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Research Article

On the Exact Analytical Solutions of Certain Lambert Transcendental Equations

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The subject matter of this paper is the Special Trans Functions Theory (STFT) in finding the exact analytical closed form solutions (the new special tran functions) of certain Lambert transcendental equations. Note that these Lambert transcendental equations frequently appear in applied physics and engineering domain. The structure of the theoretical derivations, proofs, and numerical results confirms validity and basic principles of the STFT. Undoubtedly, the proposed exact analytical approach implies the qualitative improvements of the conventional methods and techniques.

1. Introduction

The subject of the theoretical analysis presented here is a broad family of Lambert transcendental equations of the form

$$\Psi(\zeta)^{\alpha(\zeta)} - \Psi(\zeta)^{\beta(\zeta)} = \lambda(\zeta)\Psi(\zeta)^{\alpha(\zeta)+\beta(\zeta)}, \tag{1.1}$$

where $\lambda(\zeta) = (\alpha(\zeta) - \beta(\zeta))\vartheta(\zeta)$, $\Psi(\zeta) \in R$, $\alpha(\zeta) \in R^+$, $\beta(\zeta) \in R^+$, $\vartheta(\zeta) \in R^+$, and where we will restrict ourselves to the one-dimensional case, when α, β, ϑ, and Ψ are functions of the arbitrary variable ζ in the real domain.

The present paper considers possibilities for finding some exact analytical closed form solutions of the transcendental Lambert equations (1.1), by the Special Trans Functions Theory (S. M. Perovich) [1–13], since the former methods [14, 15] et al. cannot express solutions in the exact analytical closed form. Equation (1.1) has been firstly proposed by Lambert in 1758 [15] and studied by Euler. In these classical research works Lambert obtained analytical solutions for the special case of (1.1). Namely, (1.1) was reduced to the form $\ln(\Psi) = \vartheta\Psi^{\beta}$, for $\alpha \to \beta$, and solved by Lambert's W functions [14, 15]. It is to be pointed that this special case of (1.1) was solved exactly in an analytical closed form by the Special Trans Functions Theory [1–3, 10]. The comparisons between Lambert's W functions and Trans functions are given in some detail in [1, 10]. Though, the transcendental Lambert equation (1.1) has not been solved yet, there is a motivation for solving it analytically, by the STF Theory.

The needs for the exact analytical analysis of transcendental Lambert equations (1.1) arise in variety of different disciplines. In other words, the problem of obtaining the results from an analytical study of certain Lambert transcendental equations (1.1) is the important one, in both the theoretical sense and the practical purpose [1], since there is a broad class of the problems appearing in the applied physics and engineering domain being mathematically described by (1.1).

The simple mathematical analysis of (1.1) implies that it has nontrivial, real solutions: $\Psi_{<}$ (for $\Psi < (\alpha/\beta)^{1/(\alpha-\beta)}$), and $\Psi_{>}$ (for $\Psi > (\alpha/\beta)^{1/(\alpha-\beta)}$)). The condition for solutions existence in real domain takes the form $\lambda < (b-1)^{b-1}/b^{b}$, where $b = \alpha/(\alpha-\beta)$.

Remark 1.1. The essential for the Special Trans Functions Theory in solving transcendental equations within the real domain is that it always gives the least solution in terms of absolute values. This is general structural characteristic of the STFT, caused, most probably, by convergence dynamics [1, 4, 13].

We remark that the Lambert transcendental equations (1.1) are applicable, for instance, within families of the RC diode nonlinear circuits theoretical analysis [1, 4]. Analogically, the nonlinear system of equations for electrical parameters determination for different unknown materials (fluid and solid) is reduced to (1.1) [1, 16]. Within the previous analysis the observed Lambert transcendental equation has been solved by the application of some interative procedures based on the Special Trans Function Theory. Upon the results obtained in this article, it becomes possible to solve the mentioned problems from the engineering domain-exactly, that is, in the analytically closed form.

2. The Solution of Lambert Transcendental Equation (1.1) when $\Psi < (\alpha/\beta)^{1/(\alpha-\beta)}$

In this section, we attempt to find an exact analytical closed form solution of Lambert equation (1.1), $\Psi_{<}$, when $\Psi < (\alpha/\beta)^{1/(\alpha-\beta)}$, for arbitrary, real, and positive values of α, β, and ϑ.

Theorem 2.1. *Lambert transcendental equation* (1.1) *for* $\lambda < (b-1)^{b-1}/b^{b}$, *where* $b = \alpha/(\alpha-\beta)$, *and for* $\Psi < (\alpha/\beta)^{1/(\alpha-\beta)}$ *has an exact analytical closed form solution* $\Psi = \Psi_{<} = \mathrm{tran}_{L<}(\alpha,\beta,\vartheta)$, *where* $\mathrm{tran}_{L<}(\alpha,\beta,\vartheta)$ *is a new special trans function defined as*

$$\mathrm{tran}_{L<}(\alpha,\beta,\vartheta) = \lim_{x\to\infty}\left(\frac{\Psi_{L<}(x-1,\alpha,\beta,\vartheta)-\sigma_{<}}{\Psi_{L<}(x,\alpha,\beta,\vartheta)-\sigma_{<}}\right), \quad \sigma_{<} = -\frac{\Psi_{o<}}{\lambda}, \tag{2.1}$$

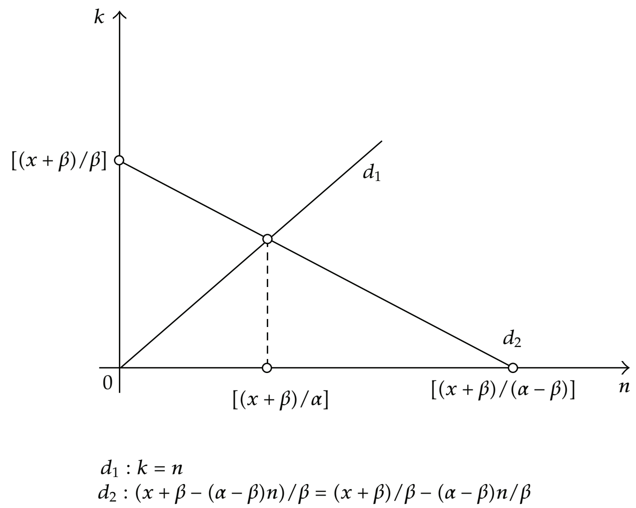


Figure 1: Graphical presentation for sumlimit determination in (2.4).

where

$$\alpha = \alpha(\zeta), \qquad \beta = \beta(\zeta), \qquad \vartheta = \vartheta(\zeta), \qquad \lambda = \lambda(\zeta). \tag{2.2}$$

$\Psi_{o<}$ *denotes a positive constant.* $\Psi_{L<}(x, \alpha, \beta, \vartheta)$ *is a function defined as*

$$\Psi_{L<}(x, \alpha, \beta, \vartheta) = \Psi_{o<} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{k+1} \lambda^k n!}{(n-k)!k!} H(x + \beta - (\alpha - \beta)n - \beta k), \tag{2.3}$$

where $H(x + \beta - (\alpha - \beta)n - \beta k)$ *is a Heaviside's unit function. Or, after sumlimit determination, the latter expression for* $\Psi_{L<}(x, \alpha, \beta, \vartheta)$ *takes the form*

$$\Psi_{L<}(x, \alpha, \beta, \vartheta) = -\Psi_{o<} \left(\sum_{n=0}^{[(x+\beta)/\alpha]} (1-\lambda)^n - \sum_{n=[(x+\beta)/\alpha]+1}^{[(x+\beta)/(\alpha-\beta)]} \sum_{k=0}^{[(x+\beta-(\alpha-\beta)n)/\beta]} \frac{(-1)^{k+1} \lambda^k n!}{(n-k)!k!} \right), \tag{2.4}$$

where $[x]$ *denotes the greatest integer less or equal to* x*.*

Remark 2.2. The sumlimit determination implies the following reason: from Heaviside's unit function, we have $x + \beta - (\alpha - \beta)n - \beta k > 0$ or, $k = (x + \beta)/\beta - ((\alpha - \beta)/\beta)n$. Thus, the graphical presentation takes the form (Figure 1).

