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Research Article

Hamilton-Poisson Realizations for the Lü System

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The Hamilton-Poisson geometry has proved to be an interesting approach for a lot of dynamics arising from different areas like biology (Gümral and Nutku, 1993), economics (Dănăiasă et al., 2008), or engineering (Ginoux and Rossetto, 2006). The Lü system was first proposed by Lü and Chen (2002) as a model of a nonlinear electrical circuit, and it was studied from various points of view. We intend to study it from mechanical geometry point of view and to point out some of its geometrical and dynamical properties.

1. Introduction

The original Lü system of differential equations on $\mathbb{R}^3$ has the following form

$$\begin{aligned} \dot{x} &= a(y-x), \\ \dot{y} &= -xz + by, \\ \dot{z} &= -cz + xy, \end{aligned} \tag{1.1}$$

where $a, b, c \in \mathbb{R}$.

The goal of our paper is to find the relations between a, b, and c parameters, for which the system (1.1) admits a Hamilton-Poisson realization. The Hamilton-Poisson realization offers us the tools to study the Lü system from mechanical geometry point of view.

To do this, one needs first to find the constants of the motion of our system. Due to the numerous parameters of the system and trying to simplify the computation, we will focus on finding only constants of motion being polynomials of degree at most three of the system (1.1).

Proposition 1.1. *The following smooth real functions H are three degree polynomial constants of the motion defined by the system* (1.1).

(i) *If $a \in \mathbb{R}^*$, $b = c = 0$ the system becomes:*

$$\begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= -xz, \\ \dot{z} &= xy, \end{aligned} \tag{1.2}$$

$$H(x, y, z) = \alpha\left(y^2 + z^2\right) + \beta, \quad \alpha, \beta \in \mathbb{R}. \tag{1.3}$$

(ii) *If $a = 0$, $b, c \in \mathbb{R}^*$ the system becomes:*

$$\begin{aligned} \dot{x} &= 0, \\ \dot{y} &= -xz + by, \\ \dot{z} &= xy - cz, \end{aligned} \tag{1.4}$$

$$H(x, y, z) = f(x), \quad f \in C^1(\mathbb{R}). \tag{1.5}$$

(iii) *If $a \in \mathbb{R}^*$, $b = c \in \mathbb{R}$ the system becomes:*

$$\begin{aligned} \dot{x} &= 0, \\ \dot{y} &= -xz + by, \\ \dot{z} &= xy - bz, \end{aligned} \tag{1.6}$$

$$H(x, y, z) = \alpha\left(xy^2 - 2byz + xz^2\right) + f(x), \quad \alpha \in \mathbb{R}, \ f \in C^1(\mathbb{R}). \tag{1.7}$$

(iv) *If $a = b = c = 0$ the system becomes:*

$$\begin{aligned} \dot{x} &= 0, \\ \dot{y} &= -xz, \\ \dot{z} &= xy, \end{aligned} \tag{1.8}$$

$$H(x, y, z) = \alpha\left(y^2 + z^2\right) + \beta\left(xy^2 + xz^2\right) + f(x), \quad \alpha, \beta \in \mathbb{R}, \ f \in C^1(\mathbb{R}). \tag{1.9}$$

Proof. It is easy to see that $dH = 0$ for each case mentioned above. □

2. Hamilton-Poisson Realizations for the System (1.2)

Let us take for the system (1.2) the Hamiltonian function given by:

$$H(x, y, z) = \frac{1}{2}\left(y^2 + z^2\right). \tag{2.1}$$

To find the Poisson structure in this case, we will use a method described by Haas and Goedert (see [5] for details). Let us consider the skew-symmetric matrix given by:

$$\Pi := \begin{bmatrix} 0 & p_1(x, y, z) & p_2(x, y, z) \\ -p_1(x, y, z) & 0 & p_3(x, y, z) \\ -p_2(x, y, z) & -p_3(x, y, z) & 0 \end{bmatrix}. \tag{2.2}$$

We have to find the real smooth functions $p_1, p_2, p_3 : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \Pi \cdot \nabla H, \tag{2.3}$$

that is, the following relations hold:

$$\begin{aligned} yp_1(x, y, z) + zp_2(x, y, z) &= a(y - x), \\ zp_3(x, y, z) &= -xz, \\ -yp_3(x, y, z) &= xy. \end{aligned} \tag{2.4}$$

