

Research Article

Almost Sure Stability and Stabilization for Hybrid Stochastic Systems with Time-Varying Delays

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The problems of almost sure (a.s.) stability and a.s. stabilization are investigated for hybrid stochastic systems (HSSs) with time-varying delays. The different time-varying delays in the drift part and in the diffusion part are considered. Based on nonnegative semimartingale convergence theorem, Hölder's inequality, Doob's martingale inequality, and Chebyshev's inequality, some sufficient conditions are proposed to guarantee that the underlying nonlinear hybrid stochastic delay systems (HSDSs) are almost surely (a.s.) stable. With these conditions, a.s. stabilization problem for a class of nonlinear HSDSs is addressed through designing linear state feedback controllers, which are obtained in terms of the solutions to a set of linear matrix inequalities (LMIs). Two numerical simulation examples are given to show the usefulness of the results derived.

1. Introduction

In the past decades, the problems of stability analysis and stabilization synthesis of stochastic systems have received significant attentions, and many results have been reported; see, for example [1–7] and the references therein. Commonly, the above problems can be solved not only in moment sense [8–10] but also in a.s. sense [11, 12]. However, in recent years, much interest has been focused on a.s. stability problems for stochastic systems; see, for example [8, 13] and the references therein.

It is well known that a lot of dynamical systems have variable structures subject to abrupt changes in their parameters, which are usually caused by abrupt phenomena such as component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances. The HSSs, which are regarded as the stochastic systems with

Markovian switching in this paper, have been used to model the previous phenomena; see, for example [14–18] and the references therein. The HSSs combine a part of the state $x(t)$ that takes values in \mathbb{R}^n continuously and another part of the state $r(t)$ that is a Markov chain taking discrete values in a finite space $S = \{1, 2, \dots, N\}$. One of the important issues in the study of HSSs is the analysis of stability. In particular, it is not necessary for the stable HSSs to require every subsystem to be stable; in other words, even all the subsystems are unstable; as the result of Markovian switching, the HSSs may be stable. These reveal that the Markovian jumps play an important role in the stability analysis of HSSs. Therefore, in the past few decades, a great deal of literature has appeared on the topic of stability analysis and stabilization synthesis of HSSs; see, for example [2, 13, 14, 19, 20].

On the other hand, time delays are frequently encountered in a variety of dynamic systems, such as nuclear reactors, chemical engineering systems, biological systems, and population dynamics models. They are often a source of instability and poor performance of systems. So the problems of stability analysis and stabilization synthesis of HSDSs have been of great importance and interest. The classical efforts can be classified into two categories, namely, moment sense criteria, see, for example [21–23], and a.s. sense criteria, see, for example [24, 25]. Among the existing results, in [25], based on the techniques proposed in [26] which were developed via the results of [11], a.s. stability and stabilization of HSDSs were studied. In [24], the a.s. stability analysis problem for a general class of HSDSs was derived from extending the results in [25] to HSSs with mode-dependent interval delays. However, to the author's best knowledge, when the different time-varying delays in the drift part and in the diffusion part are considered, the a.s. stability analysis and stabilization synthesis problems for nonlinear HSDSs have not been adequately addressed and remain an interesting and challenging research topic. This situation motivates the present study.

In this paper, we are concerned with a.s. stability analysis and stabilization synthesis problems for HSDSs. The purpose of stability is to develop conditions such that the underlying systems are a.s. stable. Following the same idea as in dealing with the stability problem, linear state feedback controllers are designed such that the special nonlinear or linear closed-loop systems are a.s. stable. The explicit expressions for the desired state feedback controllers are given by means of the solutions to a set of LMIs. Two numerical simulation examples are exploited to verify the effectiveness of the theoretical results. The main contribution of this paper is mainly twofold: (1) *the different time-varying delays in the drift part and in the diffusion part are considered for nonlinear HSDSs*; (2) *for a class of nonlinear HSDSs, the stabilization synthesis problem is investigated in the a.s. sense*.

This paper is organized as follows. In Section 2, we formulate some preliminaries. In Section 3, we investigate the a.s. stability for the hybrid stochastic systems with time-varying delays. In Section 4, the results of Section 3 are then applied to establish a sufficient criterion for the stabilization. In Section 5, two examples are discussed for illustration. Finally, conclusions are drawn in Section 6.

Notation 1. The notation used here is fairly standard unless otherwise specified. \mathbb{R}^n and $\mathbb{R}^{n \times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices, and let $\mathbb{R}_+ = [0, +\infty)$. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous, and \mathcal{F}_0 contains all \mathbb{P} -null sets). If x, y are real numbers, then $x \vee y$ stands for the maximum of x and y , and $x \wedge y$ the minimum of x and y . M^T represents the transpose of the matrix M . $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ denote the largest and smallest eigenvalue of M , respectively. $|\cdot|$ denotes the Euclidean norm in \mathbb{R}^n . $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation. $\mathbb{P}\{\cdot\}$ means the probability. $C([-\tau, 0]; \mathbb{R}^n)$

denotes the family of all continuous \mathbb{R}^n -valued function φ on $[-\tau, 0]$ with the norm $|\varphi| = \sup\{|\varphi(\theta)| : -\tau \leq \theta \leq 0\}$. $C_{\mathcal{F}_0}^b([-\tau, 0]; \mathbb{R}^n)$ being the family of all \mathcal{F}_0 -measurable bounded $C([-\tau, 0]; \mathbb{R}^n)$ -value random variables $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$. $L^1(\mathbb{R}_+; \mathbb{R}_+)$ denotes the family of functions $\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\int_0^\infty \lambda(t)dt < \infty$.

2. Problem Formulation

In this paper, let $r(t), t \geq 0$ be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \dots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$\mathbb{P}\{r(t + \Delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases} \quad (2.1)$$

where $\Delta > 0$ and $\gamma_{ij} \geq 0$ is the transition rate from mode i to mode j if $i \neq j$ while $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$. Assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. It is known that almost all sample paths of $r(\cdot)$ are right-continuous step functions with a finite number of simple jumps in any finite subinterval of $\mathbb{R}_+ := [0, \infty)$.

Let us consider a class of stochastic systems with time-varying delays:

$$dx(t) = f(x(t), x(t - \tau_1(t)), t, r(t))dt + g(x(t), x(t - \tau_2(t)), t, r(t))dB(t) \quad (2.2)$$

with initial data $x_0 = \{x(\theta) : -\tau \leq \theta \leq 0\} = \xi \in C_{\mathcal{F}_0}^b([-\tau, 0]; \mathbb{R}^n)$ and $r(0) = r_0 \in S$, where $\tau \triangleq \max\{\tau_1, \tau_2\}$, τ_1 and τ_2 are positive constant and $\tau_1(t)$ and $\tau_2(t)$ are nonnegative differential functions which denote the time-varying delays and satisfy

$$\begin{aligned} 0 \leq \tau_1(t) \leq \tau_1, \quad \dot{\tau}_1(t) \leq d_{\tau_1} < 1, \\ 0 \leq \tau_2(t) \leq \tau_2, \quad \dot{\tau}_2(t) \leq d_{\tau_2} < 1. \end{aligned} \quad (2.3)$$

The nonlinear functions $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \rightarrow \mathbb{R}^{n \times m}$ satisfy the local Lipschitz condition in (x, y, z) ; that is, for any $K > 0$, there is $L_K > 0$ such that

$$\begin{aligned} & |f(x, y, t, i) - f(\bar{x}, \bar{y}, t, i)| \vee |g(x, z, t, i) - g(\bar{x}, \bar{z}, t, i)| \\ & \leq L_K(|x - \bar{x}| + |y - \bar{y}| + |z - \bar{z}|), \end{aligned} \quad (2.4)$$

for all $|x| \vee |y| \vee |z| \vee |\bar{x}| \vee |\bar{y}| \vee |\bar{z}| \leq K$, $t \geq 0$ and $i \in S$, and moreover, $\sup_{t \geq 0, i \in S} \{|f(0, 0, t, i)| \vee |g(0, 0, t, i)| : t \geq 0, i \in S\} \leq K_0$ with some nonnegative number K_0 .