Proof. The transcendental Lambert equation (1.1) can be identified with an equation for identification (EQID) of type

$$\Psi_{L<}(x - \alpha) - \Psi_{L<}(x - \beta) - \lambda \Psi_{L<}(x - \alpha - \beta) = \Psi_{o<}, \tag{2.5}$$

where $\Psi_{L<}(x) = \Psi_{L<}(x,\alpha(\zeta),\beta(\zeta),\vartheta(\zeta))$ is an arbitrary real function, for $x > 0$, and $\Psi_{L<}(x,\alpha(\zeta),\beta(\zeta),\vartheta(\zeta)) = 0$, for $x < 0$. Equation (2.5) will be solved in the set of originals of the Laplace Transform. Thus, after taking the Laplace Transform, (2.5) takes the following form

$$P_{L<}(s,\alpha,\beta,\vartheta)\left[e^{-\alpha s}-e^{-\beta s}-\lambda e^{-(\alpha+\beta)s}\right]=\frac{\Psi_{o<}}{s}, \tag{2.6}$$

where $P_{L<}(s,\alpha,\beta,\vartheta) = L[\Psi_{L<}(x,\alpha,\beta,\vartheta)]$. Therefore,

$$P_{L<}(s,\alpha,\beta,\vartheta)e^{-\beta s}=-\frac{\Psi_{o<}}{s\left(1-e^{-s(\alpha-\beta)}\left(1-\lambda e^{-\beta s}\right)\right)} \tag{2.7}$$

since $\alpha > \beta$. By expanding, we get

$$P_{L<}(s,\alpha,\beta,\vartheta)e^{-\beta s}=-\frac{\Psi_{o<}}{s}\sum_{n=0}^{\infty}e^{-ns(\alpha-\beta)}\left(1-\lambda e^{-\beta s}\right)^{n}. \tag{2.8}$$

The series (2.8) converges for $|e^{-s(\alpha-\beta)}-\lambda e^{-\alpha s}| \ll 1$. Now, we can invert term by term to obtain the original

$$\Psi_{L<}(x,\alpha,\beta,\vartheta)=\Psi_{o<}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{k+1}\lambda^{k}n!}{(n-k)!k!}H(x+\beta-(\alpha-\beta)n-\beta k). \tag{2.9}$$

Finally, applying the sumlimit determination, the above equation takes the form

$$\Psi_{L<}(x,\alpha,\beta,\vartheta)=-\Psi_{o<}\left(\sum_{n=0}^{[(x+\beta)/\alpha]}(1-\lambda)^{n}-\sum_{n=[(x+\beta)/\alpha]+1}^{[(x+\beta)/(\alpha-\beta)]}\sum_{k=0}^{[(x+\beta-(\alpha-\beta)n)/\beta]}\frac{(-1)^{k+1}\lambda^{k}n!}{(n-k)!k!}\right), \tag{2.10}$$

where $\Psi_{L<}(x,\alpha,\beta,\vartheta) = \Psi_{L<}(x,\alpha(\zeta),\beta(\zeta),\vartheta(\zeta))$. The functional series (2.10) is the unique analytical closed form solution of (2.5) according to Lerch's theorem. □

Next, we need the following lemma.

Lemma 2.3. *For any $x > \alpha(\zeta)$, the functional series* (2.9) *satisfies* (2.5).

Proof. When we substitute (2.9) into (2.5), the following is obtained:

$$\begin{aligned}&\sum_{n=0}^{n=\infty}\sum_{k=0}^{n}\frac{(-1)^{k+1}\lambda^k n!}{(n-k)!k!}H(x-\alpha+\beta-(\alpha-\beta)n-\beta k)\\&\quad-\sum_{n=0}^{n=\infty}\sum_{k=0}^{n}\frac{(-1)^{k+1}\lambda^k n!}{(n-k)!k!}H(x-(\alpha-\beta)n-\beta k)\\&\quad-\sum_{n=0}^{n=\infty}\sum_{k=0}^{n}\frac{(-1)^{k+1}\lambda^{k+1} n!}{(n-k)!k!}H(x-\alpha-(\alpha-\beta)n-\beta k)=1.\end{aligned}\tag{2.11}$$

After simple modification, the above equation takes the form:

$$\begin{aligned}&\sum_{n=0}^{n=\infty}\sum_{k=0}^{n}\frac{(-1)^{k+1}\lambda^k n!}{(n-k)!k!}H(x-(\alpha-\beta)(n+1)-\beta k)\\&\quad-\sum_{n=0}^{n=\infty}\sum_{k=0}^{n}\frac{(-1)^{k+1}\lambda^k n!}{(n-k)!k!}H(x-(\alpha-\beta)n-\beta k)\\&\quad-\sum_{n=0}^{n=\infty}\sum_{k=0}^{n}\frac{(-1)^{k+1}\lambda^{k+1} n!}{(n-k)!k!}H(x-(\alpha-\beta)(n+1)-\beta(1+k))=1,\end{aligned}\tag{2.12}$$

or the form

$$\begin{aligned}&\left(\sum_{n=1}^{n=\infty}\sum_{k=0}^{n-1}\frac{(-1)^{k+1}\lambda^k n!}{(n-k)!k!}-\sum_{n=0}^{n=\infty}\sum_{k=0}^{n}\frac{(-1)^{k+1}\lambda^k n!}{(n-k)!k!}-\sum_{n=1}^{n=\infty}\sum_{k=1}^{n}\frac{(-1)^{k+1}\lambda^{k+1}(n-1)!}{(n-k)!(k-1)!}\right)\\&\quad\cdot H(x-(\alpha-\beta)n-\beta k)=1.\end{aligned}\tag{2.13}$$

The latter expression can be written in the form

$$\begin{aligned}&\sum_{n=1}^{n=\infty}\sum_{k=1}^{n-1}\frac{(-1)^{k+1}\lambda^k(n-1)!}{(n-k-1)!(k-1)!}\left(\frac{1}{k}-\frac{n}{(n-k)k}+\frac{1}{(n-k)}\right)H(x-(\alpha-\beta)n-\beta k)\\&\quad+(-1)H(x-\alpha+\beta)-(-1)(H(x))+H(x-\alpha)=1\end{aligned}\tag{2.14}$$

since $(1/k)-n/(n-k)k+1/(n-k)=(n-k-n+k)/(n-k)k=0$, and $x>\alpha$.

Thus, we have finished our proof. □

2.1. On the Particular Solution to (2.5)

Intuitively, we can see that the particular solution of the form

$$\Psi_{L<P}(x,\alpha,\beta,\vartheta) = (\Psi_{<}(\zeta))^{-x} + \sigma_{<} \tag{2.15}$$

satisfies (2.5) under the condition that $\Psi_{<}(\zeta)$ satisfies (1.1). Namely, after substitution of (2.15) into (2.5), we have:

$$(\Psi_{<})^{-(x-\alpha)} + \sigma_{<} - (\Psi_{<})^{-(x-\beta)} - \sigma_{<} - \lambda(\Psi_{<})^{-(x-\alpha-\beta)} - \lambda\sigma_{<} = \Psi_{o<} \tag{2.16}$$

for

$$\Psi_{<}^{\alpha} - \Psi_{<}^{\beta} = \lambda\Psi^{\alpha+\beta} \tag{2.17}$$

and for $\sigma_{<} = -\Psi_{o<}/\lambda$. This means that the particular solution (2.15) satisfies (2.5) under condition that $\Psi_{<}$ (for $\Psi < (\alpha/\beta)^{1/(\alpha-\beta)}$) satisfies (1.1). Let us note, that it is beyond the scope of this paper to have indepth theoretical knowledge of the differential form of the particular (or asymptotic) solutions of the EQIDs in the Special Trans Functions Theory [1]. But, it is worth to be mentioned that EQIDs for the broad family of the transcendental Lambert equations of type (1.1) for asymptotic solutions have the functions of the form $\Psi_{<}(\zeta)^{-x} + \sigma_{<}$.

2.2. On the Genesis of the Special Tran Function, $\mathrm{tran}_{L<}(\alpha,\beta,\vartheta)$

Concerning the uniqueness of the inverse Laplace Transform (Lerch's Theorem) (2.10) is an unique analytical closed form solution of (2.5). On the other hand, (2.15) is a particular solution of (2.5). Easily, by (2.10) and (2.15), the following asymptotic relation appears

$$\lim_{x\to\infty}\left(\frac{\Psi_{L<}(x,\alpha,\beta,\vartheta)}{\Psi_{L<P}(x,\alpha,\beta,\vartheta)}\right) = 1. \tag{2.18}$$

It is based on the functional theory, convergence dynamics and, first of all, on the condition that particular solution (2.15) is an asymptotic solution of EQID. Accordingly, we have the following.