It is easy to see that $p_3(x, y, z) = -x$. Let us denote now $p_1(x, y, z) = p$; from the second equation we obtain

$$p_2(x, y, z) = a\frac{y - x}{z} - \frac{y}{z}p. \tag{2.5}$$

Our goal now is to insert p_1, p_2, p_3 into Jacobi identity and to find the function $p(x, y, z)$. In the beginning, let us denote:

$$\begin{aligned} v_1 &:= a(y - z), \\ v_2 &:= -xz, \\ v_3 &:= xy. \end{aligned} \tag{2.6}$$

The function p is the solution of the following first order ODE (see [5] for details):

$$v_1 \frac{\partial p}{\partial x} + v_2 \frac{\partial p}{\partial y} + v_3 \frac{\partial p}{\partial z} = A \cdot p + B, \tag{2.7}$$

where

$$A = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} - \frac{(\partial v_1/\partial z)(\partial H/\partial x) + (\partial v_2/\partial z)(\partial H/\partial y) + (\partial v_3/\partial z)(\partial H/\partial z)}{\partial H/\partial z},$$
$$B = \frac{v_1(\partial v_2/\partial z) - v_2(\partial v_1/\partial z)}{(\partial H/\partial z)}. \tag{2.8}$$

Equation (2.7) becomes:

$$a(y-x)\frac{\partial p}{\partial x} - xz\frac{\partial p}{\partial y} + xy\frac{\partial p}{\partial z} = \left(-a + \frac{xy}{z}\right)p - a(y-x)\frac{x}{z}. \tag{2.9}$$

If $a = 0$, then (2.9) has the solution $p(x,y,z) = xz$.
If $a \neq 0$, then finding the solution of (2.9) remains an open problem.
Now, one can reach the following result.

Proposition 2.1. *If $a = 0$, the system* (1.2) *has the Hamilton-Poisson realization:*

$$\left(\mathbb{R}^3, \Pi := \left[\Pi^{ij}\right], H\right), \tag{2.10}$$

where

$$\Pi = \begin{bmatrix} 0 & xz & -xy \\ -xz & 0 & -x \\ xy & x & 0 \end{bmatrix},$$
$$H(x,y,z) = \frac{1}{2}\left(y^2 + z^2\right). \tag{2.11}$$

Remark 2.2. There exists only one functionally independent Casimir of our Poisson configuration, given by $C : \mathbb{R}^3 \to \mathbb{R}$,

$$C(x,y,z) = 2x - y^2 - z^2. \tag{2.12}$$

Proof. Indeed, one can easily check that:

$$\Pi \cdot \nabla C = 0. \tag{2.13}$$

As the rank of Π equals 2, it follows from the general theory of PDEs that C is the only functionally independent Casimir function of the configuration (see, e.g., [6] for details). □

The phase curves of the dynamics (1.2) are the intersections of the surfaces:

$$\begin{aligned} H &= \text{const.} \\ C &= \text{const,} \end{aligned} \tag{2.14}$$

see Figure 1.

Remark 2.3. If $a = b = c = 0$, then the system (1.2) becomes:

$$\begin{aligned} \dot{x} &= 0, \\ \dot{y} &= -xz, \\ \dot{z} &= xy. \end{aligned} \tag{2.15}$$

For the specific case $a = b = c = 0$, we extended the results presented in Proposition 2.1 to the following one.

Proposition 2.4 (Alternative Hamilton-Poisson structures). *The system* (2.15) *may be modeled as an Hamilton-Poisson system in an infinite number of different ways, that is, there exists infinitely many different (in general nonisomorphic) Poisson structures on* $\mathbb{R}^3$ *such that the system* (2.15) *is induced by an appropriate Hamiltonian.*

Proof. The triplets:

$$\left(R^3\{.,.\}_{\alpha\beta}, H_{\gamma\delta}\right), \tag{2.16}$$

where

$$\begin{aligned} &\{f,g\}_{\alpha\beta} = \nabla C_{\alpha\beta} \cdot (\nabla f \times \nabla g), \quad \forall f,g \in C^{\infty}\left(\mathbb{R}^3, \mathbb{R}\right), \\ &C_{\alpha\beta} = \alpha C + \beta H, \quad H_{\gamma\delta} = \gamma C + \delta H, \quad \alpha,\beta,\gamma,\delta \in \mathbb{R}, \quad \alpha\delta - \beta\gamma = 1, \\ &H = \frac{1}{2}\left(2x - y^2 - z^2\right), \qquad C = \frac{1}{2}x^2, \end{aligned} \tag{2.17}$$

define Hamilton-Poisson realizations of the dynamics (2.15).