Remark 2.1. It should be pointed out that the systems (2.2) can be seen as the specialization of multiple time-varying delays systems which are of the form

$$\begin{aligned} dx(t) = & f(x(t), x(t - \tau_1(t)), x(t - \tau_2(t)), t, r(t))dt \\ & + g(x(t), x(t - \tau_1(t)), x(t - \tau_2(t)), t, r(t))dB(t). \end{aligned} \quad (2.5)$$

But it is easy to see that the results in this paper can be applied to the systems (2.5) by the similar assumption in (2.4).

Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ denote the family of all nonnegative functions $V(x, t, i)$ on $\mathbb{R}^n \times \mathbb{R}_+ \times S$ that are twice continuously differentiable in x and once in t . If $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$, define an operator \mathcal{L} associated with (2.2) from $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S$ to \mathbb{R} by

$$\begin{aligned} \mathcal{L}V(x, y, z, t, i) &= V_t(x, t, i) + V_x(x, t, i)f(x, y, t, i) \\ &+ \frac{1}{2}\text{trace}\left[g^T(x, z, t)V_{xx}(x, t, i)g(x, z, t, i)\right] + \sum_{j=1}^N \gamma_{ij}V(x, t, j). \end{aligned} \quad (2.6)$$

Remark 2.2. $\mathcal{L}V$ is thought as a single notation and is defined on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S$ while V is defined on $\mathbb{R}^n \times [-\tau, \infty) \times S$.

Definition 2.3. The system (2.2) is said to be a.s. stable if for all $\xi \in C_{\mathcal{F}_0}^b([-\tau, 0]; \mathbb{R}^n)$ and $r_0 \in S$

$$\mathbb{P}\left(\lim_{t \rightarrow \infty} x(t; \xi, r_0) = 0\right) = 1. \quad (2.7)$$

3. Main Results

Theorem 3.1. Assume that there exist nonnegative functions $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$, $\lambda \in L^1(\mathbb{R}_+; \mathbb{R}_+)$, $\omega_1, \omega_2, \omega_3 \in C(\mathbb{R}^n; \mathbb{R}_+)$ such that

$$\mathcal{L}V(x, y, z, t, i) \leq \lambda(t) - k_1\omega_1(x) + k_2\omega_2(y) + k_3\omega_3(z), \quad (3.1)$$

$$\forall(x, y, z, t, i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S,$$

$$\omega_1(x) > \omega_2(x) + \omega_3(x), \quad \forall x \neq 0, \quad (3.2)$$

$$\lim_{|x| \rightarrow \infty} \inf_{t \geq 0, i \in S} V(x, t, i) = \infty, \quad (3.3)$$

where k_1, k_2 and k_3 are positive numbers satisfying $k_1 \geq \max\{k_2/(1 - d_{\tau_1}), k_3/(1 - d_{\tau_2})\}$. Then system (2.2) is almost surely stable.

To prove this theorem, let us present the following lemmas.

Lemma 3.2 (see [24, 25]). If $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$, then for any $t \geq 0$, the generalized Itô's formula is given as

$$\begin{aligned} dV(x(t), t, r(t)) &= \mathcal{L}V(x(t), x(t - \tau_1(t)), x(t - \tau_2(t)), t, r(t))dt \\ &+ V_x(x(t), t, r(t))g(x(t), x(t - \tau_2(t)), t, r(t))dB(t) \\ &+ \int_{\mathbb{R}} [V(x(t), t, r(t) + l(r(t), \alpha)) - V(x(t), t, r(t))] \times \mu(dt, d\alpha), \end{aligned} \quad (3.4)$$

where function $l(\cdot, \cdot)$ and martingale measure $\mu(\cdot, \cdot)$ are defined as, for example, (2.6) and (2.7) in [25].

Lemma 3.3 (see [27]). Let $A_1(t)$ and $A_2(t)$ be two continuous adapted increasing processes on $t \geq 0$ with $A_1(0) = A_2(0) = 0$ a.s., let $M(t)$ be a real-valued continuous local martingale with $M(0) = 0$ a.s., and let ζ be a nonnegative \mathcal{F}_0 -measurable random variable such that $\mathbb{E}\zeta < \infty$. Denote $X(t) = \zeta + A_1(t) - A_2(t) + M(t)$ for all $t \geq 0$. If $X(t)$ is nonnegative, then

$$\left\{ \lim_{t \rightarrow \infty} A_1(t) < \infty \right\} \subset \left\{ \lim_{t \rightarrow \infty} X(t) < \infty \right\} \cap \left\{ \lim_{t \rightarrow \infty} A_2(t) < \infty \right\} \quad \text{a.s.}, \quad (3.5)$$

where $C \subset D$ a.s. means $\mathbb{P}(C \cap D^c = 0) = 0$. In particular, if $\lim_{t \rightarrow \infty} A_1(t) < \infty$ a.s., then,

$$\lim_{t \rightarrow \infty} X(t) < \infty, \quad \lim_{t \rightarrow \infty} A_2(t) < \infty, \quad -\infty < \lim_{t \rightarrow \infty} M(t) < \infty \quad \text{a.s.} \quad (3.6)$$

That is, all of the three processes $X(t)$, $A_2(t)$, and $M(t)$ converge to finite random variables with probability one.

Lemma 3.4 (see [25]). Under the conditions of Theorem 3.1, for any initial data $\{x(\theta) : -\tau \leq \theta \leq 0\} = \xi \in C_{\mathcal{F}_0}^b([- \tau, 0]; \mathbb{R}^n)$ and $r(0) = i_0 \in S$, (2.2) has a unique global solution.

Proof. Fix any initial data ξ , r_0 , and let β be the bound for ξ . For each integer $k \geq \beta$, define

$$f^{(k)}(x, y, t, i) = f\left(\frac{|x| \wedge k}{|x|}x, \frac{|y| \wedge k}{|y|}y, t, i\right), \quad (3.7)$$

where we set $(|x| \wedge k/|x|)x = 0$ when $x = 0$. Define $g^{(k)}(x, z, t, i)$ similarly. By (2.4), we can observe that $f^{(k)}$ and $g^{(k)}$ satisfy the global Lipschitz condition and the linear growth condition. By the known existence-and-uniqueness theorem, there exists a unique global solution $x_k(t)$ on $t \in [-\tau, \infty)$ to the equation

$$\begin{aligned} dx_k(t) &= f^{(k)}(x_k(t), x_k(t - \tau_1(t)), t, r(t))dt \\ &\quad + g^{(k)}(x_k(t), x_k(t - \tau_2(t)), t, r(t))dB(t) \end{aligned} \quad (3.8)$$

with initial data $\{x_k(\theta) : -\tau \leq \theta \leq 0\} = \xi$ and $r(0) = r_0$.

