

## Research Article

# $H_\infty$ Neural-Network-Based Discrete-Time Fuzzy Control of Continuous-Time Nonlinear Systems with Dither

Zhi-Ren Tsai<sup>1</sup> and Jiing-Dong Hwang<sup>2</sup>

<sup>1</sup> Department of Computer Science and Information Engineering, Asia University, No. 500 Lioufeng Road, Wufeng Distinguish, Taichung City 41354, Taiwan

<sup>2</sup> Department of Electronic Engineering, Jinwen University of Science and Technology, No. 99 Anchung Road, Xindian Distinguish, New Taipei City 23154, Taiwan

Correspondence should be addressed to Jiing-Dong Hwang, ditherman@just.edu.tw

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This study presents an effective approach to stabilizing a continuous-time (CT) nonlinear system using dithers and a discrete-time (DT) fuzzy controller. A CT nonlinear system is first discretized to a DT nonlinear system. Then, a Neural-Network (NN) system is established to approximate a DT nonlinear system. Next, a Linear Difference Inclusion state-space representation is established for the dynamics of the NN system. Subsequently, a Takagi-Sugeno DT fuzzy controller is designed to stabilize this NN system. If the DT fuzzy controller cannot stabilize the NN system, a dither, as an auxiliary of the controller, is simultaneously introduced to stabilize the closed-loop CT nonlinear system by using the Simplex optimization and the linear matrix inequality method. This dither can be injected into the original CT nonlinear system by the proposed injecting procedure, and this NN system is established to approximate this dithered system. When the discretized frequency or sampling frequency of the CT system is sufficiently high, the DT system can maintain the dynamic of the CT system. We can design the sampling frequency, so the trajectory of the DT system and the relaxed CT system can be made as close as desired.

## 1. Introduction

During the past decade, fuzzy control [1, 2] has attracted great attention from both the academic and industrial communities, and there have been many successful applications. Despite this success, it has become evident that many basic and important issues [3] remain to be further addressed. These stability analysis and systematic designs are among the most important issues for fuzzy control systems [4] and  $H_\infty$  control theories [1, 2, 5–10], and there has been significant research on these issues (see [4, 11, 12]). In addition, fuzzy

controller has been suggested as an alternative approach to conventional control techniques for complex control systems [1, 2]. Moreover, Neural-Network- (NN-) based modeling has become an active research field because of its unique merits in solving complex nonlinear system identification and control problems (see [12]). Neural networks (NNs) are composed of simple elements operating in parallel, inspired by biological nervous systems. As a result, we can train a neural network to represent a particular function by adjusting the weights between elements [13]. As the discrete-time (DT) controller is cheaper and more flexible than continuous-time (CT) controller, the DT control problem for CT plant is worth studying in this paper. However, an NN-model-based design method with dither has not yet been developed to adjust the parameters of a discrete-time (DT) fuzzy controller such that the original continuous-time (CT) system is uniformly ultimately bounded (UUB) stable.

Therefore, to solve this problem, this paper proposes a less conservative DT control design methodology for a CT nonlinear system with dither based on using an NN model, then these problems of the systematic control design are overcome using the simplex optimization [14] and the LMI method [3, 11]. Our design approach is to approximate a DT nonlinear system with a multilayer perceptron of which the transfer functions. Then, an LDI state-space representation [12] is established for the dynamics of the NN system. Finally, a DT fuzzy controller is designed to stabilize the CT nonlinear system. According to this approach, if the closed-loop DT system cannot be stabilized, a dither is injected into the original CT nonlinear system as an auxiliary of the controller (see Figure 1).

A dither [15] is a high-frequency signal injected into a CT nonlinear system in order to augment stability, quench undesirable limit cycles, eliminate jump phenomena, and reduce nonlinear distortion. Zames and Shneydor [16] rigorously examined the effect of a dither depending on its amplitude distribution function. Mossaheb [17] showed that when the dither frequency is high enough, the output of the smoothed system and the dithered system may be as close as desired. Desoer and Shahrz [18] studied the effect of dither in nonlinear control systems involving backlash or hysteresis. A rigorous analysis of stability in a general CT nonlinear system with a dither control was given in [19]. Based on these articles, we suggest that the trajectory of a DT system can be predicted rigorously by establishing a corresponding system the CT relaxed system, provided that the dither's frequency and the discretized frequency (or sampling frequency) are sufficiently high. This enables us to obtain a rigorous prediction of the stability of the closed-loop DT system by establishing the stability of the NN system. On the other hand, some parameters of membership functions for fuzzy controller could not be optimized by the LMI method; hence, we use the simplex optimization method [14] to search these parameters quickly. A simulation example is given to illustrate the proposed design method.

## 2. System Description

Consider an open-loop CT nonlinear system  $f_{CT}$  described by the following equation:

$$\dot{x}(t) = f_{CT}(x(t), u, D(\xi, \varpi)) = f_{CT}(\xi), \quad (2.1)$$

where  $D(\xi, \varpi)$  is the dither signal and  $\xi$  is the maximum dither amplitude;  $\varpi$  is dither's lower-bound frequency [15] when  $D = 0$ , (2.1) is a common CT nonlinear system without dither, otherwise, it is a CT nonlinear dithered system;  $x(t)$  is a CT state vector,  $u$  is a DT control input vector, and  $f_{CT}$  is a vector-valued function that satisfies the assumptions of

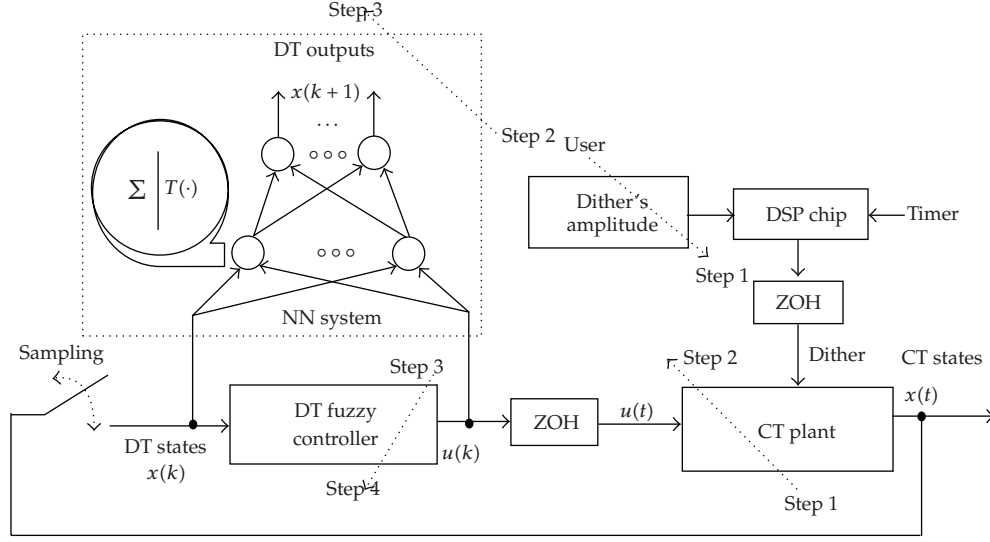


Figure 1: CT to DT fuzzy control system with dither effect and design flow chart.

