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Research Article

Integral and Variational Formulations for the Helmholtz Equation Inverse Source Problem

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The purpose of this paper is to explore the Hilbert space functional structure of the Helmholtz equation inverse source problem. An integral equation for the sources reconstruction based on the composition of the trace and Green's function operators is introduced and compared with the reciprocity source reconstruction methodologies. An equivalence theorem comparing the integral inverse source equation with the variational weak reciprocity gap functional equation is then demonstrated. Some examples on applications to the unitary disk are presented.

1. Introduction

The inverse source problem for the Helmholtz Dirichlet equation is a basic tool for the investigation of transient source problems [1–4]. In order to investigate this class of problems, let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary Γ. Let κ be a complex number, $g \in H^{1/2}(\Gamma)$, and $f \in L^2(\Omega)$. The direct problem with the Helmholtz operator: to find a regular field u that satisfy the system

$$\begin{aligned} -\Delta u - \kappa^2 u &= f \quad \text{in } \Omega, \\ \gamma_0 u &= g \quad \text{on } \Gamma, \end{aligned} \tag{1.1}$$

is well posed and has a unique solution $u \in H^1(\Omega)$ when κ^2 is not an eigenvalue of the Laplacian. In this paper we consider $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ the trace operator and simplify the notation by calling the boundary data g of problem (1.1) $\gamma_0 u$. The trace theorem [5] assures the existence of a function $g_{\nu_x} \in H^{-1/2}(\Gamma)$ which is the normal trace, that is,

$$\gamma_1 u = \partial_{\nu_x} u. \tag{1.2}$$

When κ is a real positive or an imaginary number we have, respectively, the proper or the modified Helmholtz equation. When $\kappa = 0$, we obtain the Laplace equation. For the complete setting of complex values, we consider the problem as the Helmholtz equation direct problem.

The inverse source problem consist, in knowing the Cauchy data in the boundary Γ, that is, the Dirichlet to Neumann map in at least one Dirichlet datum g, to recover the source f. It may be formally posed as follows:

$$\begin{aligned}
&\text{given a Cauchy data set } \left\{\left(g, g_{\nu_x}\right)\right\} \subset H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma),\\
&\text{with compatibility condition } \frac{\partial g}{\partial \nu_x} = g_{\nu_x} \quad \text{on } \Gamma,\\
&\text{for the equation model in the system (1.1)},\\
&\text{to find } \left(u, f\right) \in H^1(\Omega) \times L^2(\Omega).
\end{aligned} \tag{1.3}$$

The two problems, direct and inverse, can also be formulated with only one system of equations: to find $(u, f) \in H^1(\Omega) \times L^2(\Omega)$ such that

$$\begin{aligned}
-\Delta u - k^2 u &= f \quad \text{in } \Omega,\\
\gamma_0 u &= g \quad \text{on } \Gamma,\\
\gamma_1 u &= g_{\nu_x} \quad \text{on } \Gamma.
\end{aligned} \tag{1.4}$$

Since in the inverse problem the Cauchy data are known, we may associate these data with a fourth-order Dirichlet problem with the Bilaplacian operator

$$\begin{aligned}
\Delta^2 u - \kappa^4 u &= h \quad \text{in } \Omega,\\
\gamma_0 u &= g \quad \text{on } \Gamma,\\
\gamma_1 u &= g_{\nu_x} \quad \text{on } \Gamma,
\end{aligned} \tag{1.5}$$

where $h \in H^{-2}(\Omega)$ is an arbitrary given function. This problem is well posed and has a unique solution $u \in H^2(\Omega)$ when κ^4 is not an eigenvalue of the Bilaplacian. This motivates the following naive existence result

Remark 1.1. Suppose that a Cauchy data pair $(g, g_{\nu_x}) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ is more regular than normal case and $\kappa^4 \notin \Sigma_4$ is not an eigenvalue of the Bilaplacian, then there exists a solution to inverse source problem (1.3).

Proof. Since data is regular, we may consider the fourth-order direct Dirichlet problem (1.5) with the given Cauchy data and some $h \in H^{-2}(\Omega)$. The inverse source solution will be $f = -\Delta u + \kappa^2 u \in L^2(\Omega)$. □

To obtain the existence of global solution for problems (1.1), and (1.5), please see [6]. For more information about linear integral equations, see [7]. For the inverse source problems, see [8]. For functional analysis, please see [9].

In Section 2 we develop a Hilbert space functional framework to the problem based on special $L^2(\Omega)$ decomposition. The analysis of the Dirichlet to Newman map and of the Source to Neumann maps is done in Sections 2.1 and 2.2, respectively. It is based only on the analysis of the direct problem structure. The analysis of the Adjoint Source to Neumann map is done in Section 2.3. In Section 3 we use the Green function operator to put together the results found in Section 2. There in Section 3.1 we present an integral equation for the inverse problem based on the relative Dirichlet to Newman map. The reciprocity gap functional is introduced in Section 4, where an equivalence theorem between this and the integral formulation is proved. In Section 5 some particular results for the unitary disk in $\mathbb{R}^2$ are presented. Finally, we conclude the paper in Section 6.

2. The Dirichlet and the Source to Neumann Map

For $l \in \mathbb{R}$, the space $H^l(\Omega)$ is the Sobolev class of the functions of the spatial variable x. For more information, see [5]. Let us consider for future use the following sets of eigenvalues:

$$\Sigma_2 := \left\{ \lambda \in \mathbb{C} : -\Delta u = \lambda u \text{ in } H^2(\Omega) \cap H_0^1(\Omega) \right\},$$
$$\Sigma_4 := \left\{ \lambda \in \mathbb{C} : \Delta^2 u = \lambda u \text{ in } H^4(\Omega) \cap H_0^2(\Omega) \right\}. \tag{2.1}$$

Definition 2.1. One says that a function is metaharmonic when it is in the set

$$H_{-\Delta-\kappa^2}(\Omega) = \left\{ v \in L^2(\Omega);\ -\Delta v - \kappa^2 v = 0 \right\}, \tag{2.2}$$

where $\kappa^2 \notin \Sigma_2$ and $\kappa^4 \notin \Sigma_4$.

2.1. The Dirichlet to Neumann Map

Definition 2.2. Consider problem (1.1) with zero source, that is, $f = 0$ and $g \in H^{1/2}(\Gamma)$. This problem has a solution $w^0 \in H^1(\Omega)$. One defines the Dirichlet to Neumann map for the Helmholtz equation as the operator

$$\Lambda^0 : H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma),$$
$$\Lambda^0[g](x) = \frac{\partial w^0}{\partial \nu_x}(x), \quad x \in \Gamma. \tag{2.3}$$

By the trace theorem [5], it is well defined, linear, and continuous.

Remark 2.3. We alternatively can define the Dirichlet to Neumann map (2.2) for the problem (1.1) as an operator with a nonzero source problem $\Lambda^f : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ defined by the graph $(g, \Lambda^f g) := (u, \partial_\nu u)|_{\partial\Omega}$.