Define the stopping time

$$\sigma_k = \inf\{t \geq 0 : |x_k(t)| \geq k\}, \quad (3.9)$$

where we set $\inf \emptyset = \infty$ as usual. It is easy to show that $x_k(t) = x_{k+1}(t)$ if $0 \leq t \leq \sigma_k$, which implies that σ_k is increasing in k . Letting $\sigma = \lim_{k \rightarrow \infty} \sigma_k$, the property above also enables us to define $x(t)$ for $t \in [-\tau, \sigma)$ as $x(t) = x_k(t)$ if $-\tau \leq t \leq \sigma_k$.

It is clear that $x(t)$ is a unique solution of (2.2) for $t \in [-\tau, \sigma)$. To complete the proof, we only need to show $\mathbb{P}\{\sigma = \infty\} = 1$. By Lemma 3.2, we have that for any $t > 0$,

$$\begin{aligned} \mathbb{E}V(x_k(t \wedge \sigma_k), t \wedge \sigma_k, r(t \wedge \sigma_k)) &= \mathbb{E}V(x_k(0), 0, r(0)) \\ &\quad + \mathbb{E} \int_0^{t \wedge \sigma_k} \mathcal{L}^{(k)} V(x_k(s), x_k(s - \tau_1(s)), x_k(s - \tau_2(s)), s, r(s)) ds, \end{aligned} \quad (3.10)$$

where operator $\mathcal{L}^{(k)} V$ is defined similarly as $\mathcal{L} V$ was defined by (2.6). By the definitions of $f^{(k)}$ and $g^{(k)}$, if $0 \leq s \leq t \wedge \sigma_k$, we hence observe that

$$\begin{aligned} \mathcal{L}^{(k)} V(x_k(s), x_k(s - \tau_1(s)), x_k(s - \tau_2(s)), s, r(s)) \\ = \mathcal{L} V(x_k(s), x_k(s - \tau_1(s)), x_k(s - \tau_2(s)), s, r(s)). \end{aligned} \quad (3.11)$$

By the conditions of (3.1) and (3.2), we derive that

$$\begin{aligned} &\mathbb{E}V(x_k(t \wedge \sigma_k), t \wedge \sigma_k, r(t \wedge \sigma_k)) \\ &\leq V(\xi(0), 0, r_0) + \mathbb{E} \int_0^t [-k_1 \omega_1(x(s)) + k_2 \omega_2(x(s - \tau_1(s))) + k_3 \omega_3(x(s - \tau_2(s)))] ds \\ &\quad + \int_0^t \lambda(s) ds \\ &\leq V(\xi(0), 0, r_0) + \mathbb{E} \int_0^t -k_1 \omega_1(x(s)) ds + \mathbb{E} \int_{-\tau_1}^{t - \tau_1(t)} \left(\frac{k_2}{1 - d_{\tau_1}} \right) \omega_2(s) ds \\ &\quad + \mathbb{E} \int_{-\tau_2}^{t - \tau_2(t)} \left(\frac{k_3}{1 - d_{\tau_2}} \right) \omega_3(s) ds + \int_0^t \lambda(s) ds \\ &\leq V(\xi(0), 0, r_0) + \mathbb{E} \int_{-\tau}^0 k_1 [\omega_2(\xi(\theta)) + \omega_3(\xi(\theta))] d\theta \\ &\quad - \mathbb{E} \int_0^t k_1 (\omega_1(s) - \omega_2(s) - \omega_3(s)) ds + \int_0^t \lambda(s) ds \\ &\leq V(\xi(0), 0, r_0) + \mathbb{E} \int_{-\tau}^0 k_1 [\omega_2(\xi(\theta)) + \omega_3(\xi(\theta))] d\theta + \int_0^t \lambda(s) ds. \end{aligned} \quad (3.12)$$

On the other hand,

$$\begin{aligned} \mathbb{E}V(x_k(t \wedge \sigma_k), t \wedge \sigma_k, r(t \wedge \sigma_k)) &\geq \int_{\{\sigma_k \leq t\}} V(x_k(t \wedge \sigma_k), t \wedge \sigma_k, r(t \wedge \sigma_k)) d\mathbb{P} \\ &\geq \mathbb{P}\{\sigma_k \leq t\} \inf_{|x| \geq k, t \geq 0, i \in S} V(x, t, i). \end{aligned} \quad (3.13)$$

This yields

$$\mathbb{P}\{\sigma_k \leq t\} \leq \frac{V(\xi(0), 0, r_0) + \mathbb{E} \int_{-\tau}^0 k_1 [\omega_2(\xi(\theta)) + \omega_3(\xi(\theta))] d\theta + \int_0^t \lambda(s) ds}{\inf_{|x| \geq k, t \geq 0, i \in S} V(x, t, i)}. \quad (3.14)$$

Letting $k \rightarrow \infty$ and using (3.3), we obtain $\mathbb{P}(\sigma \leq t) = 0$. Since t is arbitrary, we must have $\mathbb{P}(\sigma = \infty) = 1$. The proof is therefore complete. \square

Let us now begin to prove our main result.

Proof. Let $\omega(x) = \omega_1(x) - \omega_2(x) - \omega_3(x)$ for all $x \in \mathbb{R}^n$. Inequality (3.2) implies $\omega(x) > 0$ whenever $x \neq 0$. Fix any initial value ξ and any initial state r_0 , and for simplicity write $x(t; \xi, r_0) = x(t)$.

By Lemma 3.2 and condition (3.1), we have

$$\begin{aligned} V(x(t), t, r(t)) &= V(\xi(0), 0, r_0) + \int_0^t \mathcal{L}V(x(s), x(s - \tau_1(s)), x(s - \tau_2(s)), s, r(s)) ds \\ &\quad + \int_0^t V_x(x(s), s, r(s)) g(x(s), x(s - \tau_2(s)), s, r(s)) dB(s) \\ &\quad + \int_0^t \int_{\mathbb{R}} [V(x(s), s, r_0 + l(r(s), \alpha)) - V(x(s), s, r(s))] \mu(ds, d\alpha) \\ &\leq V(\xi(0), 0, r_0) + \int_0^t \lambda(s) ds - \int_0^t k_1 \omega_1(x(s)) \\ &\quad + \int_0^t [k_2 \omega_2(x(s - \tau_1(s))) + k_3 \omega_3(x(s - \tau_2(s)))] ds \\ &\quad + \int_0^t V_x(x(s), s, r(s)) g(x(s), x(s - \tau_2(s)), s, r(s)) dB(s) \\ &\quad + \int_0^t \int_{\mathbb{R}} [V(x(s), s, r_0 + l(r(s), \alpha)) - V(x(s), s, r(s))] \mu(ds, d\alpha) \\ &\leq V(\xi(0), 0, r_0) + \int_0^t \lambda(s) ds + k_1 \int_{-\tau}^0 [\omega_2(x(s)) + \omega_3(x(s))] ds \\ &\quad - k_1 \int_0^t \omega(x(s)) ds + \int_0^t V_x(x(s), s, r(s)) g(x(s), x(s - \tau_2(s)), s, r(s)) dB(s) \\ &\quad + \int_0^t \int_{\mathbb{R}} [V(x(s), s, r_0 + l(r(s), \alpha)) - V(x(s), s, r(s))] \mu(ds, d\alpha). \end{aligned} \quad (3.15)$$