boundedness given in [19]. Then  $f_{CT}$  is discretized by setting the sampling time or sampling period  $\bar{T}$  (sec), and  $t = k \cdot \bar{T}$ ,  $k$  is a positive integer, to the following a DT nonlinear system  $f_{DT}$ :

$$x\left((k+1) \cdot \bar{T}\right) = f_{DT}\left(x\left(k \cdot \bar{T}\right), u\left(k \cdot \bar{T}\right)\right), \quad \text{or} \quad x(k+1) = f_{DT}(x(k), u(k)), \quad (2.2)$$

where  $k$  indicates the signal sequence, the DT state vector is  $x(k \cdot \bar{T})$ , and the DT control input vector is  $u(k \cdot \bar{T})$ .

In this study, an NN system is first established to approximate a CT nonlinear system (2.1). An LDI state-space representation is then established for the dynamics of the NN system. Finally, a DT fuzzy controller is designed to stabilize the CT nonlinear system.

## 2.1. Neural Network System

An NN system with  $S$  layers each having  $R^e$  ( $r = 1^e, 2^e, \dots, R^e$ ;  $e = 1, 2, \dots, S$ ) neurons is established to approximate a CT nonlinear system  $f_{CT}$ , as shown in Figure 1. Superscript text is used to distinguish these layers. Specifically, we append the number of the layer as superscript to the names of these variables. Thus, the weight matrix for the  $e$ th ( $e = 1, 2, \dots, S$ ) layer is written as  $W^e(\bar{T})$ . Moreover, it is assumed that  $v_r$  ( $r = 1^e, 2^e, \dots, R^e$ ) is the net input and that all the transfer functions  $T_S(v_r)$  of units in the NN system are described by the following sigmoid function:

$$T_S(v_r) = \lambda \cdot \left( \frac{2}{1 + \exp(-v_r/q)} - 1 \right), \quad (2.3)$$

where  $q$  and  $\lambda$  are the positive parameters associated with the sigmoid function. The transfer function vector of the  $e$ th layer is defined as

$$\Psi^e(v_r) \equiv [T_S(v_{1^e}), T_S(v_{2^e}), \dots, T_S(v_{R^e})]^T, \quad e = 1, 2, \dots, S, \quad (2.4)$$

where  $T_S(v_r)$  ( $r = 1^e, 2^e, \dots, R^e$ ) is a transfer function of the  $r$ th neuron. The final outputs of the NN system can then be inferred as follows:

$$x(k+1) = \psi^S \left( W^S \psi^{S-1} \left( W^{S-1} \psi^{S-2} \left( \dots \psi^2 \left( W^2 \psi^1 \left( W^1 Z(k) \right) \right) \dots \right) \right) \right), \quad (2.5)$$

where  $Z^T(k) = [x(k), u(k)]$ . In next section, an LDI state-space representation is established in order to deal with the stability problem of the NN system.

## 2.2. Linear Difference Inclusion (LDI)

An LDI system can be described in the state-space representation as [12]:

$$y(k+1) = A(a(k))y(k), \quad A(a(k)) = \sum_{i=1}^l h_i(a(k))A_i, \quad (2.6)$$

where  $l$  is a positive integer,  $a$  is a vector signifying the dependence of  $h_i$  on its elements,  $A_i$  ( $i = 1, 2, \dots, l$ ), and  $y(k) = [y_1(k), y_2(k), \dots, y_c(k)]^T$ . Moreover, it is assumed that  $h_i \geq 0$ ,  $\sum_{i=1}^l h_i = 1$ . From the properties of LDI, without loss of generality, we can use  $h_i(k)$  instead of  $h_i(a(k))$ . In the following, a procedure is taken to represent the dynamics of the NN system (2.5) by LDI state-space representation [12]:

$$g(T_S(v_r)) = \begin{cases} g_1 = \min_{v_r} \frac{dT_S(v_r)}{dv_r} = 0, \\ g_2 = \max_{v_r} \frac{dT_S(v_r)}{dv_r} = \frac{\lambda}{2q}, \end{cases} \quad (2.7)$$

where  $g_1$  and  $g_2$  are the minimum and the maximum of the derivative of  $T_S$ , respectively. The min-max matrix  $G^e$  is defined as

$$G^e = \text{diag}(g), \quad e = 1, 2, \dots, S; \quad r = 1^e, 2^e, \dots, R^e. \quad (2.8)$$