Definition 2.4. One has

$$\mathcal{M} := \left\{ (g, g_{\nu_x}) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma);\ g_{\nu_x} - \Lambda^0 g \in H^{1/2}(\Gamma) \right\}, \tag{2.4}$$

where Λ^0 is a Dirichlet to Neumann map.

Theorem 2.5. *Let $(g, g_{\nu_x}) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$. A function $f \in H_{-\Delta+\kappa^2}(\Omega)$is a solution to the inverse source problem* (1.3) *if and only if $(g, g_{\nu_x}) \in \mathcal{M}$*

Proof. Suppose $(g, g_{\nu_x}) \in \mathcal{M}$. Consider the following fourth-order problem:

$$\begin{gathered} \left(\Delta^2 - k^4\right) w = 0 \quad \text{in } \Omega, \\ \gamma_0 w = 0 \quad \text{on } \Gamma, \\ \gamma_1 w = g_{\nu_x} - \Lambda^0 g \quad \text{on } \Gamma. \end{gathered} \tag{2.5}$$

Let us use this problem to define a source $f_0 = (-\Delta - k^2)w$, where $w \in H^2(\Omega)$ is solution of problem (2.5). Note that $f_0 \in H_{-\Delta+\kappa^2}(\Omega)$. Let $(u, f) \in H^1(\Omega) \times L^2(\Omega)$ be a solution of the inverse source problem (1.4). Then $u = w^0 + w$, where w^0 is the solution of the homogeneous source problem in the definition of the Dirichlet to Neumann map (2.2) and $f = f_0$. The sufficiency is proved.

To proof necessity, suppose that there exists a $(u, f) \in H^1(\Omega) \times H_{-\Delta+\kappa^2}(\Omega)$ which is solution of the inverse problem (1.4).

Consider the second-order problem with homogeneous boundary Dirichlet data

$$\begin{gathered} \left(-\Delta - k^2\right) w_0 = f, \quad x \in \Omega, \\ \gamma_0 w_0 = 0, \quad x \in \Gamma, \end{gathered} \tag{2.6}$$

with a unique solution $w_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. By the trace theorem, we have $\gamma_1 w_0 \in H^{1/2}(\Gamma)$. Note that $g_{\nu_x} = \gamma_1 w_0 + \gamma_1 w^0$, where $\gamma_1 w^0 = \Lambda^0 g$. So, $\gamma_1 w_0 = g_{\nu_x} - \Lambda^0 g$. □

Remark 2.6. We have proved the existence and uniqueness of solution to (1.4) in $H^1(\Omega) \times H_{-\Delta+\kappa^2}(\Omega)$. However, this does not mean that when we do the search in a larger space $H^1(\Omega) \times L^2(\Omega)$, we will continue having uniqueness. In fact, we will prove in the next proposition that $f - f_0 \perp v \in H_{-\Delta-\kappa^2}(\Omega)$ and consequently

$$f \in f_0 + \left[H_{-\Delta-\kappa^2}(\Omega)\right]^{\perp}, \tag{2.7}$$

where $f \in L^2(\Omega)$ and $f_0 \in H_{-\Delta+\kappa^2}(\Omega)$. This leads us to conclude that the solution of (1.4) is actually a class of functions, where we can "reconstruct" or "observe" only part of the solution in $H_{-\Delta+\kappa^2}(\Omega)$.

Proposition 2.7. *If $\kappa^2 \notin \Sigma_2$ and $\kappa^4 \notin \Sigma_4$, then*

$$L^2(\Omega) = [H_{-\Delta-\kappa^2}(\Omega)]^{\perp} \oplus H_{-\Delta+\kappa^2}(\Omega). \tag{2.8}$$

Proof. Consequence of Lemmas 2.8 and 2.9. □

From now on, we are supposing always that $\kappa^2 \notin \Sigma_2$ and $\kappa^4 \notin \Sigma_4$.

Lemma 2.8. *One has $L^2(\Omega) = H_{-\Delta+\kappa^2}(\Omega) \oplus (-\Delta - \kappa^2)[H_0^2(\Omega)]$.*

Proof. For an arbitrarily given $f \in L^2(\Omega)$, the problem (1.1) with zero Dirichlet datum $g = 0$ is well posed and has an $w_0 \in H^2(\Omega)$ with normal derivative trace $\Lambda^f = \partial_\nu w_0 \in H^{1/2}(\Gamma)$. Also, since $\kappa^4 \notin \Sigma_4$, the fourth-order problem (1.5) with Cauchy datum $(0, \Lambda^f) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ and zero source has solution v and is well posed in $H^2(\Omega)$. These two $H^2(\Omega)$ solutions may be used to define a function

$$w := w_0 - v \in H_0^2(\Omega), \tag{2.9}$$

and since by problem (1.1)

$$f = \left(-\Delta - \kappa^2\right) w_0 = \left(-\Delta - \kappa^2\right) v + \left(-\Delta - \kappa^2\right) w, \tag{2.10}$$

we obtain that arbitrary $L^2(\Omega)$ function f is the sum of a function in $H_{-\Delta-\kappa^2}(\Omega)$ and a function in $(-\Delta - \kappa^2)[H_0^2(\Omega)]$. If we show that

$$H_{-\Delta+\kappa^2}(\Omega) \cap \left(-\Delta - \kappa^2\right)\left[H_0^2(\Omega)\right] = \{0\}, \tag{2.11}$$

we prove that $L^2(\Omega) = H_{-\Delta+\kappa^2} \oplus (-\Delta - \kappa^2)[H_0^2(\Omega)]$. For this, take a u in the intersection $H_{-\Delta-\kappa^2} \cap (-\Delta + \kappa^2)[H_0^2(\Omega)]$. Then

$$\left(-\Delta - \kappa^2\right) u = 0, \qquad u = \left(-\Delta + \kappa^2\right) v, \quad \text{for some } v \in H_0^2(\Omega) \tag{2.12}$$

which means that v is a solution of completely homogeneous fourth-order problem (1.5), with no source and zero Cauchy datum. Since $\kappa^4 \notin \Sigma_4$, the unique solution is trivial $v = 0$, and $u = (-\Delta + \kappa^2)v = 0$. We have proved that $L^2(\Omega) = H_{-\Delta-\kappa^2}(\Omega) \oplus (-\Delta + \kappa^2)[H_0^2(\Omega)]$. □

Lemma 2.9. *One has $(-\Delta - \kappa^2)[H_0^2(\Omega)] = [H_{-\Delta-\kappa^2}(\Omega)]^{\perp}$.*

Proof. Using the second Green theorem with $f \in H_{-\Delta-\kappa^2}(\Omega)$ and $v = (-\Delta - \kappa^2)w$ for some $w \in H_0^2(\Omega)$, we obtain

$$\begin{aligned} \int_\Omega f v \, dx &= \int_\Omega f\left(-\Delta - \kappa^2\right) w \, dx = \int_\Omega f(-\Delta w) dx - \kappa^2 \int_\Omega f w \, dx \\ &= \int_\Omega (-\Delta f) w \, dx - \kappa^2 \int_\Omega f w \, dx = 0, \end{aligned} \tag{2.13}$$

and since the scalar product with arbitrary $h \in (-\Delta - \kappa^2)[H_0^2(\Omega)]$ is zero, the inclusion $(-\Delta - \kappa^2)[H_0^2(\Omega)] \subset H^{\perp}_{-\Delta-\kappa^2}(\Omega)$ follows.