Since $\int_0^\infty \lambda(s)ds < \infty$, applying Lemma 3.3 we obtain that

$$\lim_{t \rightarrow \infty} \int_0^t \omega(x(s))ds = \int_0^\infty \omega(x(s))ds < \infty \quad \text{a.s.}, \quad (3.16)$$

$$\limsup_{t \rightarrow \infty} V(x(t), t, r(t)) < \infty \quad \text{a.s.} \quad (3.17)$$

Define $\beta : R_+ \rightarrow R_+$ as $\beta(r) = \inf_{|x| \geq r, 0 \leq t < \infty, i \in S} V(x, t, i)$. Then, it is obvious to see from (3.17) that

$$\sup_{0 \leq t < \infty} \beta(|x(t)|) \leq \sup_{0 \leq t < \infty} V(x(t), t, r(t)) < \infty \quad \text{a.s.} \quad (3.18)$$

On the other hand, by (3.3) we have $\sup_{0 \leq t < \infty} |x(t)| < \infty$ a.s.. It is easy to find an integer k_0 such that $|\xi| < k_0$ a.s. because of $\xi \in C_{\mathcal{F}_0}^b([-\tau, 0]; \mathcal{R}^n)$. Furthermore, for any integer $k > k_0$, we can define the stopping time

$$\rho_k = \inf\{t \geq 0 : |x(t)| \geq k\}, \quad (3.19)$$

where $\inf \emptyset = \infty$ as usual. Clearly, $\rho_k \rightarrow \infty$ a.s. as $k \rightarrow \infty$. Moreover, for any given $\varepsilon > 0$, there is $k_\varepsilon \geq k_0$ such that $\mathbb{P}\{\rho_k < \infty\} \leq \varepsilon$ for any $k \geq k_\varepsilon$.

It is straightforward to see from (3.16) that $\lim_{t \rightarrow \infty} \inf \omega(x(t)) = 0$ a.s.; then we claim that

$$\lim_{t \rightarrow \infty} \omega(x(t)) = 0 \quad \text{a.s.} \quad (3.20)$$

The rest of the proof is carried out by contradiction. That is, assuming that (3.20) is false, we have

$$\mathbb{P}\left\{\limsup_{t \rightarrow \infty} \omega(x(t)) > 0\right\} > 0. \quad (3.21)$$

Furthermore, there exist $\varepsilon_0 > 0$ and $\varepsilon > \varepsilon_1 > 0$ such that

$$\mathbb{P}(\sigma_{2j} < \infty : j \in Z) \geq \varepsilon_0, \quad (3.22)$$

where Z is a set of natural numbers and $\{\sigma_j\}_{j \geq 1}$ are a sequence of stopping times defined by

$$\begin{aligned} \sigma_1 &= \inf\{t \geq 0 : \omega(x(t)) \geq 2\varepsilon_1\}, \\ \sigma_{2j} &= \inf\{t \geq \sigma_{2j-1} : \omega(x(t)) \leq \varepsilon_1\}, \quad j = 1, 2, \dots, \\ \sigma_{2j+1} &= \inf\{t \geq \sigma_{2j} : \omega(x(t)) \leq 2\varepsilon_1\}, \quad j = 1, 2, \dots \end{aligned} \quad (3.23)$$

By the local Lipschitz condition (2.4), for any given $k > 0$, there exists $L_k > 0$ such that

$$|f(x, y, t, i)| \vee |g(x, z, t, i)| \leq L_k, \quad (3.24)$$

for all $|x| \vee |y| \vee |z| \leq k, t \geq 0$ and $i \in S$.

For any $j \in Z$, let $T < \sigma_{2j} - \sigma_{2j-1}$; by Hölder's inequality and Doob's martingale inequality, we compute

$$\begin{aligned} & \mathbb{E} \left\{ \mathbb{I}_{\{\sigma_{2j} < \rho_k\}} \sup_{0 \leq t \leq T} |x(\sigma_{2j-1} + t) - x(\sigma_{2j-1})|^2 \right\} \\ &= \mathbb{E} \left\{ \mathbb{I}_{\{\sigma_{2j} < \rho_k\}} \sup_{0 \leq t \leq T} \left| \int_{\sigma_{2j-1}}^{\sigma_{2j-1}+t} f(x(s), x(s - \tau_1(s)), s, r(s)) ds \right. \right. \\ & \quad \left. \left. + \int_{\sigma_{2j-1}}^{\sigma_{2j-1}+t} g(x(s), x(s - \tau_2(s)), s, r(s)) dB(s) \right|^2 \right\} \\ &\leq 2\mathbb{E} \left\{ \mathbb{I}_{\{\sigma_{2j} < \rho_k\}} \sup_{0 \leq t \leq T} \left| \int_{\sigma_{2j-1}}^{\sigma_{2j-1}+t} f(x(s), x(s - \tau_1(s)), s, r(s)) ds \right|^2 \right\} \\ & \quad + 8\mathbb{E} \left\{ \mathbb{I}_{\{\sigma_{2j} < \rho_k\}} \sup_{0 \leq t \leq T} \int_{\sigma_{2j-1}}^{\sigma_{2j-1}+t} |g(x(s), x(s - \tau_2(s)), s, r(s))|^2 ds \right\} \\ &\leq 2L_k^2 T(T + 4), \end{aligned} \quad (3.25)$$

where \mathbb{I}_A is the indicator of set A .

Since $\omega(x)$ is continuous in \mathbb{R}^n , it must be uniformly continuous in the closed ball $\bar{S}_k = \{x \in \mathbb{R}^n : |x| \leq k\}$. For any given $b > 0$, we can choose $c_b > 0$ such that $|\omega(x) - \omega(y)| < b$ whenever $x, y \in \bar{S}_k$ and $|x - y| < c_b$. Furthermore, let us choose

$$\varepsilon = \frac{\varepsilon_0}{3}, \quad k \geq k_\varepsilon, \quad b = \varepsilon_1. \quad (3.26)$$

By inequality (3.25) and Chebyshev's inequality, we have

$$\begin{aligned} & \mathbb{P}(\{\rho_k \leq \sigma_{2j}\}) + \mathbb{P} \left(\{\sigma_{2j} < \rho_k\} \cap \left\{ \sup_{0 \leq t \leq T} |\omega(x(\sigma_{2j-1} + t)) - \omega(x(\sigma_{2j-1}))| \geq \varepsilon_1 \right\} \right) \\ &\leq \mathbb{P}(\{\rho_k \leq \sigma_{2j}\} \cap \{\sigma_{2j} = \infty\}) + \mathbb{P}(\{\rho_k \leq \sigma_{2j}\} \cap \{\sigma_{2j} < \infty\}) \\ & \quad + \mathbb{P} \left(\{\sigma_{2j} < \rho_k\} \cap \left\{ \sup_{0 \leq t \leq T} |x(\sigma_{2j-1} + t) - x(\sigma_{2j-1})| \geq c_{\varepsilon_1} \right\} \right) \\ &\leq \frac{2L_k^2 T(T + 4)}{c_{\varepsilon_1}^2} + 1 - 2\varepsilon. \end{aligned} \quad (3.27)$$

Meanwhile, we can also choose $T = T(\varepsilon, \varepsilon_1, k)$ sufficiently small for

$$\frac{2L_k^2 T(T+4)}{c_{\varepsilon_1}^2} \leq \varepsilon. \quad (3.28)$$

And then, (3.27) and (3.28) yield

$$\mathbb{P}(\{\sigma_{2j} < \rho_k\} \cap \Omega_j) \geq \varepsilon, \quad (3.29)$$

where $\Omega_j = \{\sup_{0 \leq t \leq T} |\omega(x(\sigma_{2j-1} + t)) - \omega(x(\sigma_{2j-1}))| < \varepsilon_1\}$.