Next, using the interpolation method and (2.5), we can obtain

$$\begin{aligned}
x(k+1) &= \sum_{v_1=1}^2 \sum_{v_2=1}^2 \cdots \sum_{v_{R^S}=1}^2 h_{v_1}^S(k) h_{v_2}^S(k) \cdots h_{v_{R^S}}^S(k) G^S \\
&\quad \times \left( W^S \left[ \cdots \left[ \sum_{v_1=1}^2 \sum_{v_2=1}^2 \cdots \sum_{v_{R^2}=1}^2 h_{v_1}^2(k) h_{v_2}^2(k) \cdots h_{v_{R^2}}^2(k) G^2 \right. \right. \right. \\
&\quad \left. \left. \left. \times \left( W^2 \left[ \sum_{v_1=1}^2 \sum_{v_2=1}^2 \cdots \sum_{v_{R^1}=1}^2 h_{v_1}^1(k) h_{v_2}^1(k) \cdots h_{v_{R^1}}^1(k) G^1 (W^1 Z(k)) \right] \right) \right] \right] \cdots \right) \\
&= \sum_{\Omega^S} \tilde{h}_{\Omega^S}^S(k) G^S W^S \cdots \sum_{\Omega^2} \tilde{h}_{\Omega^2}^2(k) G^2 W^2 \sum_{\Omega^1} \tilde{h}_{\Omega^1}^1(k) G^1 W^1 Z(k) = \sum_{\Omega} \hat{h}_{\Omega}(k) E_{\Omega} Z(k),
\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}
\sum_{\Omega^e} \tilde{h}_{\Omega^e}^e(k) &\equiv \sum_{v_1=1}^2 \sum_{v_2=1}^2 \cdots \sum_{v_{R^e}=1}^2 h_{v_1}^e(k) h_{v_2}^e(k) \cdots h_{v_{R^e}}^e(k), \quad \text{for } e = 1, 2, \dots, S; \\
h_{v_r}^e(k) &\in [0, 1], \quad \sum_{v_r=1}^2 h_{v_r}^e(k) = 1, \quad \text{for } r = 1^e, 2^e, \dots, R^e,
\end{aligned} \tag{2.10}$$

$$E_{\Omega} \equiv G^S W^S \cdots G^2 W^2 G^1 W^1, \quad \sum_{\Omega} \hat{h}_{\Omega}(k) \equiv \sum_{\Omega^S} \cdots \sum_{\Omega^2} \sum_{\Omega^1} \tilde{h}_{\Omega^S}^S(k) \cdots \tilde{h}_{\Omega^2}^2(k) \tilde{h}_{\Omega^1}^1(k).$$

Finally, using (2.6), we can rewrite the dynamics of the NN system (2.9) in the following LDI state-space representation:

$$x(k+1) = \sum_{i=1}^l \hat{h}_i(k) E_i Z(k), \tag{2.11}$$

where  $\hat{h}_i(k) \geq 0$ ,  $\sum_{i=1}^l \hat{h}_i(k) = 1$ ,  $l$  is a positive integer, and  $E_i$  is a constant matrix with appropriate dimension associated with  $E_{\Omega}$ . The LDI state-space representations (2.11) can be further rearranged as follows:

$$x(k+1) = \sum_{i=1}^l \hat{h}_i(k) [A_i x(k) + B_i u(k)], \tag{2.12}$$

where  $A_i$  and  $B_i$  are the partitions of  $E_i$  corresponding to the partitions of  $Z(k)$ . Furthermore, we obtain the LDI relaxed representations of the CT dithered system, as shown in the next section.

### 2.3. LDI Form of the Dithered System

The LDI form of a CT nonlinear system with dither includes the dither's maximum amplitude  $\xi$  and can be obtained by replacing  $\hat{h}_i$ ,  $A_i$ ,  $B_i$  in (2.12) with the relaxed parameters  $\check{h}_i$ ,  $\check{A}_i(\xi)$ ,  $\check{B}_i(\xi)$ , respectively. For relaxed theory and its application for dithered systems, refer to [15]. Hence, we directly obtain the LDI state-space relaxed representation of this CT dithered system as

$$x(k+1) = \sum_{i=1}^l \check{h}_i(k) [\check{A}_i(\xi)x(k) + \check{B}_i(\xi)u(k)], \quad (2.13)$$

where  $u(k)$  is a Takagi-Sugeno (T-S) DT fuzzy controller, as shown in the following section.

### 3. T-S DT Fuzzy Controller

Here, a Takagi-Sugeno (T-S) DT fuzzy controller is synthesized to stabilize the NN system (2.13). The DT fuzzy controller is in the following form.

*Rule j.* IF  $x_1(k)$  is  $M_{j1}$ , and ... and  $x_\varsigma(k)$  is  $M_{j\varsigma}$ , THEN  $u(k) = -F_j x(k)$ , where  $j = 1, 2, \dots, \delta$  and  $\delta$  is the number of IF-THEN rules and  $M_{j\mu}$  ( $\mu = 1, 2, \dots, \varsigma$ ) are the fuzzy sets. Hence, the final output of this DT fuzzy controller is inferred as follows:

$$u(k) = \frac{-\sum_{j=1}^{\delta} \omega_j(x) F_j x(k)}{\sum_{j=1}^{\delta} \omega_j(x) F_j x(k)} = -\sum_{j=1}^{\delta} \bar{h}_j(x) F_j x(k) \quad (3.1)$$

with

$$\omega_j(x) = \prod_{\mu=1}^{\varsigma} M_{j\mu}(x_\mu(k)), \quad \bar{h}_j(x) = \frac{\omega_j(x)}{\sum_{j=1}^{\delta} \omega_j(x)}, \quad (3.2)$$

in which  $M_{j\mu}$  is the grade of membership of  $x_\mu$  in  $M_{j\mu}$ . In this study, it is also assumed that  $\omega_j(x) \geq 0$ ,  $\sum_{j=1}^{\delta} \omega_j(x) > 0$ ,  $j = 1, 2, \dots, \delta$ , and  $k = 1, 2, \dots, K$ . Therefore,  $\bar{h}_j(x) \geq 0$ ,  $\sum_{j=1}^{\delta} \bar{h}_j(x) = 1$  for all  $k$ . Substituting (3.1) into (2.12), we have

$$x(k+1) = \sum_{i=1}^l \sum_{j=1}^{\delta} \check{h}_i \bar{h}_j(x) [\check{A}_i(\xi) - \check{B}_i(\xi) F_j] x(k) = \sum_{i=1}^l \sum_{j=1}^{\delta} \check{h}_i \bar{h}_j \check{H}_{ij}(\xi) x(k), \quad (3.3)$$

where  $\check{H}_{ij} = \check{A}_i - \check{B}_i F_j$ .

Furthermore, we consider the CT system (2.1) by using the above NN system (3.3) and modeling error  $e_{\text{mod}}(k)$  as follows:

$$x(k+1) = \sum_{i=1}^l \sum_{j=1}^{\delta} \check{h}_i \bar{h}_j \check{H}_{ij}(\xi) x(k) + e_{\text{mod}}(k), \quad \text{for } i = 1, 2, \dots, l; m = 1, 2, \dots, \tau; j = 1, 2, \dots, \delta, \quad (3.4)$$

where  $e_{\text{mod}}(k) \equiv f_{\text{CT}}(\xi) - \sum_{i=1}^l \sum_{j=1}^{\delta} \check{h}_i \bar{h}_j [\check{A}_i(\xi) - \check{B}_i(\xi) F_j] x(k)$ .