To prove the other inclusion, let $f \in L^2(\Omega)$, suppose f is orthogonal to v for all $v \in H_{-\Delta-\kappa^2}(\Omega)$, and take a function $w \in H^2(\Omega)$ such that w is solution of (1.1) with source f and Dirichlet datum $g = 0$.

By using the second Green formulas for $v \in H_{-\Delta-\kappa^2}(\Omega)$, we have

$$0 = \int_\Omega f v \, dx = \int_\Omega \left(-\Delta - \kappa^2\right) w v \, dx = \int_\Omega w \left(-\Delta - \kappa^2\right) v \, dx = \int_\Gamma v \partial_{\nu_x} w \, d\sigma_x. \tag{2.14}$$

Note that, since the test functions are dense on $H_{-\Delta-\kappa^2}(\Omega)$, we may consider a variational formulation using the dual system $(D(\Omega), D(\Omega)^*)$. Because $v \in L^2(\Omega)$, by trace theorem $v|_{\partial\Omega} \in H^{-1/2}(\Gamma)$ and

$$\int_\Gamma v \partial_{\nu_x} w \, d\sigma_x = \langle v, \partial_{\nu_x} w \rangle_{(D(\Omega),[D(\Omega)]^*)} = 0 \Longrightarrow \partial_{\nu_x} w = 0 \quad \text{on} \quad H^{1/2}(\Gamma). \tag{2.15}$$

So, if an arbitrary $L^2(\Omega)$ function is orthogonal to all functions in $H_{-\Delta-\kappa^2}(\Omega)$, then it is in $(-\Delta - \kappa^2)[H_0^2(\Omega)]$ and the reverse inclusion follows. □

2.2. The Source to Neumann Map

Let us consider the simultaneous solution of the direct and inverse source problem (1.4) and search for a solution $(u, f) \in H^1(\Omega) \times L^2(\Omega)$. It follows from Theorem 2.5 that if we restrict the source search to the subspace $H_{-\Delta+\kappa^2}(\Omega) \subset L^2(\Omega)$, we will find a unique solution.

Definition 2.10. Consider problem (1.1) with zero Dirichlet data, that is, $g = 0$ and $f \in H_{-\Delta+\kappa^2}(\Omega) \subset L^2(\Omega)$ and solution $w_0^f \in H^2(\Omega) \cap H_0^1(\Omega)$. One defines the Source to Neumann map for the Helmholtz equation as the operator

$$\begin{gathered} \Lambda_0 : H_{-\Delta+\kappa^2}(\Omega) \longrightarrow H^{1/2}(\Gamma), \\ \Lambda_0[f](x) = \frac{\partial w_0^f}{\partial \nu_x}(x) = \gamma_1\left[w_0^f\right](x), \quad x \in \Gamma. \end{gathered} \tag{2.16}$$

By the trace theorem [5], it is well defined, linear, and continuous.

Theorem 2.11. $\Lambda_0 : H_{-\Delta+\kappa^2}(\Omega) \to H^{1/2}(\Gamma)$*, is an isomorphism.*

Remark 2.12. If we consider problem (1.1) with Dirichlet data $g \in H^{3/2}(\Gamma)$ and $f \in H_{-\Delta+\kappa^2}(\Omega) \subset L^2(\Omega)$ and solution $w_g^f \in H^2(\Omega) \cap H_0^1(\Omega)$, we can define the Source to Neumann map for the Helmholtz equation as the operator

$$\begin{gathered} \Lambda_g : H_{-\Delta+\kappa^2}(\Omega) \longrightarrow H^{-1/2}(\Gamma), \\ \Lambda_g[f](x) = \frac{\partial w_g^f}{\partial \nu_x}(x) = \gamma_1\left[w_g^f\right](x), \quad x \in \Gamma. \end{gathered} \tag{2.17}$$

Note that this more general situation will produce results such as

$$\Lambda_g[0](x) = \Lambda^0[g](x) = \gamma_1[g] = \frac{\partial w_g^0}{\partial \nu_x}(x). \tag{2.18}$$

As consequence,

(i) a functional such as a Source-Dirichlet to Neumann map may be defined

$$\begin{gathered}\Lambda[\cdot,\cdot] : H_{-\Delta+\kappa^2}(\Omega) \times H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma),\\ [f, g] \longmapsto \Lambda[f, g] = \gamma_1\left[w_g^f\right],\end{gathered} \tag{2.19}$$

(ii) and restrieted to be a functional such as a Dirichlet to Neumann map

$$\begin{gathered}\Lambda[0,\cdot] : \{0\} \times H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma),\\ g \longmapsto \Lambda[0, g] \equiv \Lambda^0[g] = \gamma_1\left[w_g^0\right],\end{gathered} \tag{2.20}$$

(iii) or to be a functional such as a Source to Neumann map

$$\begin{gathered}\Lambda[\cdot,0] : H_{-\Delta+\kappa^2}(\Omega) \times \{0\} \longrightarrow H^{-1/2}(\Gamma),\\ f \longrightarrow \Lambda[0, f] \equiv \Lambda_0[f] = \gamma_1\left[w_0^f\right].\end{gathered} \tag{2.21}$$

It is important to note that in this more general definition it is not possible to prove that $\Lambda[\cdot,\cdot]$ is an isomorphism, since the fact that the trace $\gamma_0[w_g^f] = 0$ is used in the proof.