In the following, we can obtain from (3.16) and (3.29) that

$$\begin{aligned} \infty &> \mathbb{E} \int_0^\infty \omega(x(t)) dt \\ &\geq \sum_{j=1}^\infty \mathbb{E} \left[\mathbb{I}_{\{\sigma_{2j} < \rho_k\}} \int_{\sigma_{2j-1}}^{\sigma_{2j}} \omega(x(t)) dt \right] \\ &\geq \sum_{j=1}^\infty \varepsilon_1 \mathbb{E} \left[\mathbb{I}_{\{\sigma_{2j} < \rho_k\}} (\sigma_{2j} - \sigma_{2j-1}) \right] \\ &\geq \sum_{j=1}^\infty T \varepsilon_1 \mathbb{P}(\{\sigma_{2j} < \rho_k\} \cap \Omega_j) \\ &\geq \sum_{j=1}^\infty T \varepsilon_1 \varepsilon = \frac{1}{3} \sum_{j=1}^\infty T \varepsilon_0 \varepsilon_1 = \infty. \end{aligned} \quad (3.30)$$

This is a contradiction. So there is an $\bar{\Omega} \in \Omega$ with $\mathbb{P}(\bar{\Omega}) = 1$ such that

$$\lim_{t \rightarrow \infty} \omega(x(t, \omega)) = 0, \quad \sup_{0 \leq t < \infty} |x(t, \omega)| < \infty, \quad \forall \omega \in \bar{\Omega}. \quad (3.31)$$

Finally, any fixed $\omega \in \bar{\Omega}$, $\{x(t, \omega)\}_{t \geq 0}$ is bounded in \mathbb{R}^n . By Bolzano-Weierstrass theorem, there is an increasing sequence $\{t_i\}_{i \geq 1}$ such that $\{x(t, \omega)\}_{i \geq 1}$ converges to some $z \in \mathbb{R}^n$ with $|z| < \infty$. Since $\omega(x) > 0$ whenever $x \neq 0$, we must have $\omega(x) = 0$ if and only if $x = 0$. This implies that the solution of (2.2) is a.s. stable, and the proof is therefore completed. \square

Remark 3.5. The techniques proposed in Theorem 3.1 can be used to deal with the a.s. stability problem for other HSDSs, such as the ones in [25]. In a very special case when $\tau_1(t) = \tau_2(t) = \tau$ for all $t \geq 0$ and $i \in S$, it is easy to see that $\hat{\tau}_1(t) = \hat{\tau}_2(t) = 0$, and Theorem 3.1 is exactly Theorem 2.1 in [25]. Similarly, Theorem 2.2 in [25] can be generalized to system (2.2) as a LaSalle-type theorem (see [24, 26]) for HSSs with multiple time-varying delays.

4. Almost Sure Stabilization of Nonlinear HSDSs

Consider the following nonlinear HSDSs:

$$\begin{aligned} dx(t) = & [A(r(t))x(t) + A_d(r(t))x(t - \tau_1(t)) + f(x(t), x(t - \tau_1(t)), t, r(t)) + B_u(r(t))u(t)]dt \\ & + g(x(t), x(t - \tau_2(t)), t, r(t))dB(t), \end{aligned} \quad (4.1)$$

where $B_u(r(t))$ are known constant matrices with appropriate dimensions and $B(t)$ represents a scalar Brownian motion (Wiener process) on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ that is independent of Markov chain $r(t)$ and satisfies:

$$E\{dB(t)\} = 0, \quad E\{dB(t)^2\} = dt, \quad (4.2)$$

f and g are both functions from $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S$ to \mathbb{R}^n which satisfy local Lipschitz condition and the following assumptions:

$$\begin{aligned} & |f(x(t), x(t - \tau_1(t)), t, r(t))|^2 \\ & \leq x^T(t)F_1(r(t))x(t) + x^T(t - \tau_1(t))F_2(r(t))x(t - \tau_1(t)), \\ & |g(x(t), x(t - \tau_2(t)), t, r(t))|^2 \\ & \leq x^T(t)G_1(r(t))x(t) + x^T(t - \tau_2(t))G_2(r(t))x(t - \tau_2(t)), \end{aligned} \quad (4.3)$$

where, for each $r(t) = j \in S$, $A(r(t))$, $A_d(r(t))$ are known constant matrices with appropriate dimensions, and $F_i(r(t)) \in \mathbb{R}^{n \times n}$, $G_i(r(t)) \in \mathbb{R}^{n \times n}$ ($i = 1, 2$) are positive definite matrices.

In the sequel, we denote the matrix associated with the i th mode by

$$\Gamma_i \triangleq \Gamma(r(t) = i), \quad (4.4)$$

where the matrix Γ could be A , A_d , B_u , F_1 , F_2 , G_1 , G_2 , G , or G_d .

As the given HSDSs (4.1) is nonlinear, we here consider the resulting systems can be stabilized only by linear state feedback controller which is of the form

$$u(t) = K(r(t))x(t), \quad (4.5)$$

where $K(r(t))$ are controller parameters to be designed.

Under control law (4.5), the closed-loop system can be given as follow:

$$\begin{aligned} dx(t) = & [A(r(t))x(t) + A_d(r(t))x(t - \tau_1(t)) + f(x(t), x(t - \tau_1(t)), t, r(t)) \\ & + B_u(r(t))K(r(t))x(t)]dt \\ & + g(x(t), x(t - \tau_2(t)), t, r(t))dB(t). \end{aligned} \quad (4.6)$$

The stabilization problem is therefore to design matrices $K(r(t))$ for the closed-loop system (4.6) to be a.s. stable. In order to guarantee the solvability of $K(r(t))$, the following theorem is given.