If the closed-loop CT system's sampling frequency is sufficiently high, according to the Nyquist sampling theory, the discretized states of this dithered system can approximate its original CT states. This permits a rigorous prediction of the stability of the CT dithered system by establishing the stability of the closed-loop NN system (3.4) with the bounded condition ( $e_U$ ). The modeling error  $e_{\text{mod}}(k)$  satisfies the following bounded condition:

$$e_{\text{mod}}^T(k) e_{\text{mod}}(k) \leq e_U^T e_U. \quad (3.5)$$

Moreover, according to the Lyapunov approach, the following Theorem 3.1 is given to guarantee the uniformly ultimately bounded (UUB) stability of the closed-loop CT system (3.4).

**Theorem 3.1.** *The closed-loop CT system (3.4) is UUB stable in the large if there exists a common positive definite matrix  $P > 0$ ,  $Q > 0$ ,  $\bar{\kappa}^2$  and  $\xi \geq 0$  such that*

$$\check{H}_{ij}^T P \cdot \check{H}_{ij} - P + \bar{\kappa}^2 \check{H}_{ij}^T P^T P \check{H}_{ij} \leq -Q, \quad (3.6)$$

where  $\check{H}_{ij}(\xi) = \check{A}_i(\xi) - \check{B}_i(\xi) F_j$ .

*Proof.* See the appendix. □

According to the stability conditions addressed in Theorem 3.1, the closed-loop NN system is classified into two conditions. Condition 1: if there exists a common positive definite matrix  $P$  to satisfy the stability conditions in Theorem 3.1, then the fuzzy controller can stabilize this closed-loop NN system without dither. Condition 2: if there does not exist a common positive definite matrix  $P$  to satisfy the stability conditions in Theorem 3.1, then the DT fuzzy controller and the dither (as an auxiliary of the T-S fuzzy controller) are simultaneously introduced to stabilize the closed-loop CT nonlinear system. Therefore, the rest of this paper focuses on the robust stability analysis of Condition 2.

#### 4. T-S DT Fuzzy Controller and Dither Design Algorithm

An illustration of the flow chart in Figure 1 for DT fuzzy controller and dither design algorithm is as follows.

*Step 1.* If the DT fuzzy controller cannot stabilize the NN system, a dither, as an auxiliary of this controller, is simultaneously introduced to stabilize the closed-loop CT system. This study suggests users add the dither's amplitude from zero, and go to Step 2 until Step 4 has a stable solution.

*Step 2.* The CT nonlinear system with dither is discretized by setting the sampling time  $\bar{T}$ , and go to Step 3.

*Step 3.* Collect DT training input data:  $x(k)$  and  $u(k)$ , output data:  $x(k+1)$ , and the NN system can be obtained by a Levenberg-Marquardt backpropagation (LM-BP) algorithm [13] and the LDI relaxed representation, then go to Step 4.

*Step 4.* According to the LDI relaxed representation in Step 3, a T-S DT fuzzy controller (3.1) can be designed by the linear matrix inequality (LMI) method. Finally, we can adjust the dither's amplitude in Step 1 and verify the stability condition of a system with this DT fuzzy controller in Step 4.

## 5. Case Study

The above T-S DT fuzzy controller and dither design algorithm discussed in the preceding section is illustrated below by the numerical example of a van der Pol control system:

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + 2 \cdot (1 - x_1^2)x_2 + u,\end{aligned}\tag{5.1}$$

where the initial states  $x_1(0) = -3$ ,  $x_2(0) = 0$ .

In this example, we use the dither method [19] in Case 2 to compare with our method in Case 1 as follows.

*Case 1.* Stability of the NN model in Figure 2 of the dithered system with a fuzzy controller, and demonstrate the control performance of dither plus fuzzy controller in the van der Pol system.

*Case 2.* Demonstrate the control performance of the dithering system [19] in the van der Pol system.

First, the neural-network structure 3-3-1 of Case 1 has 3 inputs, 3 sigmoid neurons of (2.3) in a hidden layer, 1 sigmoid neuron in an output layer, and their weights are 12, as shown in Figure 2.

*Step 1.* According to Theorem 3.1, the T-S DT fuzzy controller cannot stabilize the NN system of (5.2) without dither. Hence, we use a dither and DT fuzzy controller and rebuild the NN system of (5.2) with dither to stabilize the following closed-loop CT system with



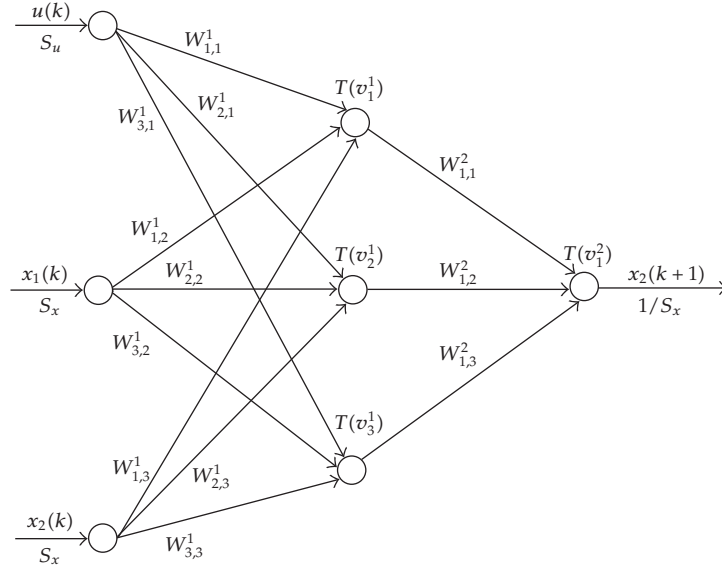


Figure 2: NN model of CT dithered system.