Lemma 2.13. *$H_{-\Delta+\kappa^2}(\Omega)$ is a Hilbert space with $L^2(\Omega)$ norm.*

Proof. Let us consider the canonical projection

$$\pi_2 : L^2(\Omega) = H_{-\Delta+\kappa^2}(\Omega) \oplus H^{\perp}_{-\Delta-\kappa^2}(\Omega) \longrightarrow H^{\perp}_{-\Delta-\kappa^2}(\Omega) \quad \left[f, f^{\perp}\right] \longmapsto f^{\perp}. \tag{2.22}$$

Note that π_2 is continuous, $\pi_2^{-1}[0]$ is closed. Since

$$\pi_2^{-1}[0] = [f, 0] \tag{2.23}$$

for all $f \in H_{-\Delta+\kappa^2}(\Omega)$, it follows that $\{0\} \times H_{\Delta+\kappa^2}(\Omega)$ is closed. Consequently $H_{-\Delta+\kappa^2}(\Omega)$ is closed subspace of the Banach space $L^2(\Omega)$. So, it is Banach and, consequently, is a Hilbert space with the scalar product induced by $L^2(\Omega)$. □

Proof of Theorem 2.11. (i) $\Lambda_0 : H_{-\Delta+\kappa^2}(\Omega) \to H^{1/2}(\Gamma)$ *is continuous.*

Note that $\Lambda_0 = \gamma_1 \circ i$ is a composition of the normal trace $\gamma_1 : H^2(\Omega) \to H^{1/2}(\Gamma)$ and the canonical embedding $i : H_{-\Delta+\kappa^2}(\Omega) \to H^2(\Omega)$. The normal trace is continuous by trace theorem. The canonical embedding is also continuous since $H_{-\Delta+\kappa^2}(\Omega)$ is closed by Lemma 2.13. So, Λ_0 is continuous.

(ii) $\Lambda_0 : H_{-\Delta+\kappa^2}(\Omega) \to H^{1/2}(\Gamma)$ *is one to one.*

Take some arbitrary $f \in \text{Ker}(\Lambda_0)$. Then $\Lambda_0[f] = \gamma_1[w_0] = 0$ is the normal derivative of the problem (1.1) with $g = 0$. By hypotheses, $f \in H_{-\Delta+\kappa^2}(\Omega)$ and, consequently, $(-\Delta+\kappa^2)f = (-\Delta+\kappa^2)(-\Delta-\kappa^2)w_0^f = 0$. The fourth-order problem

$$\begin{aligned} \left(\Delta^2 - k^4\right)w_0^f &= 0 \quad \text{in } \Omega, \\ \gamma_0 w_0^f &= 0 \quad \text{on } \Gamma, \\ \gamma_1 w_0^f &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{2.24}$$

is well posed and has a unique $w_0^f = 0$. Then $f = (-\Delta-\kappa^2)w_0^f = 0$. Since f is arbitrary, $\text{Ker}(\Lambda_0) = \{0\}$ and the injectivity is proved.

(iii) $\Lambda_0 : H_{-\Delta+\kappa^2}(\Omega) \to H^{1/2}(\Gamma)$ *is onto.*

Consider an arbitrary $g_{\nu_x} \in H^{1/2}(\Gamma)$ and $\kappa \neq 0$, where g_{ν_x} does not necessarily satisfy the compatibility condition. Let $w_{g_{\nu_x}} \in H_{-\Delta+\kappa^2}(\Omega)$ be a solution of the well-posed Neumann data problem,

$$\begin{aligned} \left(-\Delta+\kappa^2\right)w_{g_{\nu_x}} &= 0 \quad \text{on } \Omega, \\ \gamma_1 w_{g_{\nu_x}} &= g_{\nu_x} \quad \text{in } \Gamma. \end{aligned} \tag{2.25}$$

Note that with $f = w_{g_{\nu_x}}$ we obtain $\Lambda_0[f] = \gamma_1[w_{g_{\nu_x}}] = g_{\nu_x}$. So, Λ_0 is surjective.

It remains to prove the following.

(iv) $\Lambda_0^{-1} : H^{1/2}(\Gamma) \to H_{-\Delta+\kappa^2}(\Omega)$ *is continuous.*

In fact, this is a consequence of the Banach open map theorem, since.

$$\Lambda_0 : H_{-\Delta+\kappa^2}(\Omega) \longrightarrow H^{1/2}(\Gamma) \tag{2.26}$$

is a linear continuous bijective application between Banach spaces. □

2.3. The Adjoint Source to Neumann Operator

Definition 2.14. Consider again the problem (1.1) with zero Dirichlet data, that is, $g = 0$ and $f \in L^2(\Omega)$ and solution $w_0^f \in H^2(\Omega) \cap H_0^1(\Omega)$. One defines the extension of the Source to Neumann map for the Helmholtz equation as the operator

$$\overline{\Lambda_0} : L^2(\Omega) \longrightarrow H^{1/2}(\Gamma),$$
$$\overline{\Lambda_0}[f](x) = \frac{\partial w_0^f}{\partial \nu_x}(x) = \gamma_1\left[w_0^f\right](x), \quad x \in \Gamma. \tag{2.27}$$

Remark 2.15. By the trace theorem [5], it is well defined, linear, and continuous. As an extension of Λ_0, surjectivity is preserved. Also $\overline{\Lambda_0} = \Lambda_0 \circ \pi_1$.

Corollary 2.16. *The quotient of $L^2(\Omega)$ by $H^{\perp}_{-\Delta-\kappa^2}(\Omega)$ is a copy of $H^{1/2}(\Gamma)$.*

Proof. Consider the following chain of embeddings:

$$\begin{array}{ccc} L^2(\Omega) = H_{-\Delta+\kappa^2}(\Omega) \oplus [H_{-\Delta-\kappa^2}(\Omega)]^{\perp} & \overset{\overline{\Lambda_0}}{\to} & H^{1/2}(\Gamma) \\ \downarrow c_0 & & \uparrow \Lambda_0 \\ \dfrac{L^2(\Omega)}{[H_{-\Delta-\kappa^2}(\Omega)]^{\perp}} & \overset{c_1}{\to} & H_{-\Delta+\kappa^2}(\Omega), \end{array} \tag{2.28}$$

where c_0 is a canonic embedding and c_1 is an isomorphism by the Banach isomorphism theorem. Since Λ_0 also is an isomorphism by Theorem 2.11, the corollary is proved, that is, $L^2(\Omega)/H^{\perp}_{-\Delta+\kappa^2}(\Omega) \cong H^{1/2}(\Gamma)$. □

Corollary 2.17. $\mathrm{Ker}(\overline{\Lambda_0}) = H^{\perp}_{-\Delta-\kappa^2}(\Omega)$ *is a closed subspace of $L^2(\Omega)$.*

Remark 2.18. Since $\overline{\Lambda_0} \in \mathcal{L}(L^2(\Omega); H^{1/2}(\Gamma))$ is bounded, then its adjoint operator $\overline{\Lambda_0}^* \in \mathcal{L}(H^{-1/2}(\Gamma); L^2(\Omega))$ is well defined and continuous and one to one.

Corollary 2.19. $\overline{\Lambda_0}^* : H^{-1/2}(\Gamma) \to H_{-\Delta-\kappa^2}(\Omega)$ *is an isomorphism.*

Proof. (i) $\overline{\Lambda_0}^* : H^{-1/2}(\Gamma) \to H_{-\Delta-\kappa^2}(\Omega)$ is well defined.