Lemma 2.4. *The solution* (2.15) *is an asymptotic solution of the EQID* (2.5).

Proof. For (2.9) as unique analytical closed form solution of EQID (2.5), for $(1-\lambda) < 1$, we have

$$\lim_{x\to\infty}\left(\Psi_{L<}(x,\alpha,\beta,\vartheta)\right) = \lim_{x\to\infty}\left(\Psi_{o<}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{k+1}\lambda^k n!}{(n-k)!k!}H(x+\beta-(\alpha-\beta)n-\beta k)\right)$$
$$= \left(\Psi_{o<}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{k+1}\lambda^k n!}{(n-k)!k!}\right) = -\Psi_{o<}\sum_{n=0}^{\infty}(1-\lambda)^n \tag{2.19}$$
$$= -\Psi_{o<}\frac{1}{1-(1-\lambda)} = -\frac{\Psi_{o<}}{\lambda}.$$

On the other hand, for particular solution (2.15), we have

$$\lim_{x\to\infty}\left(\Psi_{LP<}(x,\alpha,\beta,\vartheta)\right) = \lim_{x\to\infty}\left(\Psi_{<}(\zeta)^{-x}+\sigma_{<}\right) = \sigma_{<} = -\frac{\Psi_{o<}}{\lambda},$$
$$\lim_{x\to\infty}\left(\Psi_{L<}(x,\alpha,\beta,\vartheta)\right) = \lim_{x\to\infty}\left(\Psi_{LP<}(x,\alpha,\beta,\vartheta)\right) = -\frac{\Psi_{o<}}{\lambda}. \tag{2.20}$$

Thus, our proof is completed. □

According to the unique solution principle and Lemma 2.4, following appears

$$\lim_{x\to\infty}\left(\frac{\Psi_{L<}(x,\alpha,\beta,\vartheta)}{\Psi_{L<}(x+1,\alpha,\beta,\vartheta)}\right) = \lim_{x\to\infty}\left(\frac{\Psi_{L<P}(x,\alpha,\beta,\vartheta)}{\Psi_{L<P}(x+1,\alpha,\beta,\vartheta)}\right). \tag{2.21}$$

It is remarkable that in the STFT, the EQID, on the one hand, has the unique exact analytical closed form solution, and, on the other hand, it has the asymptotic (particular) solution. Thus, asymptotic equalizations (2.18) and (2.21) can be established.

Finally, from (2.15), (2.18) and (2.21), we have

$$\lim_{x\to\infty}\left(\frac{\Psi_{L<}(x,\alpha,\beta,\vartheta)-\sigma_{<}}{\Psi_{L<}(x+1,\alpha,\beta,\vartheta)-\sigma_{<}}\right) = \frac{(\Psi_{<}(\zeta))^{-x}}{(\Psi_{<}(\zeta))^{-(x+1)}} = \Psi_{<}(\zeta) = \Psi_{<} \tag{2.22}$$

and, $\Psi_{<} = \mathrm{tran}_{L<}(\alpha,\beta,\vartheta)$, where $\mathrm{tran}_{L<}(\alpha,\beta,\vartheta)$ is a new special tran function defined as

$$\mathrm{tran}_{L<}(\alpha,\beta,\vartheta) = \lim_{x\to\infty}\left(\frac{\Psi_{L<}(x,\alpha,\beta,\vartheta)-\sigma_{<}}{\Psi_{L<}(x+1,\alpha,\beta,\vartheta)-\sigma_{<}}\right). \tag{2.23}$$

More explicitly, after substitution (2.10) in (2.23), we have

$$\Psi_{<} = \mathrm{tran}_{L<}(\alpha, \beta, \vartheta)$$
$$= \lim_{x \to \infty} \left(\frac{-\sum_{n=0}^{[(x+\beta)/\alpha]} (1-\lambda)^n + \sum_{n=[(x+\beta)/\alpha]+1}^{[(x+\beta)/(\alpha-\beta)]} \sum_{k=0}^{[(x+\beta-(\alpha-\beta)n)/\beta]} \mathcal{M} + (1/\lambda)}{-\sum_{n=0}^{[(x+1+\beta)/\alpha]} (1-\lambda)^n + \sum_{n=[(x+1+\beta)/\alpha]+1}^{[(x+1+\beta)/(\alpha-\beta)]} \sum_{k=0}^{[(x+1+\beta-(\alpha-\beta)n)/\beta]} \mathcal{M} + (1/\lambda)} \right), \tag{2.24}$$

where $\mathcal{M}$ denotes $(((-1)^{k+1}\lambda^k n!)/(n-k)!k!)$ and $\Psi_{<} = \Psi_{<}(\zeta) = \mathrm{tran}_{L<}(\alpha(\zeta), \beta(\zeta), \vartheta(\zeta))$. The formulae (2.24) is a new special tran function, $\mathrm{tran}_{L<}(\alpha(\zeta), \beta(\zeta), \vartheta(\zeta))$ presented in some detail.

Note that an essential part of the Special Trans Functions Theory is the equalization (2.18) [1]. Due to the previous analysis it becomes clear that by applying unique solution principle [1–13], we obtain the equalization of type (2.18). Now, we have completed our proof.

3. Concerning a Formula for Practical Applicability of $\mathrm{tran}_{L<}(\alpha, \beta, \vartheta)$

For practical analysis and numerical calculations the formula (2.18) takes the following form:

$$\frac{\langle \Psi_{L<}(x, \alpha, \beta, \vartheta) \rangle_{[P_<]}}{\langle \Psi_{L<P}(x, \alpha, \beta, \vartheta) \rangle_{[P_<]}} = 1, \tag{3.1}$$

where, $\langle \Psi_{L<}(x, \alpha, \beta, \vartheta) \rangle_{[P_<]}$ denotes the numerical value of the function $\Psi_{L<}(x, \alpha, \beta, \vartheta)$ given with $[P_<]$ accurate digits. $[P_<]$ is defined as $[P_<] = [\ln(|G_<|)]$, where the error function $G_<$ is defined like $G_< = \Psi_<^{\alpha} - \Psi_<^{\beta} - \lambda \Psi_<^{\alpha+\beta}$.

So, we get from (2.15), (2.18), (2.21) and(3.1)

$$\left\langle \frac{\Psi_{L<}(x, \alpha, \beta, \vartheta) - \sigma_<}{\Psi_{L<}(x+1, \alpha, \beta, \vartheta) - \sigma_<} \right\rangle_{[P_<]} = \left\langle \frac{\Psi_{L<P}(x, \alpha, \beta, \vartheta) - \sigma_<}{\Psi_{L<P}(x+1, \alpha, \beta, \vartheta) - \sigma_<} \right\rangle_{[P_<]}, \tag{3.2}$$

$$\langle \Psi_< \rangle_{[P_<]} = \left\langle \frac{\Psi_{L<}(x, \alpha, \beta, \vartheta) - \sigma_<}{\Psi_{L<}(x+1, \alpha, \beta, \vartheta) - \sigma_<} \right\rangle_{[P_<]}. \tag{3.3}$$

More explicitly, for fixed variable x, the $\langle \Psi_< \rangle_{[P_<]}$ takes the form

$$\langle \Psi_< \rangle_{[P_<]} = \left\langle \frac{-\sum_{n=0}^{[(x+\beta)/\alpha]} (1-\lambda)^n + \sum_{n=[(x+\beta)/\alpha]+1}^{[(x+\beta)/(\alpha-\beta)]} \sum_{k=0}^{[(x+\beta-(\alpha-\beta)n)/\beta]} \mathcal{M} + (1/\lambda)}{-\sum_{n=0}^{[(x+1+\beta)/\alpha]} (1-\lambda)^n + \sum_{n=[(x+1+\beta)/\alpha]+1}^{[(x+1+\beta)/(\alpha-\beta)]} \sum_{k=0}^{[(x+1+\beta-(\alpha-\beta)n)/\beta]} \mathcal{M} + (1/\lambda)} \right\rangle_{[P_<]}. \tag{3.4}$$

From a theoretical point of view, the solution (3.4) for $\langle\Psi_<\rangle_{[P_<]}$ can be found with an arbitrary order of accuracy by taking an appropriate value of $[x]$. We remark that for these and other matters related to the numerical accuracy we refer to [1, 3, 10].