dither:

$$\dot{x}_1 = x_2,$$

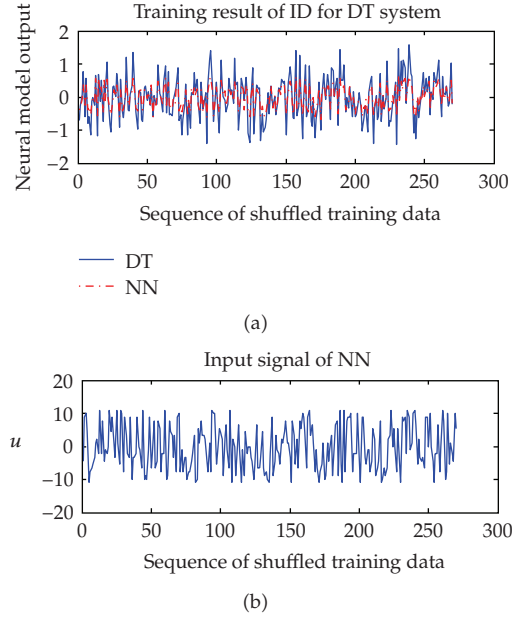
$$\dot{x}_2 = -x_1 + 2 \cdot (1 - x_1^2)x_2 + u_D + u(t) = -x_1 + 2 \left[ 1 - (x_1 + D(\xi, \varpi))^2 \right] x_2 + u(t), \quad (5.2)$$

where  $u_D = -2[2x_1D(\xi, \varpi) + D^2(\xi, \varpi)]x_2$ . A periodic symmetrical square-wave dither  $D(\xi, \varpi)$  with sufficiently high frequency is added in front of  $u(t)$ . The lower-bound dither's maximum amplitude  $\xi = 0.5$ .

Step 2. The CT nonlinear system with dither is discretized as the following equations by setting the sampling time  $\bar{T} = 0.05$  sec, and go to Step 3,

$$\begin{aligned} x_1(k+1) &= x_1(k) + x_2(k)\bar{T}, \\ x_2(k+1) &= x_2(k) + \left\{ -x_1(k) + 2 \left[ 1 - (x_1(k) + D(\varpi))^2 \right] x_2(k) + u(k) \right\} \bar{T}. \end{aligned} \quad (5.3)$$

Step 3. Collect and shuffle DT training input data:  $x(k)$  and  $u(k)$ , output data:  $x(k+1)$  to avoid most of local optimal weight values, due to the NN system is obtained by a LM-BP algorithm [13]. The training result is shown in Figure 3.



**Figure 3:** The training result of NN for system identification for Case 1.

The weight matrices of the hidden and the output layer are denoted by  $W^1$  and  $W^2$ . After training via the LM-BP algorithm, the weights can be obtained as follows:

$$\begin{aligned}
 W_{1,1}^1 &= 0.2304, & W_{2,1}^1 &= 0.6039, & W_{3,1}^1 &= 0.6846, \\
 W_{1,2}^1 &= -0.1446, & W_{2,2}^1 &= 2.4534, & W_{3,2}^1 &= -0.1107, \\
 W_{1,3}^1 &= 1.8985, & W_{2,3}^1 &= -0.1123, & W_{3,3}^1 &= 0.8417, \\
 W_{1,1}^2 &= 0.2853, & W_{1,2}^2 &= -0.0002, & W_{1,3}^2 &= 0.0267.
 \end{aligned} \tag{5.4}$$

If the symbol  $v_b^a$  denotes the net input of the  $b$ th neuron of the  $a$ th layer, then

$$v_r^1 = W_{r,1}^1 S_u u(k) + W_{r,2}^1 S_x x_1(k) + W_{r,3}^1 S_x x_2(k), \quad r = 1, 2, 3, \tag{5.5}$$

where  $S_x = 1/3$  and  $S_u = 1/11$  are the scaling constants to limit the range of inputs  $x$ ,  $u$  in the NN model; respectively, and

$$v_1^2 = W_{1,1}^2 T_S(v_1^1) + W_{1,2}^2 T_S(v_2^1) + W_{1,3}^2 T_S(v_3^1), \quad x(k+1) = T_S(v_1^2), \tag{5.6}$$

where  $T_S(v_r^1) = 2/(1 + \exp(-v_r^1/0.5)) - 1$ ,  $r = 1, 2, 3$ ,  $T_S(v_1^2) = 2/(1 + \exp(-v_1^2/0.5)) - 1$ .

Hence,  $g_1 = 0$ ,  $g_2 = 1$ , and we can obtain  $\check{A}_i(0.5)$ ,  $\check{B}_i(0.5)$  of the LDI relaxed representation as follows:

$$\begin{aligned}
\check{A}_1 &= \begin{bmatrix} 1 & \bar{T} \\ g_2^2 W_{1,1}^2 W_{1,2}^1 & g_2^2 W_{1,1}^2 W_{1,3}^1 \end{bmatrix}, & \check{A}_2 &= \begin{bmatrix} 1 & \bar{T} \\ g_2^2 W_{1,2}^2 W_{2,2}^1 & g_2^2 W_{1,2}^2 W_{2,3}^1 \end{bmatrix}, \\
\check{A}_3 &= \begin{bmatrix} 1 & \bar{T} \\ g_2^2 W_{1,3}^2 W_{3,2}^1 & g_2^2 W_{1,3}^2 W_{3,3}^1 \end{bmatrix}, \\
\check{A}_4 &= \begin{bmatrix} 1 & \bar{T} \\ g_2^2 (W_{1,1}^2 W_{1,2}^1 + W_{1,2}^2 W_{2,2}^1) & g_2^2 (W_{1,1}^2 W_{1,3}^1 + W_{1,2}^2 W_{2,3}^1) \end{bmatrix}, \\
\check{A}_5 &= \begin{bmatrix} 1 & \bar{T} \\ g_2^2 (W_{1,1}^2 W_{1,2}^1 + W_{1,3}^2 W_{3,2}^1) & g_2^2 (W_{1,1}^2 W_{1,3}^1 + W_{1,3}^2 W_{3,3}^1) \end{bmatrix}, \\
\check{A}_6 &= \begin{bmatrix} 1 & \bar{T} \\ g_2^2 (W_{1,2}^2 W_{2,2}^1 + W_{1,3}^2 W_{3,2}^1) & g_2^2 (W_{1,2}^2 W_{2,3}^1 + W_{1,3}^2 W_{3,3}^1) \end{bmatrix}, \\
\check{A}_7 &= \begin{bmatrix} 1 & \bar{T} \\ g_2^2 (W_{1,1}^2 W_{1,2}^1 + W_{1,2}^2 W_{2,2}^1 + W_{1,3}^2 W_{3,2}^1) & g_2^2 (W_{1,1}^2 W_{1,3}^1 + W_{1,2}^2 W_{2,3}^1 + W_{1,3}^2 W_{3,3}^1) \end{bmatrix}, \\
\check{B}_1 &= \begin{bmatrix} 0 \\ g_2^2 W_{1,1}^2 W_{1,1}^1 S_u / S_x \end{bmatrix}, & \check{B}_2 &= \begin{bmatrix} 0 \\ g_2^2 W_{1,2}^2 W_{2,1}^1 S_u / S_x \end{bmatrix}, & \check{B}_3 &= \begin{bmatrix} 0 \\ g_2^2 W_{1,3}^2 W_{3,1}^1 S_u / S_x \end{bmatrix}, \\
\check{B}_4 &= \begin{bmatrix} 0 \\ g_2^2 (W_{1,1}^2 W_{1,1}^1 + W_{1,2}^2 W_{2,1}^1) S_u / S_x \end{bmatrix}, & \check{B}_5 &= \begin{bmatrix} 0 \\ g_2^2 (W_{1,1}^2 W_{1,1}^1 + W_{1,3}^2 W_{3,1}^1) S_u / S_x \end{bmatrix}, \\
\check{B}_6 &= \begin{bmatrix} 0 \\ g_2^2 (W_{1,2}^2 W_{2,1}^1 + W_{1,3}^2 W_{3,1}^1) S_u / S_x \end{bmatrix}, \\
\check{B}_7 &= \begin{bmatrix} 0 \\ g_2^2 (W_{1,1}^2 W_{1,1}^1 + W_{1,2}^2 W_{2,1}^1 + W_{1,3}^2 W_{3,1}^1) S_u / S_x \end{bmatrix}.
\end{aligned} \tag{5.7}$$