Indeed, we have:

$$\{x, H_{\gamma\delta}\}_{\alpha\beta} = \begin{vmatrix} \alpha x + \beta & -\beta y & -\beta z \\ 1 & 0 & 0 \\ \gamma x + \delta & -\delta y & -\delta z \end{vmatrix} = 0 = \dot{x};$$

$$\{y, H_{\gamma\delta}\}_{\alpha\beta} = \begin{vmatrix} \alpha x + \beta & -\beta y & -\beta z \\ 0 & 1 & 0 \\ \gamma x + \delta & -\delta y & -\delta z \end{vmatrix} = -xz = \dot{y}; \tag{2.18}$$

$$\{z, H_{\gamma\delta}\}_{\alpha\beta} = \begin{vmatrix} \alpha x + \beta & -\beta y & -\beta z \\ 0 & 0 & 1 \\ \gamma x + \delta & -\delta y & -\delta z \end{vmatrix} = xy = \dot{z}.$$

□

Let us pass now to study some geometrical and dynamical aspects of the system (2.15).

Proposition 2.5 (Lax formulation). *The dynamics* (2.15) *allows a formulation in terms of Lax pairs.*

Proof. Let us take:

$$L = \begin{bmatrix} 0 & \alpha x - \dfrac{\alpha\beta\gamma}{\sqrt{\beta^2-\gamma^2}} z + \delta & \dfrac{\alpha\gamma}{\sqrt{\beta^2-\gamma^2}} x + \alpha\beta z + \dfrac{\gamma\delta}{\sqrt{\beta^2-\gamma^2}} \\ -\alpha x + \dfrac{\alpha\beta\gamma}{\sqrt{\beta^2-\gamma^2}} z - \delta & 0 & -\dfrac{\alpha\beta^2}{\sqrt{\beta^2-\gamma^2}} y \\ -\dfrac{\alpha\gamma}{\sqrt{\beta^2-\gamma^2}} x - \alpha\beta z - \dfrac{\gamma\delta}{\sqrt{\beta^2-\gamma^2}} & \dfrac{\alpha\beta^2}{\sqrt{\beta^2-\gamma^2}} y & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & \gamma z - \dfrac{\sqrt{\beta^2-\gamma^2}\,\delta}{\alpha\beta} & -\sqrt{\beta^2-\gamma^2}\, z - \dfrac{\gamma\delta}{\alpha\beta} \\ -\gamma z + \dfrac{\sqrt{\beta^2-\gamma^2}\,\delta}{\alpha\beta} & 0 & \beta y \\ \sqrt{\beta^2-\gamma^2}\, z + \dfrac{\gamma\delta}{\alpha\beta} & -\beta y & 0 \end{bmatrix}, \tag{2.19}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $i = \sqrt{-1}$.

Then, using MATHEMATICA 7.0, we can put the system (2.15) in the equivalent form

$$\dot{L} = [L, B] \tag{2.20}$$

as desired. □

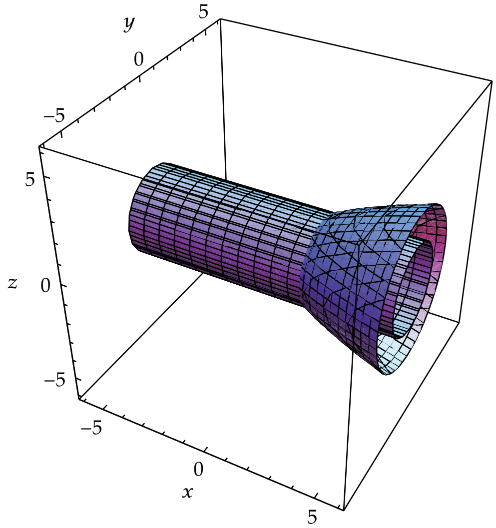

Figure 1: The phase curves of the dynamics (1.2).

Let us continue now with a discussion concerning the nonlinear stability of equilibrium states of our system (2.15) (see [7] for details).

It is obvious to see that the equilibrium points of our dynamics are given by:

$$\begin{aligned} e_1^M &= (M, 0, 0), \quad M \in \mathbb{R}, \\ e_2^M &= (0, M, N), \quad M, N \in \mathbb{R}. \end{aligned} \tag{2.21}$$

About their stability,we reached the following result.