Let us note that after some sumlimit modifications, for numerical calculations, formulae (3.4) takes the form:

$$\langle\Psi_<\rangle_{[P_<]} = \left\langle \frac{-\sum_{n=0}^{[x/b]}(1-\lambda)^n + \sum_{n=[x/b]+1}^{[x]}\sum_{k=0}^{[(x-n)/(b-1)]}\mathcal{M} + (1/\lambda)}{-\sum_{n=0}^{[(x+1)/b]}(1-\lambda)^n + \sum_{n=[(x+1)/b]+1}^{[x+1]}\sum_{k=0}^{[(x+1-n)/(b-1)]}\mathcal{M} + (1/\lambda)} \right\rangle_{[P_<]}^{1/(\alpha-\beta)}, \quad (3.5)$$

where, $b = \alpha/(\alpha-\beta)$.

4. The Special Trans Functions Theory for the Transcendental Lambert Equation (1.1) for $\Psi > (\alpha/\beta)^{1/(\alpha-\beta)}$

In this section we attempt to find an exact analytical closed form solution of the transcendental Lambert equation (1.1) for $\Psi > (\alpha/\beta)^{1/(\alpha-\beta)}$, $\Psi_>$.

After simple modification, (1.1) takes the form

$$(\Psi_>)^{-\beta} - (\Psi_>)^{-\alpha} = \lambda, \quad (4.1)$$

or, the form

$$Z_> + \lambda = (Z_>)^{\beta/\alpha}, \quad (4.2)$$

where $Z_> = Z_>(\zeta) = (1/\Psi_>(\zeta))^{\alpha(\zeta)}$, and, finally,

$$\Psi_> = Z_>^{-1/\alpha}. \quad (4.3)$$

Theorem 4.1. *The transcendental Lambert equation* (4.2) *for* $\lambda < (b-1)^{b-1}/b^b$*, where* $b = \alpha/(\alpha-\beta)$*, and for* $\Psi > (\alpha/\beta)^{1/(\alpha-\beta)}$ *has an analytical closed form solution* $Z = Z_> = \mathrm{tran}_{ZL>}(\alpha,\beta,\lambda)$*, where* $\mathrm{tran}_{ZL>}(\alpha,\beta,\lambda)$ *is a new special tran function defined as*

$$\mathrm{tran}_{ZL>}(\alpha,\beta,\lambda) = \lim_{x\to\infty}\left(\frac{\Psi_{L>}(x-1,\alpha,\beta,\lambda)-\sigma_>}{\Psi_{L>}(x,\alpha,\beta,\lambda)-\sigma_>}\right), \qquad \sigma_> = \frac{\Psi_{o>}}{\lambda}, \quad (4.4)$$

where, by $\Psi_{o>}$ *we denote a positive constant which may be different in each equality.* $\Psi_{L>}(x,\alpha,\beta,\lambda)$ *is a function, defined as* $\Psi_{L>}(x,\alpha,\beta,\lambda) = (\Psi_{o>}/\lambda)\sum_{n=0}^{\infty}\sum_{k=0}^{n}((-1)^k n!/(n-k)!k!\lambda^n)H(x-n(\beta/\alpha)-k(1-\beta/\alpha))$*, or, it can be rewritten in the form*

$$\Psi_{L>}(x,\alpha,\beta,\lambda) = \frac{\Psi_{o>}}{\lambda}\sum_{n=[x]+1}^{[x(\alpha/\beta)]}\ \sum_{k=0}^{[(x-n(\beta/\alpha))/(1-\beta/\alpha)]}\frac{(-1)^k n!}{(n-k)!k!\lambda^n}. \quad (4.5)$$

Theorem 4.1 can be proved by similar argument as in Section 3. In fact, we only need to adapt the Special Trans Functions Theory routinely for new case of the transcendental Lambert equation of type (4.2).

Proof. The transcendental Lambert equation (4.2) can be identified with an EQID of type

$$\Psi_{L>}(x-1)+\lambda\Psi_{L>}(x)-\Psi_{L>}\left(x-\frac{\beta}{\alpha}\right)=\Psi_{o>}, \tag{4.6}$$

where $\Psi_{L>}(x,\alpha,\beta,\lambda)$ is an arbitrary real function for $x>0$, and $\Psi_{L>}(x,\alpha,\beta,\lambda)=0$ for $x<0$. We will solve (4.6) in the set of originals of the Laplace Transform. Thus, after taking the Laplace Transform, (4.6) takes the following form:

$$P_{L>}(s,\alpha,\beta,\lambda)\left(e^{-s}+\lambda-e^{-\beta s/\alpha}\right)=\frac{\Psi_{o>}}{s} \tag{4.7}$$

or,

$$P_{L>}(s,\alpha,\beta,\lambda)=\frac{\Psi_{o>}}{s\lambda\left(1-\left(e^{-\beta s/\alpha}/\lambda\right)\left(1-e^{-s(1-\beta/\alpha)}\right)\right)}. \tag{4.8}$$

By expanding, we get

$$P_{L>}(s,\alpha,\beta,\lambda)=\frac{\Psi_{o>}}{s\lambda}\sum_{n=0}^{\infty}\left(\frac{e^{-\beta s/\alpha}}{\lambda}\right)^{n}\sum_{k=0}^{n}\frac{n!(-1)^{k}e^{-ks(1-(\beta/\alpha))}}{(n-k)!k!}. \tag{4.9}$$

The series (4.8) converges for $|1-e^{-s(1-(\beta/\alpha))}|\ll 1$. Now, we can invert term by term to obtain the original

$$\Psi_{L>}(x,\alpha,\beta,\lambda)=\frac{\Psi_{o>}}{\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{k}n!}{\lambda^{n}(n-k)!k!}H\left(x-n\frac{\beta}{\alpha}-k\left(1-\frac{\beta}{\alpha}\right)\right). \tag{4.10}$$

Finally, by applying the sumlimit determination, the analytical solution to (4.6) can be written in the closed form representation

$$\Psi_{L>}(x,\alpha,\beta,\lambda)=\frac{\Psi_{o>}}{\lambda}\sum_{n=[x]+1}^{[x(\alpha/\beta)]}\ \sum_{k=0}^{[(x-n(\beta/\alpha))/(1-\beta/\alpha)]}\frac{(-1)^{k}n!}{(n-k)!k!\lambda^{n}}. \tag{4.11}$$

The functional series (4.11) is the unique analytical closed-form solution of (4.6) according to Lerchs theorem. □

Consequently, we have the following lemma.

Lemma 4.2. *For any $x>1$ the functional series* (4.10) *satisfies* (4.6).

Proof. After substituting (4.10) into (4.6), we obtain

$$\begin{aligned}
&\frac{1}{\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k n!}{\lambda^n (n-k)!k!}H\left(x-1-n\frac{\beta}{\alpha}-k\left(1-\frac{\beta}{\alpha}\right)\right)\\
&\quad-\frac{1}{\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k n!\lambda}{\lambda^n (n-k)!k!}H\left(x-n\frac{\beta}{\alpha}-k\left(1-\frac{\beta}{\alpha}\right)\right)\\
&\quad=\frac{1}{\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k n!}{\lambda^n (n-k)!k!}H\left(x-(n+1)\frac{\beta}{\alpha}-k\left(1-\frac{\beta}{\alpha}\right)\right)=1.
\end{aligned}\tag{4.12}$$