We know that

$$H^{\perp}_{-\Delta-k^2}(\Omega) = \mathrm{Ker}\left(\overline{\Lambda_0}\right) = \mathrm{Rg}\left(\overline{\Lambda_0}^*\right),$$
$$\left[{}^{\perp}\mathrm{Rg}\left(\overline{\Lambda_0}^*\right)\right]^{\perp} = H_{-\Delta-k^2}(\Omega), \tag{2.29}$$

where Rg and Ker denote the operator range and kernel, respectively. It is well known that if X is a Banach space and N, that is, a subspace of its dual X^*, then $[{}^{\perp}N]^{\perp} = \overline{N}^{w^*}$ is the weak star closure of N in X^*. Applying this result to our case, we have $H_{-\Delta-\kappa^2}(\Omega) = \overline{\mathrm{Rg}(\overline{\Lambda_0}^*)}^{w^*}$ in $L^2(\Omega)$. Particularly, $\mathrm{Rg}(\overline{\Lambda_0}^*) \subset H_{-\Delta-\kappa^2}(\Omega)$ assures that $\overline{\Lambda_0}^*$ is well defined.

(ii) $\overline{\Lambda_0}^* : H^{-1/2}(\Gamma) \to H_{-\Delta-\kappa^2}(\Omega)$ *is one to one.*

Note that, $\Lambda_0 : H_{-\Delta+\kappa^2}(\Omega) \to H^{1/2}(\Gamma)$ is onto and $H_{-\Delta-\kappa^2}(\Omega)$ is a closed subspace of $L^2(\Omega)$,

(iii) $\overline{\Lambda_0}^* : H^{-1/2}(\Gamma) \to H_{-\Delta-\kappa^2}(\Omega)$ *is onto.*

Note that since, $\mathrm{Rg}(\overline{\Lambda_0})$ is closed in $L^2(\Omega)$,

$$\mathrm{Ker}\left(\overline{\Lambda_0}\right) = H^{\perp}_{-\Delta-\kappa^2}(\Omega) \Longrightarrow H_{-\Delta-\kappa^2}(\Omega) = \mathrm{Ker}\left(\overline{\Lambda_0}\right)^{\perp} \subset \mathrm{Rg}\left(\overline{\Lambda_0}^*\right) = \overline{\mathrm{Rg}\left(\overline{\Lambda_0}^*\right)}^{w^*} = H_{-\Delta-\kappa^2}(\Omega). \tag{2.30}$$

Since $\overline{\Lambda_0}^*$ is linear, continuous, and bijective from $H^{-1/2}(\Gamma)$ to $H_{-\Delta-\kappa^2}(\Omega)$, by the open mapping Banach theorem, $\overline{\Lambda_0}^*$ is an isomorphism. □

Proposition 2.20. *If* $\kappa_1 \neq \kappa_2$, *then* $H_{-\Delta-\kappa_1^2}(\Omega) \cap H_{-\Delta-\kappa_2^2}(\Omega) = \{0\}$.

Proof. Suppose that $v \in H_{-\Delta-\kappa_1^2}(\Omega) \cap H_{-\Delta-\kappa_2^2}(\Omega)$. Then

$$\begin{aligned} \left(-\Delta - \kappa_1^2\right)v = 0, &\quad \text{if } v \in H_{-\Delta-\kappa_1^2}(\Omega), \\ \left(-\Delta - \kappa_2^2\right)v = 0, &\quad \text{if } v \in H_{-\Delta-\kappa_2^2}(\Omega), \end{aligned} \tag{2.31}$$

and, consequently, $v = 0$. □

Remark 2.21. If we substitute κ by $i\kappa$ in problem (1.1) and use the same argument already used in the precedent proofs, then

(i) $L^2(\Omega) = H_{-\Delta-\kappa^2}(\Omega) \oplus (-\Delta + \kappa^2)[H_0^2(\Omega)]$,

(ii) $H^{\perp}_{-\Delta+\kappa^2}(\Omega) = (-\Delta + \kappa^2)[H_0^2(\Omega)]$,

(iii) $H_{-\Delta+\kappa^2}(\Omega)$ is a closed subspace of $L^2(\Omega)$,

(iv) $L^2(\Omega) = H_{-\Delta-\kappa^2}(\Omega) \oplus H^{\perp}_{-\Delta+\kappa^2}(\Omega)$,

(v) if $\kappa_1 \neq \kappa_2$, $H_{-\Delta-\kappa_1^2}(\Omega) \cap H_{-\Delta-\kappa_2^2}(\Omega) = \{0\}$.

3. Integral Representation

Definition 3.1. The Dirichlet Green function $G(x, \zeta)$ for the problem (1.1) is its solution with source $\delta(x - \zeta)$, $x, \zeta \in \Omega$, and homogeneous Dirichlet data, that is, $G(x, \zeta) = 0$ for x on Γ.

Definition 3.2. Let $G(x,\zeta)$ be the Green function for problem (1.1) with homogeneous Dirichlet boundary conditions. Then, the solution

$$S : L^2(\Omega)\times H^{1/2}(\Omega)\longrightarrow H^1(\Omega),\quad (f,g)\longmapsto u=S[f,g],$$
$$S[f,g](x)=\int_\Omega f(\zeta)G(x,\zeta)d\zeta+\int_\Gamma g(\zeta)\frac{\partial G(x,\zeta)}{\partial \nu_\zeta}\,d\sigma_\zeta \quad \text{for } x\in\Omega. \tag{3.1}$$

Remark 3.3. By using problem (1.1) linearity, we formally decompose the solution in two additive parts

$$u=S[f,g]:=S[f,0]+S[0,g]:L^2(\Omega)\times H^{1/2}(\Gamma)\longrightarrow H^1(\Omega), \tag{3.2}$$

where $S[f,0]:L^2(\Omega)\times\{0\}\to H^1(\Omega)$ is the homogeneous Dirichlet source auxiliary problem solution and $S[0,g]:\{0\}\times H^{1/2}(\Gamma)\to H^1(\Omega)$ is the zero source auxiliary Dirichlet problem solution. For simplicity, for a fixed $f\in L^2(\Omega)$ or $g\in H^{1/2}(\Gamma)$, we will denote

$$S^f=S[f,\cdot]:H^{1/2}(\Gamma)\longrightarrow H^1(\Omega),$$
$$S_g=S[\cdot,g]:L^2(\Omega)\longrightarrow H^1(\Omega). \tag{3.3}$$

By taking the normal trace of the solution (3.1), we obtain

$$\Lambda[f,g](x)=\gamma_1\circ S[f,g]=\int_\Omega f(\zeta)\frac{\partial G(x,\zeta)}{\partial\nu_x}\,d\zeta+\int_\Gamma g(\zeta)\frac{\partial^2 G(x,\zeta)}{\partial\nu_x\partial\nu_\zeta}\,d\sigma_\zeta\quad \text{for } x\in\Gamma \tag{3.4}$$

which is an explicit representation to the Dirichlet to Newman map with arbitrary f and g.