Proposition 2.6 (A stability result). *The equilibrium states e_1^M are nonlinearly stable for any $M \in \mathbb{R}$.*

Proof. We shall use energy-Casimir method, see [8] for details. Let

$$H_\varphi = H + \varphi(C) = \frac{1}{2}\left(y^2 + z^2\right) + \varphi\left(2x - y^2 - z^2\right) \tag{2.22}$$

be the energy-Casimir function, where $\varphi : R \to R$ is a smooth real valued function defined on R.

Now, the first variation of H_φ is given by:

$$\delta H_\varphi = y\delta y + z\delta z + \dot{\varphi}\left(2x - y^2 - z^2\right)(2\delta x - 2y\delta y - 2z\delta z). \tag{2.23}$$

This equals zero at the equilibrium of interest if and only if

$$\dot{\varphi}(2M) = 0. \tag{2.24}$$

The second variation of H_φ is given by:

$$\delta^2 H_\varphi = (\delta y)^2 + (\delta z)^2 + \ddot{\varphi} \cdot (2\delta x - 2y\delta y - 2z\delta z)^2 - 2\dot{\varphi}\Big((\delta y)^2 + (\delta z)^2\Big). \tag{2.25}$$

At the equilibrium of interest, the second variation becomes:

$$\delta^2 H_\varphi(M,0,0) = (\delta y)^2 + (\delta z)^2 + 4\ddot{\varphi} \cdot \ (\delta x)^2. \tag{2.26}$$

Having chosen φ such that:

$$\begin{gathered}\dot{\varphi}(2M) = 0, \\ \ddot{\varphi}(2M) > 0,\end{gathered} \tag{2.27}$$

we can conclude that the second variation of H_φ at the equilibrium of interest is positive defined and thus e^M is nonlinearly stable. □

As a consequence, we can reach the periodical orbits of the equilibrium points e_1^M.

Proposition 2.7 (Periodical orbits). *The reduced dynamics to the coadjoint orbit*

$$2x - y^2 - z^2 = 2M \tag{2.28}$$

has near the equilibrium point e_1^M at least one periodic solution whose period is close to

$$\frac{2\pi}{|M|}. \tag{2.29}$$

Proof. Indeed, we have successively

(i) the restriction of our dynamics (1.2) to the coadjoint orbit

$$2x - y^2 - z^2 = 2M \tag{2.30}$$

gives rise to a classical Hamiltonian system,

(ii) the matrix of the linear part of the reduced dynamics has purely imaginary roots, more exactly

$$\lambda_1 = 0, \qquad \lambda_{2,3} = \pm Mi. \tag{2.31}$$

(iii) $\operatorname{span}(\nabla C(e_1^M)) = V_0$, where

$$V_0 = \ker\big(A\big(e_1^M\big)\big), \tag{2.32}$$

(iv) the reduced Hamiltonian has a local minimum at the equilibrium state e_1^M (see the proof of Proposition 2.4).

Then our assertion follows via the Moser-Weinstein theorem with zero eigenvalue, see [9] for details. □

Remark 2.8. The nonlinear stability of the equilibrium states $e_2^{M,N}$ remains an open problem, both energy methods (energy-Casimir method and Arnold method) being inconclusive.

3. Hamilton-Poisson Realizations of the System (1.4)

As we have proved in [10], the system (1.4) admits a Hamilton-Poisson realization only in the special case $b = c$; more exactly, we have reached the following result.

Proposition 3.1. *If $a = 0$ and $b = c$, the system* (1.4) *has the Hamilton-Poisson realization*

$$\left(\mathbb{R}^3, \Pi := \left[\Pi^{ij}\right], H\right), \tag{3.1}$$

where

$$\Pi := \begin{bmatrix} 0 & xz - by & bz - xy \\ -xz + by & 0 & \frac{1}{2}(y^2 + z^2) \\ -bz + xy & -\frac{1}{2}(y^2 + z^2) & 0 \end{bmatrix}, \qquad H(x, y, z) = x. \tag{3.2}$$

Using a method described in [6], we have found the Casimir of the configuration given by.

$$C(x, y, z) = \frac{1}{2}\left(y^2 + z^2\right)x - byz, \quad b \in \mathbb{R}^*, \tag{3.3}$$

(see [10]).

Now we can broaden this result to the following one.