As a result of a simple modification, the above equation takes the form

$$\begin{aligned}
&\frac{1}{\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k n!}{\lambda^n (n-k)!k!}H\left(x-(n+1)\frac{\beta}{\alpha}-(k+1)\left(1-\frac{\beta}{\alpha}\right)\right)\\
&\quad+\frac{1}{\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k n!\lambda}{\lambda^n (n-k)!k!}H\left(x-n\frac{\beta}{\alpha}-k\left(1-\frac{\beta}{\alpha}\right)\right)\\
&\quad-\frac{1}{\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k n!}{\lambda^n (n-k)!k!}H\left(x-(n+1)\frac{\beta}{\alpha}-k\left(1-\frac{\beta}{\alpha}\right)\right)=1
\end{aligned}\tag{4.13}$$

or the form

$$\begin{aligned}
&\left(\frac{1}{\lambda}\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{(-1)^{k-1}(n-1)!}{\lambda^n (n-k)!(k-1)!}+\frac{1}{\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k n!\lambda}{\lambda^n (n-k)!k!}-\frac{1}{\lambda}\sum_{n=1}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k (n-1)!}{\lambda^n (n-1-k)!k!}\right)\\
&\quad\cdot H\left(x-n\frac{\beta}{\alpha}-k\left(1-\frac{\beta}{\alpha}\right)\right)=1.
\end{aligned}\tag{4.14}$$

Namely,

$$\begin{aligned}
&\left(\frac{1}{\lambda}\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{(-1)^{k-1}(n-1)!}{\lambda^n (n-k)!(k-1)!}+\frac{1}{\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k n!\lambda}{\lambda^n (n-k)!k!}-\frac{1}{\lambda}\sum_{n=1}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k (n-1)!}{\lambda^n (n-1-k)!k!}\right)\\
&\quad\cdot H\left(x-n\frac{\beta}{\alpha}-k\left(1-\frac{\beta}{\alpha}\right)\right)=\frac{1}{\lambda}\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{(-1)^{k-1}(n-1)!}{\lambda^{n-1}(n-1-k)!(k-1)!}\left(\frac{1}{(n-k)}-\frac{n}{(n-k)k}+\frac{1}{k}\right)\\
&\quad+H(x)+\frac{1}{\lambda}H\left(x-\frac{\beta}{\alpha}\right)-\frac{1}{\lambda}H\left(x-\frac{\beta}{\alpha}-1+\frac{\beta}{\alpha}\right)=1,
\end{aligned}\tag{4.15}$$

since

$$\left(\frac{1}{(n-k)} - \frac{n}{(n-k)k} + \frac{1}{k}\right) = \frac{k-n+n-k}{(n-k)k} = 0, \quad x > 1. \tag{4.16}$$

Thus, the proof is finished. □

4.1. *On the Particular Solution of* (4.6)

Intuitively, we can obtain that particular solution of the form

$$\Psi_{L>P}(x,\alpha,\beta,\lambda) = (Z_{>}(\zeta))^{-x} + \sigma_{>} \tag{4.17}$$

satisfies (4.6), under condition that $Z_{>}(\zeta)$ satisfies (4.2). Namely, after substitution of (4.17) into (4.6), we have

$$(Z_{>})^{-(x-1)} + \sigma_{>} - (Z_{>})^{-(x-(\beta/\alpha))} - \sigma_{<} + \lambda(Z_{>})^{-x} + \lambda\sigma_{>} = \Psi_{o>} \tag{4.18}$$

for

$$Z_{>} + \lambda = \lambda Z_{>}^{\beta/\alpha} \tag{4.19}$$

and, for $\sigma_{>} = \Psi_{o>}/\lambda$. In other words, the particular solution (4.17) satisfies (4.6), under the condition that $Z_{>}$ satisfies (4.2). The choice of the EQID is determined in a way that after substitution of the particular solution in EQID, we obtain the starting Lambert transcendental equation.

4.2. *On the Genesis of the Special Tran Function,* $\mathrm{tran}_{L>}(\alpha,\beta,\vartheta)$

Concerning the uniqueness of the inverse Laplace Transform (Lerch's Theorem) (4.11) is an unique analytical closed form solution of (4.6). On the other hand, (4.17) is a particular solution of (4.6). It is easy to verify by (4.11) and (4.17) the existance of the following asymptotic equalization

$$\lim_{x\to\infty}\left(\frac{\Psi_{L>}(x,\alpha,\beta,\lambda)}{\Psi_{L>P}(x,\alpha,\beta,\lambda)}\right) = 1. \tag{4.20}$$

Note, that the statement (4.20) is based on the simple functional theory.

Lemma 4.3. *Solution* (4.17) *is an asymptotic solution of the EQID* (4.6).

Proof. For unique analytical closed form solution of EQID (4.6), for $(1-\lambda)<1$, we have

$$\lim_{x\to\infty}\left(\Psi_{L>}(x,\alpha,\beta,\vartheta)\right)=\lim_{x\to\infty}\left(\frac{\Psi_{o>}}{\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!k!\lambda^n}H\left(x-n\frac{\beta}{\alpha}-k\left(1-\frac{\beta}{\alpha}\right)\right)\right)$$
$$=\left(\frac{\Psi_{o>}}{\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!k!\lambda^n}\right)=\frac{\Psi_{o>}}{\lambda}\sum_{n=0}^{\infty}\left(\frac{1}{\lambda}\right)^n(1-1)^n=\frac{\Psi_{o>}}{\lambda}. \tag{4.21}$$

On the other hand, for particular solution (4.17), we have

$$\lim_{x\to\infty}\left(\Psi_{LP>}(x,\alpha,\beta,\vartheta)\right)=\lim_{x\to\infty}\left(\Psi_{>}(\zeta)^{-x}+\sigma_{>}\right)=\sigma_{>}=\frac{\Psi_{o>}}{\lambda}. \tag{4.22}$$

Finally, the following appears

$$\lim_{x\to\infty}\left(\Psi_{L>}(x,\alpha,\beta,\vartheta)\right)=\lim_{x\to\infty}\left(\Psi_{LP>}(x,\alpha,\beta,\vartheta)\right)=\frac{\Psi_{o>}}{\lambda} \tag{4.23}$$

and, thus the proof is finished. □

According to the unique solution principle and Lemma 4.3, we have

$$\lim_{x\to\infty}\left(\frac{\Psi_{L>}(x,\alpha,\beta,\lambda)-\sigma_{>}}{\Psi_{L>}(x+1,\alpha,\beta,\lambda)-\sigma_{>}}\right)=\lim_{x\to\infty}\left(\frac{\Psi_{L>P}(x,\alpha,\beta,\lambda)-\sigma_{>}}{\Psi_{L>P}(x+1,\alpha,\beta,\lambda)-\sigma_{>}}\right). \tag{4.24}$$

Now, from (4.17) appears

$$\lim_{x\to\infty}\left(\frac{\Psi_{L>}(x,\alpha,\beta,\lambda)-\sigma_{>}}{\Psi_{L>}(x+1,\alpha,\beta,\lambda)-\sigma_{>}}\right)=\frac{(Z_{>}(\zeta))^{-x}}{(Z_{>}(\zeta))^{-(x+1)}}=Z_{>}(\zeta)=Z_{>} \tag{4.25}$$

and $Z_{>}=\mathrm{tran}_{ZL>}(\alpha,\beta,\lambda)$, where $\mathrm{tran}_{ZL>}(\alpha,\beta,\lambda)$ is a new special tran function defined as

$$Z_{>}=\mathrm{tran}_{ZL>}(\alpha,\beta,\lambda)=\lim_{x\to\infty}\left(\frac{\Psi_{L>}(x,\alpha,\beta,\lambda)-\sigma_{>}}{\Psi_{L>}(x+1,\alpha,\beta,\lambda)-\sigma_{>}}\right). \tag{4.26}$$

More explicitly, after substitution (4.11) into (4.26) we have

$$Z_{>}=\lim_{x\to\infty}\left(\frac{\sum_{n=[x]+1}^{[x(\alpha/\beta)]}\sum_{k=0}^{[(x-n(\beta/\alpha))/(1-\beta/\alpha)]}\mathcal{M}}{\sum_{n=[x+1]+1}^{[(x+1)(\alpha/\beta)]}\sum_{k=0}^{[((x+1)-n(\beta/\alpha))/(1-\beta/\alpha)]}\mathcal{M}}\right), \tag{4.27}$$

where $Z_> = Z_>(\zeta) = \mathrm{tran}_{ZL>}(\alpha(\zeta), \beta(\zeta), \lambda(\zeta))$. From (4.3), follows

$$\Psi_> = \mathrm{tran}_{L>}(\alpha,\beta,\lambda) = \lim_{x\to\infty}\left[\left(\frac{\sum_{n=[x]+1}^{[x(\alpha/\beta)]}\sum_{k=0}^{[(x-n(\beta/\alpha))/(1-\beta/\alpha)]}\mathcal{M}}{\sum_{n=[x+1]+1}^{[(x+1)(\alpha/\beta)]}\sum_{k=0}^{[((x+1)-n(\beta/\alpha))/(1-\beta/\alpha)]}\mathcal{M}}\right)\right]^{-1/\alpha}, \tag{4.28}$$

where $\mathrm{tran}_{L>}(\alpha,\beta,\lambda)$ is another special tran function.