By using the same notation adopted for the additive decomposition of the solution map, fixed $f\in L^2(\Omega)$ or $g\in H^{1/2}(\Gamma)$, we will denote

$$\Lambda^f=\Lambda[f,\cdot]:H^{1/2}(\Gamma)\longrightarrow H^1(\Omega),$$
$$\Lambda_g=\Lambda[\cdot,g]:L^2(\Omega)\longrightarrow H^1(\Omega). \tag{3.5}$$

Remark 3.4. Note that

$$S_0[f](x)=S[f,0](x)=\int_\Omega f(\zeta)G(x,\zeta)d\zeta,\quad x\in\Omega,$$
$$S^0[g](x)=S[0,g](x)=\int_\Gamma g(\zeta)\frac{\partial G(x,\zeta)}{\partial\nu_\zeta}\,d\sigma(\zeta),\quad x\in\Omega. \tag{3.6}$$

With this decomposition, we obtain the following explicit representation to operators in Section 2:

(1)

$$\begin{aligned}\Lambda[f,g] &= \gamma_1 \circ S[f,g](x) = \frac{\partial S[f,g]}{\partial \nu_x}(x) \\ &= \int_\Omega f(\zeta)\frac{\partial G(x,\zeta)}{\partial \nu_x}\,d\zeta + \int_\Gamma g(\zeta)\frac{\partial^2 G(x,\zeta)}{\partial \nu_x \partial \nu_\zeta}\,d\sigma(\zeta),\end{aligned} \tag{3.7}$$

with $x \in \Gamma$;

(2)

$$\Lambda^0[g](x) = \Lambda[0,g](x) = \gamma_1 \circ S[0,g](x) = \int_\Gamma g(\zeta)\frac{\partial^2 G(x,\zeta)}{\partial \nu_x \partial \nu_\zeta}\,d\sigma(\zeta), \quad x \in \Gamma, \tag{3.8}$$

is an explicit representation to the Dirichlet to Neumann map;

(3)

$$\Lambda_0[f] = (x)\Lambda[f,0](x) = \gamma_1 \circ S[f,0](x) = \int_\Omega f(\zeta)\frac{\partial G(x,\zeta)}{\partial \nu_x}\,d\zeta, \quad x \in \Gamma, \tag{3.9}$$

is an explicit representation to the Source to Neumann map.

3.1. The Inverse Source Integral Equation

Lemma 3.5. *Let u_j, $j = 1, 2$, be two solutions of problem* (1.1) *with the same source f and different Dirichlet data g_j, $j = 1, 2$, respectively. Then*

(i) $\Lambda^f[g_1]-\Lambda^0[g_1] = \Lambda^f[g_2]-\Lambda^0[g_2]$ *on Γ, that is, the relative Dirichlet to Newman operator $\Lambda^f - \Lambda^0 : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is constant operator whose functional value is independent of the Dirichlet datum g and depends only on the source function f;*

(ii) $\int_\Omega f(\zeta)(\partial G(x,\zeta)/\partial \nu_x)d\zeta = \Lambda^f[g_j] - \Lambda^0[g_j]$ *for all solutions of* (1.1) *with arbitrary Dirichlet data but the same source, that is, the integral is the function given by the relative Dirichlet to Newman map.*

Proof. The equality $\int_\Omega f(\zeta)(\partial G(x,\zeta)/\partial \nu_x)d\zeta = \Lambda^f[g_1] - \Lambda^0[g_1] = \Lambda^f[g_2] - \Lambda^0[g_2]$ in both (i) and (ii) is a trivial consequence of (3.4). Note that in this case the unique information available for source reconstruction is given by only one measurement, say that Neumann boundary measurement

$$\partial_{\nu_x} u = \Lambda^f[g], \tag{3.10}$$

corresponding to some specific Dirichlet datum g, which without loss of generality can be assumed zero. Note also that problem (1.1) with Dirichlet datum $g = 0$ and source $f \in L^2(\Omega)$

has solution $u \in H^2(\Omega)$. The normal trace of this regular solution is in $H^{1/2}(\Gamma)$. So we have proved that the range of $\Lambda^f - \Lambda^0$ is in $H^{1/2}(\Gamma)$. The domain of $\Lambda^f - \Lambda^0$ is $H^{1/2}(\Omega)$ since this is the set of nonzero Dirichlet data that gives the same function in the range. □

Definition 3.6 (strong integral equation problem). Since in the inverse source problem the exact Cauchy data pair (g, g_{ν_x}) is given, the relative Dirichlet to Newman map value for the source problem (1.1)

$$\Lambda = \left(\Lambda^f - \Lambda^0\right)[g] = g_{\nu_x}(x) - \int_\Gamma g(\zeta)\frac{\partial^2 G(x,\zeta)}{\partial\nu_x\partial\nu_\zeta}\, d\sigma_\zeta \quad \text{for } x \in \Gamma, \tag{3.11}$$

is known and Lemma 3.5 suggests the following integral equation formulation for the source reconstruction problem: to find $f \in L^2(\Omega)$ such that

$$F[f] = \Lambda \quad \text{on } \Gamma, \tag{3.12}$$

where $F := \overline{\Lambda_0} = \Lambda[\cdot, 0] : L^2(\Omega) \to H^{1/2}(\Gamma)$,

$$F[f](x) := \int_\Omega f(\zeta)\frac{\partial G(x,\zeta)}{\partial\nu_x}\, d\zeta, \tag{3.13}$$

for $x \in \Gamma$.

Remark 3.7. Note that we introduce here F as a simplified notation to the extended Source to Neumann map $\overline{\Lambda_0}$. This notation is more usual.

The following corollary resumes all that has been discussed.

Corollary 3.8. *Supposed that $\kappa^2 \notin \Sigma_2$ and $\kappa^4 \notin \Sigma_4$. Then,*

(i) *for a Cauchy datum $(v, \partial_\nu v)|_\Gamma = (0, \Lambda) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$, there exists a unique function $f \in H_{-\Delta+\kappa^2}(\Omega)$ solution of the inverse source problem* (1.3) *for the Helmholtz equation* (1.1),

(ii) *the associated mapping*

$$F^{-1} : H^{1/2}(\Gamma) \longrightarrow H_{-\Delta+\kappa^2}(\Omega) \tag{3.14}$$

defines a linear homeomorphism between these spaces,

(iii) *and is a right inverse of the mapping $F : L^2(\Omega) \to H^{1/2}(\Omega)$ defined by the strong inverse source equation* (3.12),

(iv) *the projection $Q : L^2(\Omega) \to L^2(\Omega) \setminus H_{-\Delta+\kappa^2}(\Omega)$ is well defined and constant in the level set*

$$C_h = \left\{f \in L^2 : \left(-\Delta + \kappa^2\right) f = h\right\}, \tag{3.15}$$

(v) *if the source f is known to be in the class C_h, then a single boundary measurement* $(0, \Lambda)$ *is sufficient to identify* f*,*

