513283e − 3	8.743460e − 4	1.141807e − 3	2.126268e − 4	2.918805e − 4

Table 3: The verification of the second convergence when $\upsilon = \gamma = 0.2$.

	$\|e^n(h,\tau)\|_\infty / \|e^{2n}(h/2,\tau/2)\|_\infty$			$\|\eta^n(h,\tau)\|_\infty / \|\eta^{2n}(h/2,\tau/2)\|_\infty$		
	$\tau = h = 0.1$	$\tau = h = 0.05$	$\tau = h = 0.025$	$\tau = h = 0.1$	$\tau = h = 0.05$	$\tau = h = 0.025$
$t = 1$	—	4.03347	4.19698	—	4.03408	4.19544
$t = 2$	—	4.03732	4.19596	—	4.03322	4.19539
$t = 3$	—	4.03711	4.19728	—	4.03381	4.19567
$t = 4$	—	4.03684	4.19721	—	4.03357	4.19557
$t = 5$	—	4.03656	4.19714	—	4.03121	4.19674

Table 4: The verification of the second convergence when $\upsilon = \gamma = 0.5$.

	$\|e^n(h,\tau)\|_\infty / \|e^{2n}(h/2,\tau/2)\|_\infty$			$\|\eta^n(h,\tau)\|_\infty / \|\eta^{2n}(h/2,\tau/2)\|_\infty$		
	$\tau = h = 0.1$	$\tau = h = 0.05$	$\tau = h = 0.025$	$\tau = h = 0.1$	$\tau = h = 0.05$	$\tau = h = 0.025$
$t = 1$	—	4.03879	4.19724	—	4.02977	4.19707
$t = 2$	—	4.04031	4.19807	—	4.03326	4.19667
$t = 3$	—	4.03824	4.19792	—	4.03031	4.20322
$t = 4$	—	4.03976	4.20669	—	4.01240	4.14642
$t = 5$	—	4.03871	4.19002	—	3.94200	4.11211

5. Conclusion

In this paper, we propose Crank-Nicolson nonlinear-implicit finite difference scheme of the initial-boundary problem of dissipative symmetric regularized long wave equations with damping term. The two-levels finite difference scheme is of second-order convergence and unconditionally stable, which can start by itself. From the Tables 1 and 2 we conclude that the C-N scheme is more efficient than the Scheme 2 in [21]. From the Tables 3 and 4 we conclude that the C-N scheme is of second-order convergence obviously. Figures 1–4 show that the height of wave crest is more and more low with time elapsing due to the effect of damping term and dissipative term and when υ, γ become bigger the droop of the height of wave crest is faster.
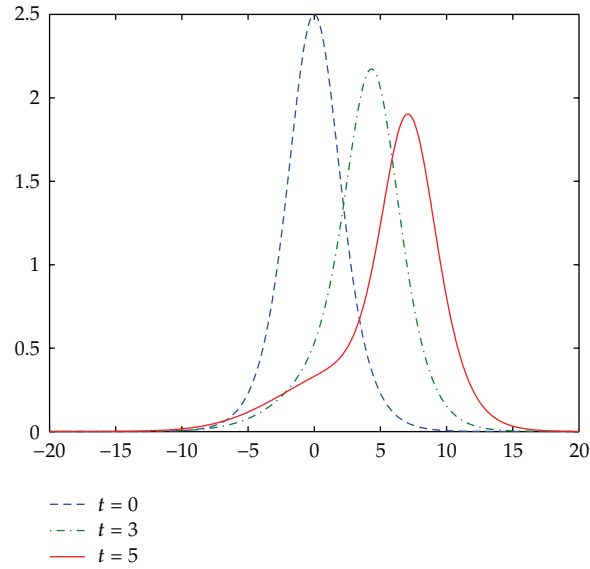
Figure 1: When $v = \gamma = 0.2$, the wave graph of u at various time.

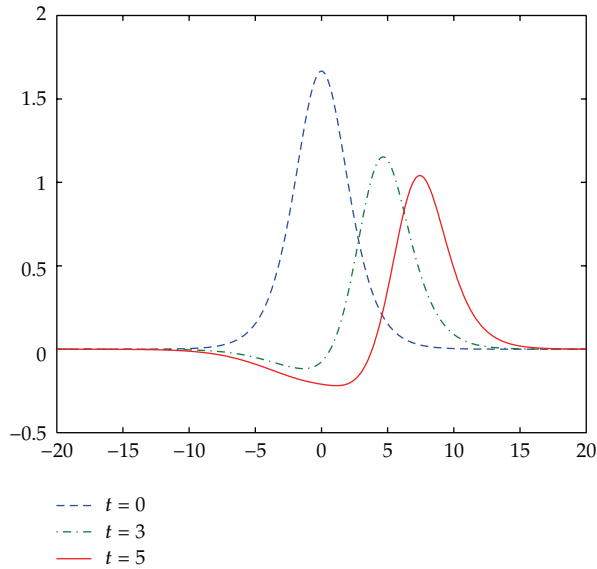**Figure 2:** When $v = \gamma = 0.2$, the wave graph of ρ at various time.
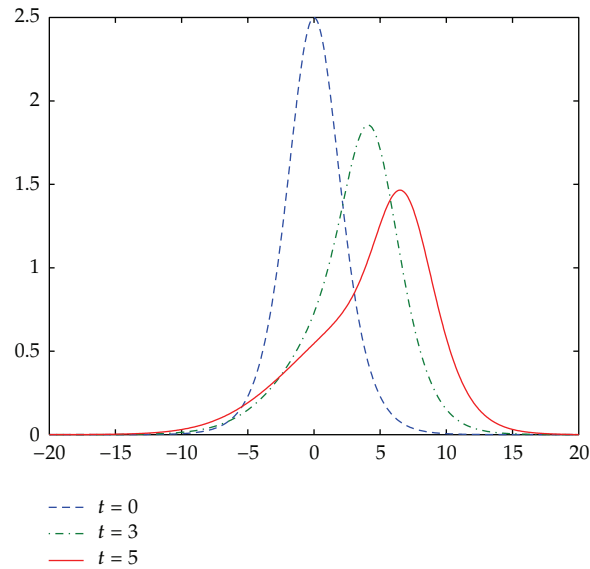
Acknowledgments

Figure 3: When $v = \gamma = 0.5$, the wave graph of u at various time.
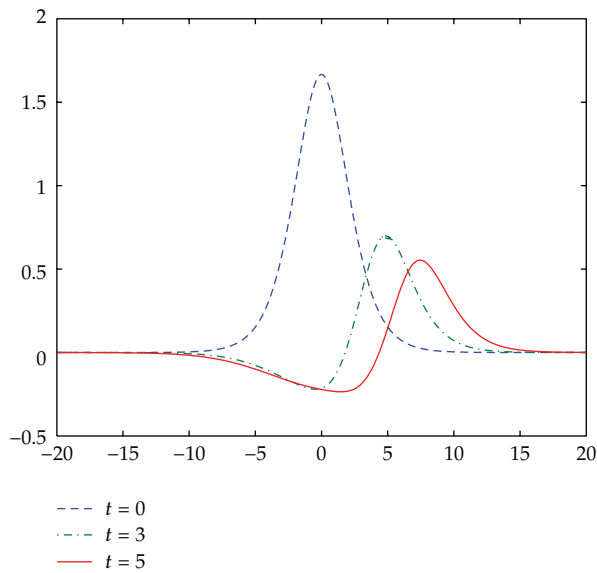

Figure 4: When $v = \gamma = 0.5$, the wave graph of ρ at various time.

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