Now, the proof is completed.

Analogically, from (4.3) and (4.28), we obtain formulae for the practical application

$$\langle\Psi_>\rangle_{[P_>]} = \left\langle\left(\frac{\sum_{n=[x]+1}^{[x(\alpha/\beta)]}\sum_{k=0}^{[(x-n(\beta/\alpha))/(1-(\beta/\alpha))]}\mathcal{M}}{\sum_{n=[x+1]+1}^{[(x+1)(\alpha/\beta)]}\sum_{k=0}^{[((x+1)-n(\beta/\alpha))/(1-\beta/\alpha)]}\mathcal{M}}\right)^{-1/\alpha}\right\rangle_{[P_>]}, \tag{4.29}$$

where $\langle\Psi_>\rangle_{[P_>]}$ denotes the numerical value of $\Psi_>$ given with $[P_>]$ accurate digits. $[P_>]$ is defined as $[P_>] = [\ln(|G_>|)]$, where the error function $G_>$ is defined as $G_> = \Psi_>^{\alpha} - \Psi_>^{\beta} - \lambda\Psi_>^{\alpha+\beta}$.

Remark 4.4. If there is more than two solutions of the transcendental Lambert equation (1.1), in real domain, than the appropriate modifications are necessary [13].

5. The Numerical Results Analysis

It is not difficult to see that analytical solutions (3.4) and (4.29) obtained by the Special Trans Functions Theory, gives highly accurate numerical results by MATHEMATICA program, suggesting that the STFT numerically works (Tables 1, 2, and 3).

Concerning the numerical results through the number of accurate digits $P_<$ and $P_>$, we deal with a sumlimit $[x]$. It may be convenient to consider the $P_<$ and $P_>$ as a functions of $[x]$. In fact, we have interest of knowing the functional dependence $P_< = P_<([x])$ (or, $P_> = P_>([x])$), which is sometimes the ordinary linear function. Of course, the linear functional form of $P_< = P_<([x])$ (or, $P_> = P_>([x])$) is frequently used when the problems in applied physics, or in engineering domain are in matter. Thus, the number of accurate digits in the numerical structure of $\Psi_<$ (or $\Psi_>$) depends on the $[x]$. An important study concerning the numbers of accurate digits in the numerical structure of the result has been formulated in [1–3, 6]. On the other hand, the number of accurate digits in the practical applications of $\Psi_<$ (or $\Psi_>$) is in accordance with physical requirements of exactness.

Finally, we will illustrate the STFT application by some simple examples of the transcendental Lambert equations (1.1). For convenience we restrict ourselves to the one dimensional case of (1.1).

Table 1

$[x]$	$\Psi_<$	$G_<$
4	1.061610	$4.58107E-07$
6	1.06161040	$7.39075E-09$
10	1.06161040584	$1.92367E-12$
12	1.0616104058422	$3.10350E-14$
15	1.061610405842267	$6.35967E-17$
18	1.061610405842267125	$1.30322E-19$
21	1.061610405842267125815	$2.67053E-22$
25	1.061610405842267125815953	$6.95090E-26$
28	1.061610405842267125815953494	$1.42437E-28$
35	1.06161040584226712581595349425165	$7.59710E-35$
37	1.06161040584226712581595349425165683	$1.22566E-36$

5.1. *The Numerical Results for Some Examples of the Transcendental Lambert Equation* (1.1) *for* $\Psi_<$

The subjects of the numerical analysis presented here are some solutions of two simple examples of (1.1), for $\Psi < (\alpha/\beta)^{1/(\alpha-\beta)}$.

5.1.1. Example 1 for $\Psi_<$

Equation (1.1) for the following values: $\alpha = 4$, $\beta = 2$, and $\lambda = (\alpha - \beta)\vartheta = 1/10$, takes the form

$$(\Psi_<)^4 - (\Psi_<)^2 = \frac{1}{10}(\Psi_<)^6. \tag{5.1}$$

According to (3.4), for above example, we obtain the solution

$$\langle \Psi_< \rangle_{[P_<]} = \left\langle \frac{-\sum_{n=0}^{[(x+2)/4]} (9/10)^n + \sum_{n=[(x+2)/4]+1}^{[(x+2)/2]} \sum_{k=0}^{[(x+2-2n)/2]} \mathcal{Y} + 10}{-\sum_{n=0}^{[(x+3)/4]} (9/10)^n + \sum_{n=[(x+3)/4]+1}^{[(x+3)/2]} \sum_{k=0}^{[(x+3-2n)/2]} \mathcal{Y} + 10} \right\rangle_{[P_<]}, \tag{5.2}$$

where $\mathcal{Y}$ denotes $((-1)^{k+1}(1/10)^k n!/(n-k)!k!)$ and $P_< = [\text{abs}(G_<)]$, and $G_< = (\Psi_<)^4 - (\Psi_<)^2 - (1/10)(\Psi_<)^6$. In Table 1, using the Special Trans Functions Theory (5.2) we obtained the following numerical results.

Remark 5.1. From (5.1), by simple quadratic equation approach we have solution of the form

$$\Psi_< = \sqrt{5 - \sqrt{15}} \approx 1.06161040584226712581595349425165683 \tag{5.3}$$

if we restrict on 36 accurate digits.

Let us note that the result in the last row in Table 1 (for $[x] = 37$), obtained by STFT, is impressive, and, consequently, it proves that STFT numerically works!

Table 2

$[x]$	$\Psi_<$	$G_<$
15	1.028316	$1.50047E-06$
25	1.02831660	$3.37631E-09$
35	1.02831660970	$7.62081E-12$
45	1.0283166097035	$1.71991E-14$
55	1.028316609703590	$3.88160E-17$
60	1.028316609703590904	$1.84401E-18$
80	1.0283166097035909049828	$9.39235E-24$
90	1.0283166097035909049828531	$2.11973E-26$
100	1.0283166097035909049828531652	$4.78394E-29$
120	1.02831660970359090498285316522474	$2.43667E-34$
125	1.02831660970359090498285316522474301	$1.15758E-35$

5.1.2. Example 2 for $\Psi_<$

Equation (1.1) for a following values: $\alpha = 12$, $\beta = 9$, and $\lambda = (\alpha - \beta)\vartheta = 1/16$, takes the form

$$(\Psi_<)^{12} - (\Psi_<)^{9} = \frac{1}{16}(\Psi_<)^{21}. \tag{5.4}$$

According to (3.4), for above example, we obtain the solution

$$\langle \Psi_< \rangle_{[P_<]} = \left\langle \frac{-\sum_{n=0}^{[(x+9)/12]} (15/16)^n + \sum_{n=[(x+9)/12]+1}^{[(x+9)/3]} \sum_{k=0}^{[(x+9-3n)/3]} \mathcal{W} + 16}{-\sum_{n=0}^{[(x+10)/12]} (15/16)^n + \sum_{n=[(x+10)/12]+1}^{[(x+10)/3]} \sum_{k=0}^{[(x+10-3n)/3]} \mathcal{W} + 16} \right\rangle_{[P_<]}, \tag{5.5}$$

where $\mathcal{W}$ denotes $((-1)^{k+1}(1/16)^k n!/(n-k)!k!)$ and $P_< = [\text{abs}(G_<)]$, and $G_< = (\Psi_<)^{12} - (\Psi_<)^{9} - (1/16)(\Psi_<)^{21}$. In Table 2, using the Special Trans Functions Theory (5.5) we obtain the following numerical results.

Remark 5.2. From (5.5), by classical approach, we have solution of the form

$$\Psi_< = \frac{2}{3}\left[\sqrt[3]{3\sqrt{33}+17} - \sqrt[3]{3\sqrt{33}-17} - 1\right] = 1.02831660970359090498285316522474301 \tag{5.6}$$

if we restrict on 35 accurate digits.