(vi) *for a Cauchy datum* $(v, \partial_\nu v)|_\Gamma = (0, \Lambda) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$*, there are many functions* $\overline{f} = f + h$*, where* $f \in H_{-\Delta+\kappa^2}(\Omega)$ *is an observed consequence of (ii) and* $h \in (-\Delta + \kappa^2)[H_0^2(\Omega)] = L^2(\Omega) \setminus H_{-\Delta+\kappa^2}(\Omega)$ *is an arbitrary nonobserved function.*

Proof. The items are trivial consequences of the results already proved. □

Remark 3.9. Given $h \in C_h$, the unique solution referred to in Section 3.1 is the unique solution of the fourth-order direct problem [1]:

$$\begin{gathered}\left(\Delta^2 - \kappa^4\right) w^f = h \quad \text{in } \Omega, \\ w^f = 0 \quad \text{on } \Gamma, \\ \partial_\nu w^f = 0 \quad \text{on } \Gamma.\end{gathered} \tag{3.16}$$

Remark 3.10. The adopted Hilbert space framework for solution of the problem may be understood as an a priori information about the criteria for selecting the observable and the nonobservable part of the source. Other Sobolev spaces that induced partitions of the pivot space $L^p(\Omega), 1 \le p \le \infty$ (in this work $p = 2$) will modify this observability relation.

Remark 3.11 (relation between star-shaped and metaharmonic functions). Let us define the set

$$U_{\text{Type}} = \{\chi_\omega : \omega \subset \Omega \text{ is of type Type}\}. \tag{3.17}$$

The $U_{\text{squares}} \subset U_{\text{star shape}} \subset U_{\text{characteristic}}$. Note that U_{squares} is dense in $L^2(\Omega)$. If, for all $\omega \subset \Omega$, there exists a family of metaharmonic functions in $H_{-\Delta+\kappa^2}(\Omega)$ that approach χ_ω, then $H_{-\Delta+\kappa^2}(\Omega)$ is dense in $L^2(\Omega)$.

Remark 3.12. The most important classes of sources that may be reconstructed uniquely from boundary data occur when f is metaharmonic or when $f \equiv f\chi_\omega$, where χ_ω is the characteristic function of an open star-shaped set $\omega \subset \Omega$ with C^2 being boundary and f a $C^2(\overline{\Omega})$ function. We will discuss these classes when establishing uniqueness.

Remark 3.13 (the adjoint integral equation). This equation may be used for the source reconstruction independent of solution of the direct problem (1.1). By substituting the explicit integral definition of F in the duality definition of adjoint

$$\langle F[f], \psi \rangle_{H^{1/2} \times H^{-1/2}} = \langle f, F^*[\psi] \rangle_{L^2 \times L^2}, \tag{3.18}$$

we obtain that $F^* : H^{-1/2}(\Gamma) \to L^2(\Omega)$ is explicitly given by

$$h(\zeta) = F^*[\psi](\zeta) = \int_\Gamma \psi(x) \frac{\partial G(x, \zeta)}{\partial \nu_x} \, d\sigma_x. \tag{3.19}$$

Remark 3.14. From these formulas, we can deduce that, for a fixed $f \in L^2(\Omega)$, the operator

$$\Lambda_g^f(x) - \Lambda_g^0(x) = F[f](x), \quad x \in \Gamma, \tag{3.20}$$

for any $g \in H^2(\Gamma)$, we will call this operator Extended Dirichlet to Neumann map.

Remark 3.15. Once we know the integral formulation to F, we can determine the integral formulation to F^*. In fact,

$$\langle F[f], \psi \rangle_{H^{1/2} \times H^{-1/2}} = \langle f, F^*[\psi] \rangle_{L^2 \times L^2}, \tag{3.21}$$

and from

$$F[f](x) = \int_\Omega (\zeta) \frac{\partial G(x,\zeta)}{\partial \nu_x} d\zeta, \quad x \in \Gamma, \tag{3.22}$$

it follows that

$$F^*[\psi](\zeta) = \int_\Omega \psi(x) \frac{\partial G(x,\zeta)}{\partial \nu_x} d\sigma(x), \quad x \in \Gamma. \tag{3.23}$$

Remark 3.16. Consider the following direct problem

$$\begin{aligned} -\Delta w + k^2 w = 0 \quad & \text{in } \Omega, \\ \gamma_0 w = \psi \quad & \text{on } \Gamma, \end{aligned} \tag{3.24}$$

with $\psi \in H^{-1/2}(\Gamma)$. This problem has a unique solution $w \in L^2(\Omega)$. Let

$$w(\zeta) = S_{1,\psi}(\zeta) = \int_\Omega \psi(x) \frac{\partial G(x,\zeta)}{\partial \nu_x} d\sigma(x) = F^*[\psi](\zeta), \tag{3.25}$$

where G is the associated Green function. This happens for each $\psi \in H^{-1/2(\Gamma)}$ since $F^*[\psi](\zeta) = S_{1,\psi}(\zeta)$. From this we deduce that the integral $S_{1,\psi}$ inherits all good properties from F^*.

4. Integral and Variational Solutions: The Equivalence Theorem

4.1. Integral Formulation

With the integral formulas (3.8), (3.2), Definition (2.2), and supposing that the compatibility condition has been verified, we obtain that

$$F[f](x) = \Lambda_g^f(x) - \Lambda^0[g](x) \tag{4.1}$$

is the integral equation

$$\int_{\Omega} f(\zeta) \frac{\partial G(x,\zeta)}{\partial \nu_x} \, d\zeta = g_{\nu_x}(x) - \int_{\Gamma} g(\zeta) \frac{\partial G(x,\zeta)}{\partial \eta(x) \partial \eta(\zeta)} \, d\sigma(\zeta). \tag{4.2}$$

4.2. Variational Formulation

Definition 4.1 (reciprocity gap functional problem). One may use the second Green theorem

$$\int_{\Omega} (u\Delta v - v\Delta u) dx = \int_{\Gamma} \left(u \frac{\partial v}{\partial \nu_x} - v \frac{\partial u}{\partial \nu_x} \right) d\sigma_x, \tag{4.3}$$

valid for all $u, v \in H^2(\Omega)$ with u a solution of problem (1.1) to formulate the reciprocity gap functional inverse problem: to find $f \in L^2(\Omega)$

$$\int_{\Omega} f v \, dx = \int_{\Gamma} \left(g_{\nu_x} v - g \frac{\partial v}{\partial \nu_x} \right) d\sigma_x, \tag{4.4}$$

for all $v \in H_{-\Delta - \kappa^2}(\Omega)$.

Note that the Lax-Milgram theorem assures the existence of a solution in this case.

4.3. The Equivalence Theorem

Theorem 4.2. *Let one consider the two inverse source problems related with problem* (1.1) *with relative Dirichlet to Newman map* $\Lambda \in L^2(\Omega)$*:*

(i) *integral equation problem given by* (3.12)*;*

(ii) *reciprocity gap functional problem given by* (4.4).