Because the case of  $\check{A}_i = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$  and  $\check{B}_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  did not have any effect on  $x(k+1)$  in the LDI relaxed representation, so the relaxed LMI conditions are not considered this case.

*Step 4.* According to [15], the lower-bound dither's frequency  $\varpi$  is 50 Hz. In this example, we choose the sampling frequency of a T-S DT fuzzy controller (3.1) to be 20 Hz, and it can be designed by the linear matrix inequality (LMI) method in Theorem 3.1. First, we design the membership functions of Rule 1, 2 as follows:

$$\begin{aligned} \text{Rule 1 : IF } x_1(k) \text{ is } M_1(x_1(k)), \quad \text{THEN } u(k) &= -F_1x(k), \\ \text{Rule 2 : IF } x_1(k) \text{ is } M_2(x_1(k)), \quad \text{THEN } u(k) &= -F_2x(k), \end{aligned} \quad (5.8)$$

where  $x(k) = [x_1(k), x_2(k)]^T$ .

Next, we design

$$\bar{h}_1 = M_1 = \exp\left(-\frac{(x_1(k) - c)^2}{(2\sigma^2)}\right), \quad \bar{h}_2 = M_2 = 1 - \bar{h}_1. \quad (5.9)$$

Then, we use the Simplex optimization method [14] to search  $\sigma = 2.2132$  and  $c = 0.00000445$ . According to (3.1), the overall T-S DT fuzzy controller is

$$u(k) = -\sum_{j=1}^2 \bar{h}_j F_j x(k). \quad (5.10)$$

Finally, we can adjust dither's amplitude in Step 1, and according to the LMI solutions:

$$P = \begin{bmatrix} 4.174 & 0.1803 \\ 0.1803 & 4.7201 \end{bmatrix}, \quad \bar{\kappa}^2 = 10^{-7}, \quad Q = \begin{bmatrix} 10^{-6} & 0 \\ 0 & 10^{-6} \end{bmatrix}, \quad (5.11)$$

we have verified the stability condition of the system with these DT fuzzy gains:

$$F_1 = [-0.0000376, 7.3814], \quad F_2 = [0.0000391, -18.8912]. \quad (5.12)$$

The DT controller of Case 1 is to check the fulfillment of (3.5). According to the recorded values shown in Figure 4, (3.5) is satisfied. Hence, the DT controller of Case 1 can stabilize the CT dithered nonlinear system (5.2), as shown in Figure 5. The DT controller of Case 1 is shown in Figure 6. However, Case 2 cannot stabilize this CT nonlinear system, as shown in Figure 7. Furthermore, the different dither's shapes did not have an effect on the stability of the system, but the system responses to different dithers' shapes of Case 1 must be clearly different.

## 6. Conclusion

This study presents an effective NN-based approach to stabilizing continuous-time (CT) nonlinear systems by a dither and a T-S discrete-time (DT) fuzzy controller. This NN system is established to approximate a nonlinear system with dither. The dynamics of the NN system

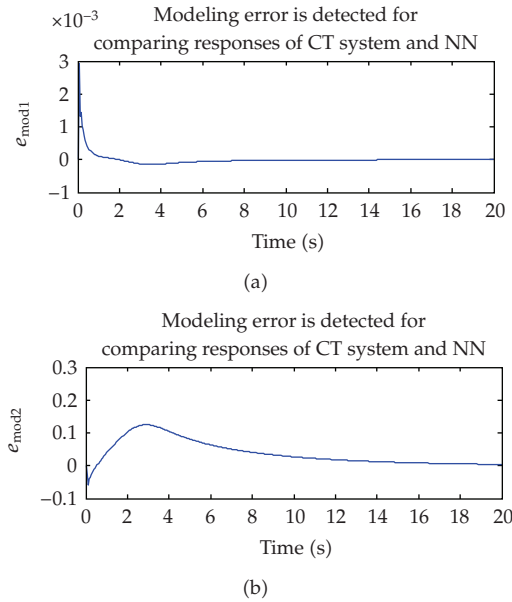


Figure 4: The result of detecting modeling error  $e_{\text{mod}} = [e_{\text{mod}1}, e_{\text{mod}2}]^T$  for Case 1.