Theorem 4.1. *If there exist sequences of scalars $\varepsilon_{1i} > 0$, $\varepsilon_{2i} > 0$, $\delta_i > 0$, positive definite matrices $X_i > 0$ and matrices Y_i such that the following LMIs*

$$\begin{bmatrix} M_{i1} & M_{i2} & M_{i4} \\ * & -M_{i3} & 0 \\ * & * & -M_{i5} \end{bmatrix} < 0 \quad \forall i, j \in S, \quad (4.7)$$

$$X_i \geq \delta_i I \quad (4.8)$$

hold, where

$$\begin{aligned} M_{i1} &= A_i X_i + X_i A_i^T + B_{ui} Y_i + Y_i^T B_{ui}^T + \varepsilon_{1i} A_{di} A_{di}^T + \varepsilon_{2i} I + \gamma_{ii} X_i, \\ M_{i2} &= [X_i, X_i, X_i, X_i, X_i], \\ M_{i3} &= \text{diag}(\varepsilon_{2i} F_{1i}^{-1}, c_1 \varepsilon_{2j} F_{2j}^{-1}, \delta_i G_{1i}^{-1}, c_2 \delta_i G_{2j}^{-1}, c_1 \varepsilon_{1j} I), \\ M_{i4} &= [\sqrt{\gamma_{i1}} X_i, \dots, \sqrt{\gamma_{i(i-1)}} X_i, \sqrt{\gamma_{i(i+1)}} X_i, \dots, \sqrt{\gamma_{iN}} X_i], \\ M_{i5} &= \text{diag}(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_N), \\ c_1 &= 1 - d_{\tau_1}, \quad c_2 = 1 - d_{\tau_2}, \end{aligned} \quad (4.9)$$

then the controlled system (4.6) is a.s. stable and the state feedback controller determined by

$$u(t) = K_i x(t), \quad K_i = Y_i X_i^{-1}, \quad i \in S. \quad (4.10)$$

Proof. Let $P_i = X_i^{-1}$ and $V(x, i) = x^T P_i x + \int_{t-\tau_1(t)}^t x^T(s) Q_1 x(s) ds + \int_{t-\tau_2(t)}^t x^T(s) Q_2 x(s) ds$. The operator $\mathcal{L}V : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times S \rightarrow \mathbb{R}$ has the form

$$\begin{aligned} \mathcal{L}V(x, y, z, i) &= x^T Q_1 x - (1 - \dot{\tau}_1(t)) y^T Q_1 y + x^T Q_2 x - (1 - \dot{\tau}_2(t)) z^T Q_2 z \\ &\quad + 2x^T P_i (A_i x + A_{di} y + f(x, y, i) + B_{ui} K_i x) \\ &\quad + g^T(x, z, i) P_i g(x, z, i) + \sum_{j=1}^N \gamma_{ij} x^T P_j x \end{aligned}$$

$$\begin{aligned}
&\leq x^T \left[Q_1 + Q_2 + P_i A_i + A_i^T P_i + P_i B_{ui} K_i + (B_{ui} K_i)^T P_i + \varepsilon_{1i} P_i A_{di} A_{di}^T P_i \right. \\
&\quad \left. + \varepsilon_{2i} P_i^2 + \sum_{j=1}^N \gamma_{ij} P_j + \varepsilon_{2i}^{-1} F_{1i} + \delta_i^{-1} G_{1i} \right] x \\
&\quad + y^T \left[\varepsilon_{1i}^{-1} I + \varepsilon_{2i}^{-1} F_{2i} - (1 - d_{\tau_1}) Q_1 \right] y + z^T \left[\delta_i^{-1} G_{2i} - (1 - d_{\tau_2}) Q_2 \right] z.
\end{aligned} \tag{4.11}$$

So

$$\mathcal{L}V(x, y, z, i) \leq -\omega_{1i}(x) + (1 - d_{\tau_1})\omega_{2i}(y) + (1 - d_{\tau_2})\omega_{3i}(z), \tag{4.12}$$

where

$$\begin{aligned}
\omega_{1i}(x) &= x^T \left[-Q_1 - Q_2 - P_i A_i - A_i^T P_i - P_i B_{ui} K_i - (B_{ui} K_i)^T P_i - \varepsilon_{1i} P_i A_{di} A_{di}^T P_i \right. \\
&\quad \left. - \varepsilon_{2i} P_i^2 - \varepsilon_{2i}^{-1} F_{1i} - \delta_i^{-1} G_{1i} - \sum_{k=1}^N \gamma_{ik} P_k \right] x, \\
\omega_{2i}(x) &= x^T \left[c_1^{-1} \varepsilon_{1i}^{-1} I + c_1^{-1} \varepsilon_{2i}^{-1} F_{2i} - Q_1 \right] x, \\
\omega_{3i}(x) &= x^T \left[c_2^{-1} \delta_i^{-1} G_{2i} - Q_2 \right] x.
\end{aligned} \tag{4.13}$$

By assumption 1, it is easy to see that we can choose Q_1 and Q_2 such that $\omega_{2i}(x) \geq 0$, $\omega_{3i}(x) \geq 0$ for all $x \in \mathbb{R}^n$, $i \in S$.

Noting that $P_i = X_i^{-1}$ and $Y_i = K_i X_i$, we can pre- and postmultiply (4.7) by $\text{diag}(P_i, \dots, P_i)$, and using Schur complements, we can obtain

$$\Phi_{ij} < 0, \tag{4.14}$$

where

$$\begin{aligned}
\Phi_{ij} &= P_i A_i + A_i^T P_i + P_i B_{ui} K_i + (B_{ui} K_i)^T P_i + \varepsilon_{1i} P_i A_{di} A_{di}^T P_i + \varepsilon_{2i} P_i^2 + \delta_i^{-1} G_{1i} \\
&\quad + \varepsilon_{2i}^{-1} F_{1i} + \sum_{k=1}^N \gamma_{ik} P_k + c_1^{-1} \varepsilon_{1j}^{-1} I + c_1^{-1} \varepsilon_{2j}^{-1} F_{2j} + c_2^{-1} \delta_j G_{2j}.
\end{aligned} \tag{4.15}$$

This implies

$$\omega_{1i}(x) > \omega_{2j}(x) + \omega_{3j}(x) \geq 0, \quad \forall x \neq 0. \tag{4.16}$$

Let $\omega_1(x) = \min_{i \in S} \omega_{1i}(x)$, $\omega_2(x) = \max_{i \in S} \omega_{2i}(x)$, and $\omega_3(x) = \max_{i \in S} \omega_{3i}(x)$.

Clearly

$$\omega_1(x) > \omega_2(x) + \omega_3(x) \geq 0, \quad \forall x \neq 0. \quad (4.17)$$

Moreover, by (4.24) we further obtain

$$\mathcal{L}V(x, y, z, i) \leq -\omega_1(x) + (1 - d_{\tau_1})\omega_2(y) + (1 - d_{\tau_2})\omega_3(z). \quad (4.18)$$

The required assertion now follows from Theorem 3.1. \square

If the systems (4.6) reduces to linear HSDSs of the form

$$\begin{aligned} dx(t) = & [A(r(t))x(t) + A_d(r(t))x(t - \tau_1(t)) \\ & + B_u(r(t))K(r(t))x(t)]dt + [G(r(t))x(t) + G_d(r(t))x(t - \tau_2(t))]dB(t), \end{aligned} \quad (4.19)$$

where $A(r(t)), A_d(r(t)), B_u(r(t)), G(r(t)),$ and $G_d(r(t))$ are known constant matrices with appropriate dimensions.

Then, the following corollary follows directly from Theorem 4.1.