Let us note that the result in the last row in Table 2 (for $[x] = 125$), obtained by STFT, is correct, and consequently, we have a confirmation that the STFT numerically works.

Table 3

$[x]$	$\Psi_>$	$G_>$
4	2.9787553	$2.03239E-06$
5	2.978755335	$3.27890E-08$
8	2.97875533506990	$1.37687E-13$
11	2.9787553350699041400	$5.78171E-19$
14	2.978755335069904140041494	$2.42784E-24$
17	2.97875533506990414004149468 2037	$1.01949E-29$
20	2.978755335069904140041494682037 63347	$4.28103E-35$

5.2. *The Numerical Results for Some Examples of* (1.1) *for* $\Psi_>$

The subject of the numerical analysis presented here are some solutions of two simple examples of (1.1) for $\Psi > (\alpha/\beta)^{1/(\alpha-\beta)}$.

5.2.1. Example 1 for $\Psi_>$

Equation (1.1) for the following values: $\alpha = 4$, $\beta = 2$, and $\lambda = (\alpha-\beta)\vartheta = 1/10$, takes the form

$$(\Psi_>)^4 - (\Psi_>)^2 = \frac{1}{10}(\Psi_>)^6. \tag{5.7}$$

According to (4.29), for above example, we obtain the solution

$$\langle\Psi_>\rangle_{[P_>]} = \left\langle\left(\frac{\sum_{n=[x]+1}^{[2x]}\sum_{k=0}^{[2x-n]}\left((-1)^k n!/(n-k)!k!(1/10)^n\right)}{\sum_{n=[x+1]+1}^{[2(x+1)]}\sum_{k=0}^{[2x+2-n]}\left((-1)^k n!/(n-k)!k!(1/10)^n\right)}\right)^{-1/4}\right\rangle_{[P_>]}, \tag{5.8}$$

where $P_> = [\text{abs}(G_>)]$, and $G_> = (\Psi_>)^4 - (\Psi_>)^2 - (1/10)(\Psi_>)^6$. In Table 3, using the Special Trans Functions Theory (5.8) we have the following numerical results.

Remark 5.3. Deriving from (5.8), by conventional approach, we get the solution of the form

$$\Psi_< = \sqrt{5+\sqrt{15}} \approx 2.97875533506990414004149468203763347 \tag{5.9}$$

if we restrict on 35 accurate digits.

Let us note that the result in the last row of the Table 3 (for $[x] = 20$) is correct, and we must declare that STFT, for this case, also works!

5.2.2. Example 2 for $\Psi_>$

The transcendental Lambert equation (1.1) for following values: $\alpha = 12$, $\beta = 9$, and $\lambda = (\alpha - \beta)\vartheta = 1/16$, takes the form

$$(\Psi_>)^{12} - (\Psi_>)^{9} = \frac{1}{16}(\Psi_>)^{21}. \tag{5.10}$$

According to (4.2), for above example, we have

$$Z_> + \frac{1}{16} = Z_>^{3/4}, \tag{5.11}$$

or, the form

$$16Z_> + 1 = 16Z_>^{3/4}. \tag{5.12}$$

After simple modification, the above equation takes the following form

$$\Upsilon + 1 = 2\Upsilon^{3/4}, \tag{5.13}$$

where

$$\Upsilon = 16Z_>. \tag{5.14}$$

Thus, by application of the STFT we have that EQID for (5.13) that takes the form

$$\varphi_>(x-1) + \varphi_>(x) - 2\varphi_>\left(x - \frac{3}{4}\right) = \varphi_{0>}. \tag{5.15}$$

Due to (2.15), the particular solution of (5.15) takes the form

$$\varphi_{>P}(x) = \Upsilon^{-x} + \sigma_>. \tag{5.16}$$

From (5.13), (5.15) and (5.16) we have

$$\sigma_> = \frac{\varphi_{0>}}{2-2} \longrightarrow \infty. \tag{5.17}$$

Consequently, from (2.23), analogically, we have for Υ

$$\Upsilon = \lim_{x\to\infty}\left(\frac{\varphi_>(x-1) - \sigma_>}{\varphi_>(x) - \sigma_>}\right) \longrightarrow 1 \tag{5.18}$$

since $\sigma_> \to \infty$. *From* (5.14) we have $Z = 1/16$, and $\Psi_> = (Z)^{-1/\alpha} = \sqrt[3]{2}$. From (5.13) we can observed that $Y = 1$ is its trivial solution $(1 + 1 = 2 \cdot 1^{3/4})$.

Remark 5.4. An example when the number of nontrivial solutions is equal three $(\theta^3 - \theta = \lambda\theta^4)$, has been presented in some detail in [13].

6. Conclusions

From the theoretical point of the analysis presented in this paper, the Special Trans Functions Theory is a consistent general approach for solving a broad class of Lambert transcendental equations (1.1), exactly, analytically, independently of particular case. This means that in the some manner we can obtain the Special Trans Functions for variety of different Lambert transcendental equations.

Accordingly, in this paper the family of the transcendental Lambert equations (1.1), is solved analytically in the closed form, by the Special Trans Functions Theory, in the real domain.

Formulae (2.24) (or (3.4)) and (4.27) (or (4.29)) derived in this paper, by using the STFT, are valid in the mathematical sense, and, are correct in the numerical sense (see Tables 1, 2, and 3), as well. Namely, the obtained analytical solutions (2.24) and (4.27), apart from the theoretical values have practical applications (3.4) and (4.29).

Let us note that the Special Trans Functions Theory is recently developed theory, and any numerical confirmation of the STFT validity is very important, because it is relevant argument for our statement concerning the STFT application validity independent on the comprehensiveness of the author‘s explanations of it.

Of course, the theoretical confirmations are comprised in: the initial idea (born in neutron slowing down analysis [1–3]), consistent theoretical structure of the Special Trans Functions genesis, correct mathematical derivations, analysis of convergence dynamics between analytical unique and asymptotic solutions of the EQID, and so forth.

Let us note that one of the advantages of the Special Trans Functions Theory, as a mathematical method, is its applicability to a broad class of transcendental Lambert functional forms.

The theoretical accuracy of the numerical structures in (3.4) and (4.29) is unlimited and extremely precise [1, 3].

The forms of the EQID are routinely applied, according to the intuitive assumptions.

Finally, by STFT application, we have the possibility to obtain different gradient parameters. It is not difficult to see that the latter statement implies the rigorous analytical analysis for any nonlinear problem solved by STFT. Thus, definition of some new analytically sensitive parameters directly follows in the forms

$$\begin{gathered}\frac{\partial\Psi_<}{\partial\alpha}, \quad \frac{\partial\Psi_<}{\partial\beta}, \quad \frac{\partial\Psi_<}{\partial\vartheta}, \quad \frac{\partial\Psi_<}{\partial\lambda}, \\ \frac{\partial\Psi_>}{\partial\alpha}, \quad \frac{\partial\Psi_>}{\partial\beta}, \quad \frac{\partial\Psi_>}{\partial\vartheta}, \quad \frac{\partial\Psi_>}{\partial\lambda}.\end{gathered} \tag{6.1}$$

Or, from (2.24) and (4.28), we have, respectively,

$$\frac{\partial \mathrm{tran}_{L<}}{\partial \alpha}, \quad \frac{\partial \mathrm{tran}_{L<}}{\partial \beta}, \quad \frac{\partial \mathrm{tran}_{L<}}{\partial \vartheta}, \quad \frac{\partial \mathrm{tran}_{L<}}{\partial \lambda},$$
$$\frac{\partial \mathrm{tran}_{L>}}{\partial \alpha}, \quad \frac{\partial \mathrm{tran}_{L>}}{\partial \beta}, \quad \frac{\partial \mathrm{tran}_{L>}}{\partial \vartheta}, \quad \frac{\partial \mathrm{tran}_{L>}}{\partial \lambda}. \tag{6.2}$$

For example, from (3.3), we have:

$$\left\langle \frac{\partial \Psi_<}{\partial \alpha} \right\rangle_{[P_<]} = \left\langle \frac{(\partial \Psi_{L<}(x)/\partial \alpha)(\Psi_{L<}(x+1) - \sigma_<) - (\partial \Psi_{L<}(x+1)/\partial \alpha)(\Psi_{L<}(x) - \sigma_<)}{(\Psi_{L<}(x+1) - \sigma_<)^2} \right\rangle_{[P_<]}, \tag{6.3}$$

or we have

$$\left\langle \frac{\partial \Psi_<}{\partial \alpha} \right\rangle_{[P_<]} = \left\langle \frac{(\partial \Psi_{L<}(x)/\partial \alpha) - (\partial \Psi_{L<}(x+1)/\partial \alpha)\langle \Psi_< \rangle_{[G]}}{(\Psi_{L<}(x+1) - \sigma_<)} \right\rangle_{[P_<]}, \tag{6.4}$$

and analogically

$$\left\langle \frac{\partial \Psi_>}{\partial \alpha} \right\rangle_{[P_>]} = \left\langle \frac{(\partial \Psi_{L>}(x)/\partial \alpha) - (\partial \Psi_{L>}(x+1)/\partial \alpha)\langle \Psi_> \rangle_{[G]}}{(\Psi_{L>}(x+1) - \sigma_>)} \right\rangle_{[P_>]}. \tag{6.5}$$

The latter expressions are a significant contribution of the Special Trans Function Theory in applied physics and engineering domain for analytical approach to theoretical processes analysis.