Proposition 3.2 (Alternative Hamilton-Poisson structures). *The system* (1.4) *may be realized as a Hamilton-Poisson system in an infinite number of different ways, that is, there exists infinitely many different (in general nonisomorphic) Poisson structures on $\mathbb{R}^3$ such that the system* (1.4) *is induced by an appropriate Hamiltonian.*

Proof. The triples:

$$\left(R^3 \{\cdot, \cdot\}_{\alpha\beta}, H_{\gamma\delta}\right), \tag{3.4}$$

where

$$\{f, g\}_{\alpha\beta} = \nabla C_{\alpha\beta} \cdot (\nabla f \times \nabla g), \quad \forall f, g \in C^{\infty}\left(\mathbb{R}^3, \mathbb{R}\right),$$
$$C_{\alpha\beta} = \alpha C + \beta H, \quad H_{\gamma\delta} = \gamma C + \delta H, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}, \quad \alpha\delta - \beta\gamma = 1, \tag{3.5}$$
$$H = x, \quad C = \frac{1}{2}x\left(y^2 + z^2\right) - byz, \quad b \in \mathbb{R},$$

define Hamilton-Poisson realizations of the dynamics (1.4).

Indeed, we have:

$$\{x, H_{\gamma\delta}\}_{\alpha\beta} = \begin{vmatrix} \alpha + \frac{\beta}{2}(y^2 + z^2) & \beta(xy - bz) & \beta(xz - by) \\ 1 & 0 & 0 \\ \gamma + \frac{\delta}{2}(y^2 + z^2) & \delta(xy - bz) & \delta(xz - by) \end{vmatrix} = 0 = \dot{x};$$

$$\{y, H_{\gamma\delta}\}_{\alpha\beta} = \begin{vmatrix} \alpha + \frac{\beta}{2}(y^2 + z^2) & \beta(xy - bz) & \beta(xz - by) \\ 0 & 1 & 0 \\ \gamma + \frac{\delta}{2}(y^2 + z^2) & \delta(xy - bz) & \delta(xz - by) \end{vmatrix} = xz - by = \dot{y}; \tag{3.6}$$

$$\{z, H_{\gamma\delta}\}_{\alpha\beta} = \begin{vmatrix} \alpha + \frac{\beta}{2}(y^2 + z^2) & \beta(xy - bz) & \beta(xz - by) \\ 0 & 0 & 1 \\ \gamma + \frac{\delta}{2}(y^2 + z^2) & \delta(xy - bz) & \delta(xz - by) \end{vmatrix} = bz - xy = \dot{z}.$$

□

Let us pass to discuss some dynamical and geometrical properties of the system (1.4).

Proposition 3.3 (Lax formulation). *The dynamics* (1.4) *allows a formulation in terms of Lax pairs.*

Proof. Let us take

$$L = \begin{bmatrix} 0 & u & v \\ -u & 0 & w \\ -v & -w & 0 \end{bmatrix}, \tag{3.7}$$

where

$$u = \alpha + \frac{\alpha(\beta^2+\gamma^2-\delta^2)i\sqrt{\beta^2+\gamma^2}}{2b\delta(\beta^2+\gamma^2)}x + \frac{\alpha\beta\gamma\delta(\beta^2+\gamma^2-\delta^2)i\sqrt{\beta^2+\gamma^2}}{2b\delta(\beta^2+\gamma^2)^2}y - \frac{\alpha\beta\gamma(\beta^2+\gamma^2-\delta^2)}{2b\delta(\beta^2+\gamma^2)}z,$$

$$v = -\frac{\alpha i\sqrt{\beta^2+\gamma^2}}{\beta} + \frac{\alpha\gamma^2 i\sqrt{\beta^2+\gamma^2}}{\beta(\beta^2+\gamma^2)} + \frac{\alpha\beta(\beta^2+\gamma^2-\delta^2)}{2b\delta(\beta^2+\gamma^2)}x + \frac{\alpha\gamma(\beta^2+\gamma^2-\delta^2)}{2b(\beta^2+\gamma^2)}y + \frac{\alpha\gamma(\beta^2+\gamma^2-\delta^2)i\sqrt{\beta^2+\gamma^2}}{2b\delta(\beta^2+\gamma^2)}z,$$

$$w = \frac{\alpha\gamma^2(\beta^2+\gamma^2-\delta^2)i\sqrt{\beta^2+\gamma^2}}{2b(\beta^2+\gamma^2)^2}y - \frac{\alpha\gamma^2(\beta^2+\gamma^2-\delta^2)}{2b\delta(\beta^2+\gamma^2)}z,$$