Corollary 4.2. *If there exist sequences of scalars $\varepsilon_{1i} > 0, \varepsilon_{2i} > 0,$ positive definite matrices $X_i > 0$ and matrices Y_i such that the following LMIs*

$$\begin{bmatrix} M_{i1} & M_{i2} & M_{i4} \\ * & -M_{i3} & 0 \\ * & * & -M_{i5} \end{bmatrix} < 0 \quad \forall i, j \in S \quad (4.20)$$

hold, where

$$\begin{aligned} M_{i1} &= A_i X_i + X_i A_i^T + B_{ui} Y_i + Y_i^T B_{ui}^T + \varepsilon_{1i} A_{di} A_{di}^T + \gamma_{ii} X_i, \\ M_{i2} &= [\sqrt{2} X_i G_i^T, X_j, X_j, \sqrt{2} X_j^T G_{dj}], \\ M_{i3} &= \text{diag}(X_i, c_1 \varepsilon_{1j} I, \varepsilon_{2j} I, X_j), \\ M_{i4} &= [\sqrt{\gamma_{i1}} X_i, \dots, \sqrt{\gamma_{i(i-1)}} X_i, \sqrt{\gamma_{i(i+1)}} X_i, \dots, \sqrt{\gamma_{iN}} X_i], \\ M_{i5} &= \text{diag}(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_N), \\ c_1 &= 1 - d_{\tau_1}, \quad c_2 = 1 - d_{\tau_2}, \end{aligned} \quad (4.21)$$

then the controlled system (4.19) is a.s. stable and the state feedback controller determined by

$$u(t) = K_i x(t), \quad K_i = Y_i X_i^{-1}, \quad i \in S. \quad (4.22)$$

Proof. Let $P_i = X_i^{-1}$ and $V(x, i) = x^T P_i x + \int_{t-\tau_1(t)}^t x^T(s) Q_1 x(s) ds + \int_{t-\tau_2(t)}^t x^T(s) Q_2 x(s) ds$.
The operator $\mathcal{L}V : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times S \rightarrow \mathbb{R}$ has the form

$$\begin{aligned}
\mathcal{L}V(x, y, z, i) &= x^T Q_1 x - (1 - \dot{\tau}_1(t)) y^T Q_1 y + x^T Q_2 x - (1 - \dot{\tau}_2(t)) z^T Q_2 z \\
&\quad + 2x^T P_i [A_i x + A_{di} y + B_{ui} K_i x] + [G_i x + G_{di} z]^T P_i [G_i x + G_{di} z] \\
&\quad + \sum_{k=1}^N \gamma_{ik} x^T P_k x \\
&\leq x^T \left[Q_1 + Q_2 + P_i A_i + A_i^T P_i + P_i B_{ui} K_i + (B_{ui} K_i)^T P_i \right. \\
&\quad \left. + \varepsilon_{1i} P_i A_{di} A_{di}^T P_i + \sum_{k=1}^N \gamma_{ik} P_k + 2G_i^T P_i G_i \right] x \\
&\quad + y^T \left[\varepsilon_{1i}^{-1} I - (1 - d_{\tau_1}) Q_1 \right] y + z^T \left[\varepsilon_{2i}^{-1} I + 2G_{di}^T P_i G_{di} - (1 - d_{\tau_2}) Q_2 \right] z.
\end{aligned} \tag{4.23}$$

So

$$\mathcal{L}V(x, y, z, i) \leq -\omega_{1i}(x) + (1 - d_{\tau_1})\omega_{2i}(y) + (1 - d_{\tau_2})\omega_{3i}(z), \tag{4.24}$$

where

$$\begin{aligned}
\omega_{1i}(x) &= x^T \left[-Q_1 - Q_2 - P_i A_i - A_i^T P_i - P_i B_{ui} K_i - (B_{ui} K_i)^T P_i \right. \\
&\quad \left. - \varepsilon_{1i} P_i A_{di} A_{di}^T P_i - \sum_{k=1}^N \gamma_{ik} P_k - 2G_i^T P_i G_i \right] x, \\
\omega_{2i}(x) &= x^T \left[c_1^{-1} \varepsilon_{1i}^{-1} I - Q_1 \right] x, \\
\omega_{3i}(x) &= x^T \left[\varepsilon_{2i}^{-1} I + 2c_2^{-1} G_{di}^T P_i G_{di} - Q_2 \right] x.
\end{aligned} \tag{4.25}$$

It is easy to see that we can choose Q_1 and Q_2 such that $\omega_{2i}(x) \geq 0$, $\omega_{3i}(x) \geq 0$ for all $x \in \mathbb{R}^n$, $i \in S$.

Noting that $P_i = X_i^{-1}$ and $Y_i = K_i X_i$, we can pre- and postmultiply (4.7) by $\text{diag}(P_i, \dots, P_i)$, and using Schur complements, we can obtain

$$\Phi_{ij} < 0, \tag{4.26}$$

where

$$\begin{aligned} \Phi_{ij} = & P_i A_i + A_i^T P_i + P_i B_{ui} K_i + (B_{ui} K_i)^T P_i + \varepsilon_{1i} P_i A_{di} A_{di}^T P_i \\ & + \sum_{k=1}^N \gamma_{ik} P_k + 2G_i^T P_i G_i + c_1^{-1} \varepsilon_{1j}^{-1} I + \varepsilon_{2j}^{-1} I + 2c_2^{-1} G_{dj}^T P_j G_{dj}. \end{aligned} \quad (4.27)$$

This implies

$$\omega_{1i}(x) > \omega_{2j}(x) + \omega_{3j}(x) \geq 0, \quad \forall x \neq 0. \quad (4.28)$$

Let $\omega_1(x) = \min_{i \in S} \omega_{1i}(x)$, $\omega_2(x) = \max_{i \in S} \omega_{2i}(x)$, and $\omega_3(x) = \max_{i \in S} \omega_{3i}(x)$.
Clearly

$$\omega_1(x) > \omega_2(x) + \omega_3(x) \geq 0, \quad \forall x \neq 0. \quad (4.29)$$

Moreover, by (4.24) we further obtain

$$\mathcal{L}V(x, y, z, i) \leq -\omega_1(x) + (1 - d_{\tau_1})\omega_2(y) + (1 - d_{\tau_2})\omega_3(z). \quad (4.30)$$

The required assertion now follows from Theorem 3.1. \square

5. Examples

In this section we will provide two examples to illustrate our results. In the following examples we assume that $B(t)$ is a scalar Brownian motion, $\gamma(t)$ is a right-continuous Markov chain independent of $B(t)$ and taking values in $S = \{1, 2\}$, and the step size $\Delta = 0.0001$. By using the YALMIP toolbox, simulations results are shown in Figures 1–3. Figure 1 gives a portion of state $\gamma(t)$ of Example 5.1 for clear display. Figure 2 simulates the numerical results for Example 5.1. The simulation results have illustrated our theoretical analysis. Following from Theorem 4.1, the simulation results for Example 5.2 can be founded in Figure 3, which verify our desired results.

Example 5.1. Let

$$\Gamma = (\gamma_{ij})_{2 \times 2} = \begin{pmatrix} -0.8 & 0.8 \\ 0.3 & -0.3 \end{pmatrix}. \quad (5.1)$$

Consider scalar nonlinear HSDSs:

$$dx(t) = f(x(t), t, r(t))dt + g(x(t), x(t - \tau_2(t)), t, r(t))dB(t), \quad (5.2)$$

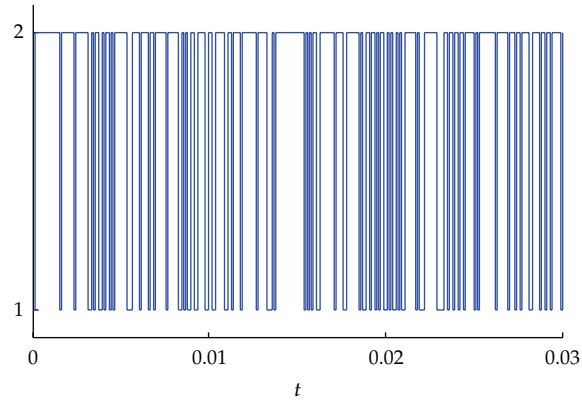


Figure 1: The state $\gamma(t)$ of Example 5.1.