According to the authors' knowledge, this is the first direct application of the STFT to the mathematical genesis of the analytical closed form solution to the broad family of the transcendental Lambert equations (1.1), independently of any physical process or phenomena.

References

[1] S. M. Perovich, *The Special Tran Functions Theory*, Research Monograph, University of Montenegro, Podgorica, Montenegro, 2004.

[2] S. M. Perovich, "The transcendental method in the theory of neutron slowing down," *Journal of Physics A*, vol. 25, no. 10, pp. 2969–2988, 1992.

[3] S. M. Perovich, "Reply to the comment on the transcendental method," *Journal of Physics A*, vol. 28, no. 14, pp. 4191–4200, 1995.

[4] S. M. Perovich and D. V. Tosic, "Transcendental method in nonlinear circuit theory," *Electronics Letters*, vol. 32, no. 16, pp. 1433–1434, 1996.

[5] S. M. Perovich, "Concerning the analytical solution of the dispersive equation in the linear transport theory," *Transport Theory and Statistical Physics*, vol. 26, no. 6, pp. 705–725, 1997.

[6] S. M. Perovich, I. Djurovic, and D. V. Tosic, "Some Comments Concerning the Discrete Eigenvalue," *Nuclear Science and Engineering*, vol. 131, no. 3, p. 439, 1999.

[7] S. M. Perovich, A. Lompar, and I. Djurovic, *The Special Trans Function Theory to the Conductive Fluid Level Estimation*, vol. 436, ISA, 2002.

[8] S. M. Perovich and S. I. Bauk, "Determination of the Plutonium temperatureusing the special trans functions theory," *Nuclear Technology and Protection*, vol. 25, no. 3, pp. 164–170, 2010.

[9] S. M. Perovich and S. I. Bauk, "The analytical analysis of hopfield neuron parameters by the application of special trans function theory," in *Adaptive and Natural Computing Algorithms*, Monograph Publications, pp. 29–32, Springer, New York, NY, USA, 2005.

[10] S. M. Perovich, S. K. Simic, D. V. Tosic, and S. I. Bauk, "On the analytical solution of some families of transcendental equations," *Applied Mathematics Letters*, vol. 20, no. 5, pp. 493–498, 2007.

[11] S. M. Perovich, S. I. Bauk, and M. D. J. Jovanovic, "Concerning an analytical solution of some families of nonlinear functional equations," in *Proceedings of the International Conference on Numerical Analysis and Applied Mathematics*, vol. 936 of *AIP Conference Proceedings*, pp. 412–415, September 2007.

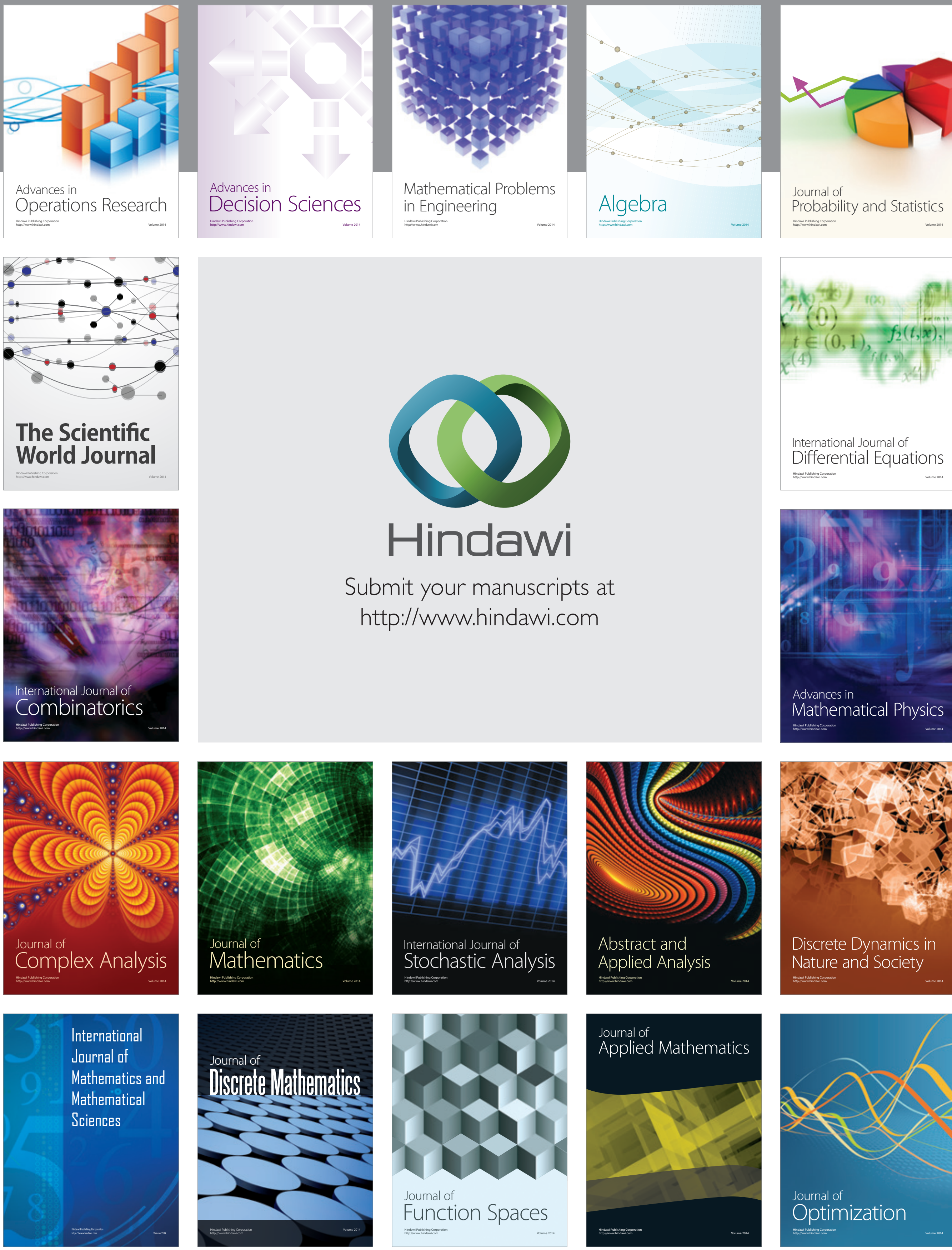
[12] S. M. Perovich and S. I. Bauk, "The STFT in an exponential function approximation," in *Proceedings of the International Conference on Numerical Analysis and Applied Mathematics*, AIP Conference Proceedings, pp. 416–419, September 2007.

[13] S. M. Perovich and S. Kordic, "The special trans functions theory and quadratic equations," in *Proceedings of the 9th International Conference "Research and Development in Mechanical Industry" (RaDMI '09)*, Kotor, Montenegro, 2009.

[14] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, "On the Lambert W function," *Advances in Computational Mathematics*, vol. 5, no. 4, pp. 329–359, 1996.

[15] J. H. Lamber, "Observations variae in Mathesin Puram," *Acta Helvetica*, vol. 3, pp. 128–168, 1758.

[16] E. Humo, *The Special Measurement*, Svjetlost, Sarajevo, Bosnia, 1981.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