$$B = \begin{bmatrix} 0 & \omega & \varphi \\ -\omega & 0 & \gamma\left(y - \dfrac{\delta}{i\sqrt{\beta^2+\gamma^2}}z\right) \\ -\varphi & -\gamma\left(y - \dfrac{\delta}{i\sqrt{\beta^2+\gamma^2}}z\right) & 0 \end{bmatrix}, \tag{3.8}$$

where

$$\omega = \frac{-bi\sqrt{\beta^2+\gamma^2}(\beta^2+\gamma^2)(\beta^2+\gamma^2+\delta^2) + \beta\gamma(\beta^2+\gamma^2-\delta^2)\left((\beta^2+\gamma^2)y + i\delta\sqrt{\beta^2+\gamma^2}z\right)}{\gamma(\beta^2+\gamma^2)(\beta^2+\gamma^2-\delta^2)},$$

$$\varphi = -\frac{b\beta(\beta^2+\gamma^2+\delta^2)}{\gamma(\beta^2+\gamma^2-\delta^2)} - i\sqrt{\beta^2+\gamma^2}y + \delta z, \quad i = \sqrt{-1} \tag{3.9}$$

and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

Then, using MATHEMATICA 7.0, we can put the system (1.4) in the equivalent form

$$\dot{L} = [L, B] \tag{3.10}$$

as desired. □

The equilibrium points of the dynamics (1.4) are given by

$$e_1^M = (M, 0, 0), \quad M \in \mathbb{R},$$
$$e_2^M = (-b, M, -M), \quad M, b \in \mathbb{R}, \tag{3.11}$$
$$e_3^M = (b, M, M), \quad M, b \in \mathbb{R}.$$

About their stability, we have proven in [10] the following result,

Proposition 3.4 (Stability problem). *If $M > b$ or $M < -b, b > 0$, then the equilibrium states e_1^M are nonlinearly stable.*

As a consequence, we can find the periodical orbits of the equilibrium points e_1^M.

Proposition 3.5 (Periodical orbits). *If $M > b$, $b > 0$, the reduced dynamics to the coadjoint orbit $x = M$ has near the equilibrium point at least one periodic solution whose period is close to*

$$\frac{2\pi}{\sqrt{M^2 - b^2}}. \tag{3.12}$$

Proof. Indeed, we have successively

(i) the restriction of our dynamics (1.4) to the coadjoint orbit

$$x = M \tag{3.13}$$

gives rise to a classical Hamiltonian system,

(ii) the matrix of the linear part of the reduced dynamics has purely imaginary roots, more exactly

$$\lambda_1 = 0, \qquad \lambda_{2,3} = \pm i\sqrt{M^2 - b^2}. \tag{3.14}$$

(iii) $\operatorname{span}(\nabla C(e_1^M)) = V_0$, where

$$V_0 = \ker\left(A\left(e_1^M\right)\right), \tag{3.15}$$

(iv) if $M > b, b > 0$, then the reduced Hamiltonian has a local minimum at the equilibrium state e_1^M (see the proof of Proposition 3.4 [10]).

Then our assertion follows via the Moser-Weinstein theorem with zero eigenvalue, see [9] for details. □

4. Conclusion

The paper presents Hamilton-Poisson realizations of a dynamical system arising from electrical engineering; due to its chaotic behavior, finding the solution of the system could be very difficult. A Hamilton-Poisson realization offers us the possibility to find this solution as the intersection of two surfaces, the surfaces equation being given by the Hamiltonian and the Casimir of our configuration. The first paragraph of the paper presents the only four cases for which the Lü system admits as Hamiltonian a three degree polynomial function. Finding another kind of function as a Hamiltonian of the Lü system remains an open problem. The first case, $a \in \mathbb{R}$, $b, c = 0$ is the subject of the second paragraph. For this specific case, we have proved that a Hamilton-Poisson realization exists if and only if $a = 0$. Lax formulation, stability problems, and the existence of the periodical orbits are discussed, too. The third part of the paper analyses the case $a = 0$, $b, c \in \mathbb{R}$. We have proved that Hamilton-Poisson realization exists only if $b = c$. The last two cases, $a = 0$, $b = c$ and $a = b = c = 0$, can be found as the first studied cases. We can conclude that the Lü system admits Hamilton-Poisson realization with a three degree polynomial function as the Hamiltonian only if $a = 0$, $b = c \in \mathbb{R}$, or $a = b = c = 0$.

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