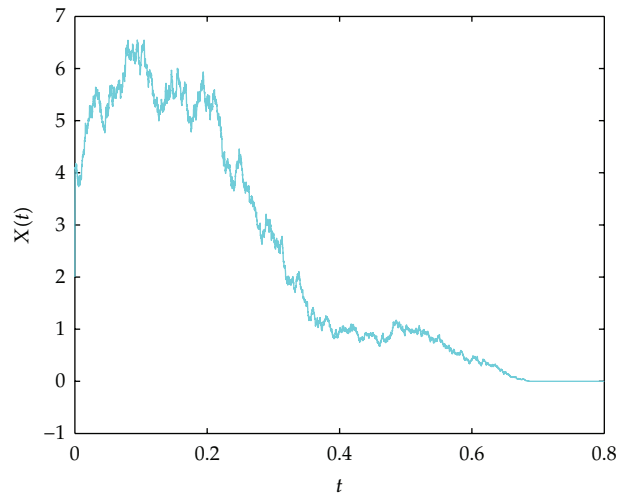


Figure 2: The state evolution of Example 5.1.

where

$$\begin{aligned}
 f(x, t, 1) &= -6\sqrt[5]{x}, \\
 g(x, z, t, 1) &= -\sqrt[5]{x^3} + 2\sqrt[5]{z^3}, \\
 f(x, t, 2) &= \frac{3}{2\sqrt[3]{1+t}} - 4\sqrt[5]{x}, \\
 g(x, z, t, 2) &= \sqrt[5]{x^3} \cos(t) + \frac{5}{4}\sqrt[5]{z^3} \sin(t),
 \end{aligned} \tag{5.3}$$

$$\tau_2(t) = 0.3 + 0.3 \sin(t).$$

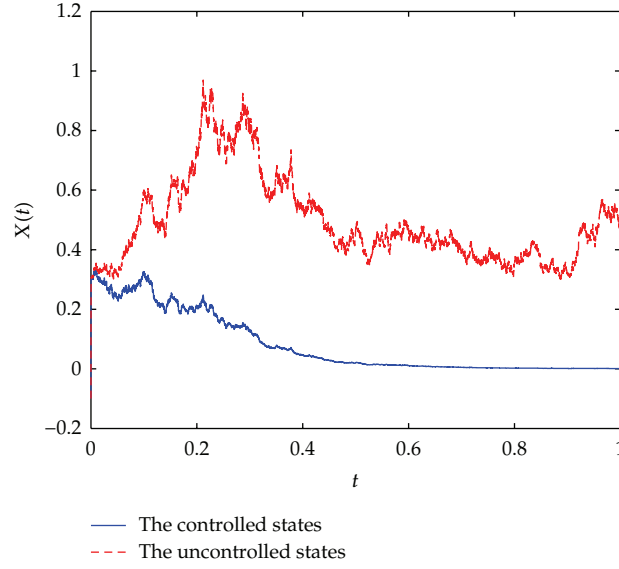


Figure 3: The state evolution of Example 5.2.

To examine the stability of system (5.2), we consider a Lyapunov function candidate $V : \mathbb{R} \times S \rightarrow \mathbb{R}_+$ as $V(x, i) = x^2$ for $i = 1, 2$. Then we have

$$\begin{aligned} \mathcal{L}V(x, z, t, 1) &\leq -10x^{6/5} + 4z^{6/5}, \\ \mathcal{L}V(x, z, t, 2) &\leq \frac{3x}{\sqrt[3]{1+t}} - 6x^{6/5} + \frac{25}{8}z^{6/5}. \end{aligned} \quad (5.4)$$

By the elementary inequality $\alpha^c \beta^{1-c} \leq c\alpha + (1-c)\beta$ for all $\alpha \geq 0$, $\beta \geq 0$, and $0 \leq c \leq 1$, we see that inequality

$$\frac{3x}{\sqrt[3]{1+t}} = \left(\frac{6}{5}\kappa x^{6/5}\right)^{6/5} \left(6\left(\frac{\kappa}{5}\right)^{-5} (1+t)^{-2}\right)^{1/6} \leq \kappa x^{6/5} + \frac{\kappa_1}{(1+t)^2} \quad (5.5)$$

holds for any $\kappa > 0$, where $\kappa_1 = (\kappa/5)^{-5}$.

From inequalities (5.4)–(5.5), we have

$$\mathcal{L}V(x, z, t, i) \leq \frac{\kappa_1}{(1+t)^2} - (6-\kappa)x^{6/5} + 4z^{6/5}, \quad (5.6)$$

for all $t \geq 0$ and $i \in S$. By $\tau_2(t) = 0.3 + 0.3 \sin(t)$, it is easy to see that $d_{\tau_2}(t) < 1/3$; then, we choose constant κ such that $0 < \kappa < (2 - 6d_{\tau_2})/(1 - d_{\tau_2})$, and hence conditions of Theorem 3.1 are satisfied.

Example 5.2. Let

$$\Gamma = (\gamma_{ij})_{2 \times 2} = \begin{pmatrix} -0.6 & 0.6 \\ 0.5 & -0.5 \end{pmatrix}. \quad (5.7)$$

Consider scalar nonlinear closed-loop HSDSs:

$$\begin{aligned} dx(t) = & [f(x(t), x(t - \tau_1(t)), t, r(t)) + B(r(t))K(r(t))x(t)]dt \\ & + g(x(t), x(t - \tau_2(t)), t, r(t))dB(t) \end{aligned} \quad (5.8)$$

with

$$f(x, y, t, 1) = x + \frac{1}{2}y + \frac{2x^3}{(|x| + 1)^2} + y \sin(t), \quad (5.9)$$

$$g(x, z, t, 1) = x \cos(t) + \frac{z^3}{(|z| + 1)^2},$$

$$f(x, y, t, 2) = -2x + y + \frac{x^3}{(|x| + 2)^2} + \frac{y^3}{(|y| + 1)^2}, \quad (5.10)$$

$$g(x, z, t, 2) = 2x \sin(t) + \frac{x^3}{2(|x| + 1)^2} + \frac{z^3}{(|z| + 2)^2},$$

$\tau_1(t) = 0.1 + 0.1 \sin(t)$, $\tau_2(t) = 0.2 + 0.2 \sin(2t)$, $B_1 = 2$, $B_2 = -3$, $A_1 = 1$, $A_2 = 2$, $A_{d1} = 1/2$, $A_{d2} = 1$, $F_{11} = 8$, $F_{12} = G_{11} = 2$, $G_{12} = F_{21} = F_{22} = 2$, $G_{21} = 1/2$, $G_{22} = 2$.

By Theorem 4.1 we can find the feasible solution $K_1 = -3$, $K_2 = 2$ for the a.s. stability.

6. Conclusions

In this paper, we have investigated the a.s. stability analysis and stabilization synthesis problems for nonlinear HSDSs. Some sufficient conditions are given to guarantee the resulting systems to be a.s. stable. Under these conditions, a.s. stabilization problem for a class of nonlinear HSDSs is solved in terms of the solutions to a set of LMIs. Finally, the results of this paper have been demonstrated by two numerical simulation examples.

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