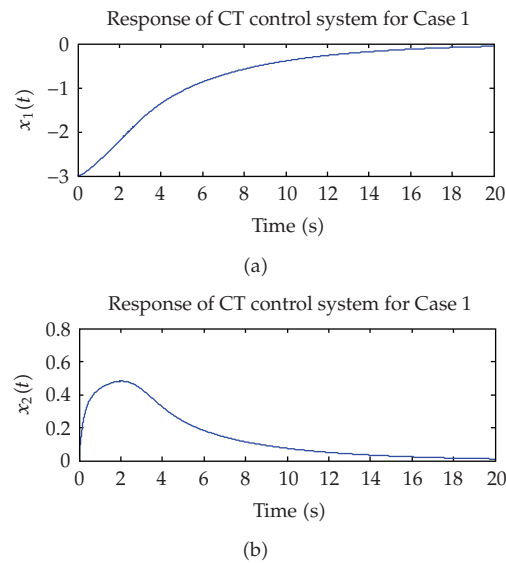


Figure 5: The detail of response of a van der Pol control system for Case 1.

are then converted into an LDI relaxed representation, and finally, a T-S DT fuzzy controller is designed to stabilize the CT nonlinear system by the LMI method. If the designed DT fuzzy controller cannot asymptotically stabilize the NN system, a dither is injected into this system. The T-S DT fuzzy controller and the dither signal are simultaneously introduced to stabilize the closed-loop CT nonlinear system. With sufficiently high dither frequency and system sampling frequency, simulation results show that the DT fuzzy controller can

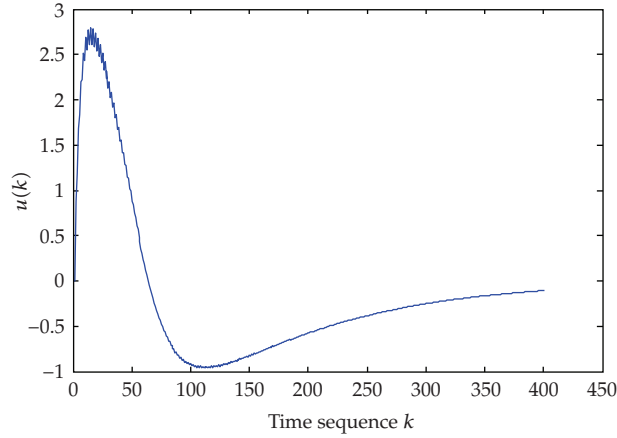


Figure 6: The digital control signal of a van der Pol CT plant for Case 1.

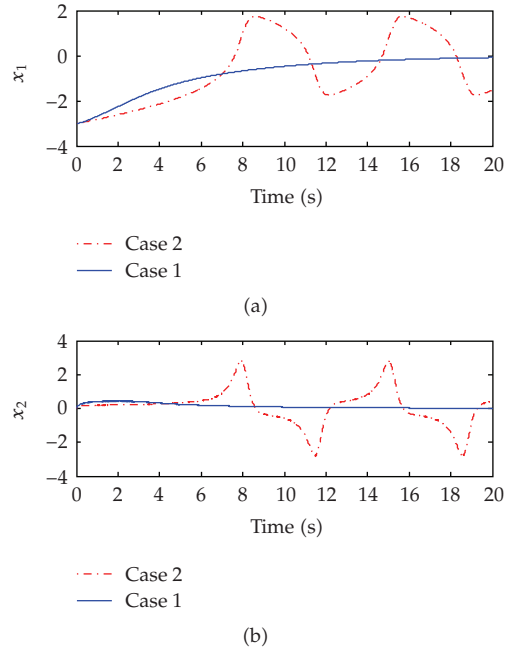


Figure 7: Comparison of results for Cases 1, 2.

stabilize the nonlinear dithered system by appropriately regulating the dither amplitude. The algorithm of LMI solver needs the special form to obtain control gains; therefore we will develop less conservative theorem by SOS algorithm for further research [20]. LM-BP has disadvantages such as getting into local minimum for the offline training stage of NN. For this reason, we will improve these disadvantages of LM-BP by studying fast convergence of genetic algorithm [14] with the multilayer perceptron (MLP) neural network [13].

## Appendix

**Lemma A.1** (see [11]). *For any matrices  $A$  and  $B$  with appropriate dimension, we have*

$$A^T B + B^T A \leq \bar{\kappa}^2 A^T A + \bar{\kappa}^{-2} B^T B. \quad (\text{A.1})$$

Let the Lyapunov function for the nonlinear system be defined as

$$V(k) = x(k)^T P x(k), \quad (\text{A.2})$$

where  $P = P^T > 0$ . We then evaluate the backward difference  $\Delta V(k)$  of  $V(k)$  to obtain

$$\begin{aligned} \Delta V(k) &= V(k+1) - V(k) = x(k+1)^T P \cdot x(k+1) - x(k)^T P \cdot x(k) \\ &= \left\{ \sum_{i=1}^l \sum_{j=1}^{\delta} \check{h}_i \bar{h}_j \check{H}_{ij}(\xi) x(k) + e_{\text{mod}}(k) \right\}^T P \left\{ \sum_{i=1}^l \sum_{j=1}^{\delta} \check{h}_i \bar{h}_j \check{H}_{ij}(\xi) x(k) + e_{\text{mod}}(k) \right\} \\ &\quad - x(k)^T P \cdot x(k) \\ &= \sum_{i=1}^l \sum_{j=1}^{\delta} \check{h}_i \bar{h}_j x(k)^T \left[ \check{H}_{ij}^T(\xi) P \check{H}_{ij}(\xi) - P \right] x(k) + \sum_{i=1}^l \sum_{j=1}^{\delta} \check{h}_i \bar{h}_j x(k)^T \check{H}_{ij}^T P \cdot e_{\text{mod}}(k) \\ &\quad + \sum_{i=1}^l \sum_{j=1}^{\delta} \check{h}_i \bar{h}_j e_{\text{mod}}(k)^T P \cdot \check{H}_{ij} x(k) + e_{\text{mod}}(k)^T P \cdot e_{\text{mod}}(k). \end{aligned} \quad (\text{A.3})$$

According to Lemma .1, we obtain

$$\begin{aligned} &x(k)^T \check{H}_{ij}^T P \cdot e_{\text{mod}}(k) + e_{\text{mod}}(k)^T P \cdot \check{H}_{ij} x(k) \\ &\leq \bar{\kappa}^2 x(k)^T \check{H}_{ij}^T P^T P \check{H}_{ij} x(k) + \bar{\kappa}^{-2} e_{\text{mod}}(k)^T e_{\text{mod}}(k). \end{aligned} \quad (\text{A.4})$$

Therefore, we obtain

$$\begin{aligned} \Delta V(k) &\leq \sum_{i=1}^l \sum_{j=1}^{\delta} \check{h}_i \bar{h}_j x(k)^T \left( \check{H}_{ij}^T P \cdot \check{H}_{ij} - P \right) x(k) \\ &\quad + \sum_{i=1}^l \sum_{j=1}^{\delta} \check{h}_i \bar{h}_j \left[ \bar{\kappa}^2 x(k)^T \check{H}_{ij}^T P^T P \cdot \check{H}_{ij} x(k) + \bar{\kappa}^{-2} e_{\text{mod}}(k)^T e_{\text{mod}}(k) \right] \\ &= \sum_{i=1}^l \sum_{j=1}^{\delta} \check{h}_i \bar{h}_j x(k)^T \left( \check{H}_{ij}^T P \cdot \check{H}_{ij} - P + \bar{\kappa}^2 \check{H}_{ij}^T P^T P \cdot \check{H}_{ij} \right) x(k) + \bar{\kappa}^{-2} e_{\text{mod}}(k)^T e_{\text{mod}}(k) \\ &\leq -x(k)^T Q \cdot x(k) + \bar{\kappa}^{-2} e_{\text{mod}}(k)^T e_{\text{mod}}(k). \end{aligned} \quad (\text{A.5})$$

In accordance with  $\check{H}_{ij}^T P \cdot \check{H}_{ij} - P + \bar{\kappa}^2 \check{H}_{ij}^T P^T P \check{H}_{ij} \leq -Q < 0$ , we have

$$\Delta V(k) \leq -x(k)^T Q \cdot x(k) + \bar{\kappa}^{-2} e_{\text{mod}}(k)^T e_{\text{mod}}(k) \leq -\lambda_{\min}(Q)x(k)^T x(k) + \bar{\kappa}^{-2} e_U^T e_U, \quad (\text{A.6})$$

where  $\lambda_{\min}(Q)$  denotes the minimum eigenvalue of  $Q$ . Whenever

$$\|x(k)\| > \frac{\bar{\kappa}^{-1} e_U}{\sqrt{\lambda_{\min}(Q)}}, \quad \Delta V(k) < 0. \quad (\text{A.7})$$

By a standard Lyapunov extension [21], this illustrates the trajectories of the closed-loop nonlinear systems are UUB stable. From  $k = 0$  to  $N$  yields

$$\begin{aligned} V(N+1) - V(0) &< - \sum_{k=0}^N x(k)^T Q \cdot x(k) + \bar{\kappa}^{-2} \sum_{k=0}^N e_{\text{mod}}(k)^T e_{\text{mod}}(k), \\ \sum_{k=0}^N x(k)^T Q \cdot x(k) &< V(0) - V(N+1) + \bar{\kappa}^{-2} \sum_{k=0}^N e_{\text{mod}}(k)^T e_{\text{mod}}(k). \end{aligned} \quad (\text{A.8})$$

Hence, we have

$$\sum_{k=0}^N x(k)^T Q \cdot x(k) < x(0)^T \tilde{P} \cdot x(0) + \bar{\kappa}^{-2} \sum_{k=0}^N e_{\text{mod}}(k)^T e_{\text{mod}}(k). \quad (\text{A.9})$$

Hence, the  $H_\infty$  control performance is achieved with a prescribed  $\bar{\kappa}^{-2}$  in (3.6). Next, recasting a control problem as an LMI problem is equivalent to finding a solution to the original problem,  $\check{H}_{ij}^T P \cdot \check{H}_{ij} - P < 0$ . The stability conditions encountered in Theorem 3.1 are expressed in the following forms of LMIs. The conditions  $\check{H}_{ij}^T P \cdot \check{H}_{ij} - P < 0$  are not jointly convex in  $F_j$  and  $P$ . Now multiplying the inequality on the left and right by  $P^{-1}$  and defining new variables  $\bar{P} = P^{-1}$  and  $\bar{M}_j = F_j \bar{P}$ , the conditions  $\check{H}_{ij}^T P \cdot \check{H}_{ij} - P < 0$  can be rewritten using



the Schur complement as follows:

$$\begin{aligned}
& \begin{bmatrix} -\bar{P} & (\check{A}_i\bar{P} - \check{B}_i\bar{M}_j)^T \\ \check{A}_i\bar{P} - \check{B}_i\bar{M}_j & -\bar{P} \end{bmatrix} < 0, \quad \text{for } i = 1, 2, \dots, l; j = 1, 2, \dots, \delta, \\
& \begin{bmatrix} -2\bar{P} & (2\check{A}_i\bar{P} - \check{B}_i\bar{M}_j - \check{B}_i\bar{M}_\beta)^T \\ 2\check{A}_i\bar{P} - \check{B}_i\bar{M}_j - \check{B}_i\bar{M}_\beta & -\bar{P} \end{bmatrix} < 0, \quad \text{for } j < \beta = 1, 2, \dots, \delta, \\
& \begin{bmatrix} -2\bar{P} & (\check{A}_i\bar{P} - \check{B}_i\bar{M}_j + \check{A}_\alpha\bar{P} - \check{B}_\alpha\bar{M}_j)^T \\ \check{A}_i\bar{P} - \check{B}_i\bar{M}_j + \check{A}_\alpha\bar{P} - \check{B}_\alpha\bar{M}_j & -\bar{P} \end{bmatrix} < 0, \quad \text{for } i < \alpha = 1, 2, \dots, l, \\
& \begin{bmatrix} -4\bar{P} & \check{M}^T \\ \check{M} & -\bar{P} \end{bmatrix} < 0, \quad \text{for } i < \alpha = 1, 2, \dots, l; j < \beta = 1, 2, \dots, \delta,
\end{aligned} \tag{A.10}$$

where  $\check{M} \equiv \check{A}_i\bar{P} - \check{B}_i\bar{M}_j + \check{A}_\alpha\bar{P} - \check{B}_\alpha\bar{M}_\beta + \check{A}_i\bar{P} - \check{B}_i\bar{M}_\beta + \check{A}_\alpha\bar{P} - \check{B}_\alpha\bar{M}_j$ .

The feedback gain  $F_j$  and a common  $P$  can be obtained as  $P = \bar{P}^{-1}$ ,  $F_j = \bar{M}_j\bar{P}^{-1}$  from the solutions  $\bar{P}$  and  $\bar{M}_j$ .

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