Then $(i) \Rightarrow (ii)$*. Suppose additionally that the relative Dirichlet to Newman data* $\Lambda \in H^{1/2}(\Gamma)$*. Then* $(ii) \Rightarrow (i)$*.*

Proof. Let us consider the inverse source problem: to find $(u, f) \in H^1(\Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} -\Delta u - k^2 u &= f \quad \text{in } \Omega, \\ \gamma_0 u &= g \quad \text{on } \Gamma, \\ \gamma_1 u &= g_{\nu_x} \quad \text{on } \Gamma, \end{aligned} \tag{4.5}$$

where $g \in H^{1/2}(\Gamma)$, $g_{\nu_x} \in H^{-1/2}(\Gamma)$ with compatibility condition $g_\nu = \partial g / \partial \nu$.

(i) $\Rightarrow$ (ii).

We start the demonstration by supposing that (i) is true; that is, there exists $f \in L^2(\Omega)$ such that

$$\int_{\Omega} f(\zeta) \frac{\partial G(x,\zeta)}{\partial \nu_x} d\zeta = g_{\nu_x} - \int_{\Gamma} g(\zeta) \frac{\partial^2 G(x,\zeta)}{\partial \nu_x \partial \nu_\zeta} d\sigma(\zeta), \tag{4.6}$$

where G is the Green function associated with the Helmholtz operator in Ω.

Let $v \in H_{\Delta-\kappa^2}(\Omega)$ be extended to the boundary of Ω. We then have the following integral representation for v:

$$v(\zeta) = \int_{\Gamma} v(x) \frac{\partial G(x,\zeta)}{\partial \nu_x}(\zeta) d\sigma(x), \quad \zeta \in \Gamma. \tag{4.7}$$

By taking the normal trace

$$\frac{\partial v}{\partial \nu_\zeta}(\zeta) = \int_{\Gamma} v(x) \frac{\partial^2 G(x,\zeta)}{\partial \nu_x \partial \nu_\zeta} d\sigma(x), \quad \zeta \in \Gamma. \tag{4.8}$$

We now multiply (4.2) by v and integrate on Γ to obtain

$$\begin{aligned}
&\int_{\Gamma} v(x) \left[\int_{\Omega} f(\zeta) \frac{\partial G}{\partial \nu_x}(x,\zeta) d\zeta \right] d\sigma(x) \\
&\quad = \int_{\Gamma} v(x) \left[\varphi(x) - \int_{\Gamma} g(\zeta) \frac{\partial^2 G(x,\zeta)}{\partial \nu_x \partial \nu_\zeta} d\sigma(\zeta) \right] d\sigma(x) \\
&\quad = \int_{\Gamma} v(x)\varphi(x) d\sigma(x) - \int_{\Gamma} v(x) \left[\int_{\Gamma} g(\zeta) \frac{\partial^2 G(x,\zeta)}{\partial \nu_x \partial \nu_\zeta} d\sigma(\zeta) \right] d\sigma(x).
\end{aligned} \tag{4.9}$$

Now applying the Fubini theorem, we obtain

$$\begin{aligned}
&\int_{\Omega} f(\zeta) \overbrace{\left[\int_{\Gamma} v(x) \frac{\partial G(x,\zeta)}{\partial \nu_x} d\sigma(x) \right]}^{v(x)} d\zeta \\
&\quad = \int_{\Gamma} v(x) \overbrace{\varphi(x)}^{\partial g(x)/\partial \nu_x} d\sigma(x) - \int_{\Gamma} g(\zeta) \overbrace{\left[\int_{\Gamma} v(x) \frac{\partial^2 G(x,\zeta)}{\partial \nu_x \partial \eta(\zeta)} d\sigma(x) \right]}^{\partial v(\zeta)/\partial \nu_\zeta} d\sigma(\zeta),
\end{aligned} \tag{4.10}$$

which implies

$$\int_{\Omega} f(\zeta) v(\zeta) d\zeta = \int_{\Gamma} \left(v(\zeta) \frac{\partial g(\zeta)}{\partial \eta(\zeta)} - g(\zeta) \frac{\partial v(\zeta)}{\partial \eta(\zeta)} \right) d\sigma(\zeta). \tag{4.11}$$

Since $v \in H_{-\Delta+\kappa^2}(\Omega)$ is arbitrary, we obtain the weak formulation. Note that this is almost expected, since the integral formulation is stronger than the variational, and, as usual, strong $\Rightarrow$ weak.

(ii) $\Rightarrow$ (i).

Let us now suppose that for all test functions $v \in H_{-\Delta+\kappa^2}(\Omega)$ we have the weak reciprocity integral equation (4.4)

$$\int_\Omega f(x)v(x)dx = \int_\Gamma g_\nu(x)v(x)d\sigma(x) - \int_\Gamma g(x)\frac{\partial v(x)}{\partial \nu}\,d\sigma(x). \tag{4.12}$$

Consider the substitution of the Green function integral representation

$$v(\zeta) = \int_\Gamma v(x)\frac{\partial G(x,\zeta)}{\partial \nu_x}\,d\sigma(x), \tag{4.13}$$

whose normal derivative is

$$\frac{\partial v(\zeta)}{\partial \nu_\zeta} = \int_\Gamma v(x)\frac{\partial^2 G(x,\zeta)}{\partial \nu_\zeta \partial \nu_x}\,d\sigma(x), \tag{4.14}$$

in (4.4) and apply Fubini's theorem to obtain

$$\begin{aligned}&\int_\Gamma v(x)\int_\Omega f(\zeta)\frac{\partial G(x,\zeta)}{\partial \nu_x}\,d\zeta\,d\sigma(x)\\ &\quad= \int_\Gamma v(x)\left[\int_\Gamma g_\nu(\zeta)\frac{\partial G(x,\zeta)}{\partial \nu_x}\,d\sigma(x) - \int_\Gamma g(\zeta)\frac{\partial^2 G(x,\zeta)}{\partial \nu_x\partial \nu_\zeta}\,d\sigma(\zeta)\right]d\sigma(x).\end{aligned} \tag{4.15}$$

Note that since

(i)

$$\int_\Gamma g(\zeta)\frac{\partial^2 G(x,\zeta)}{\partial \nu_x\partial \nu_\zeta}\,d\sigma(\zeta) = \Lambda^0[g], \tag{4.16}$$

is the Dirichlet to Neumann map,

(ii) and

$$\int_\Gamma g_\nu(\zeta)\frac{\partial G(x,\zeta)}{\partial \nu_x}\,d\sigma(x) = g_\nu(\zeta), \tag{4.17}$$

we obtain

$$\int_\Gamma v(x)\left[\int_\Omega f(\zeta)\frac{\partial G(x,\zeta)}{\nu_x}\,d\zeta - g_\nu + \Lambda^0[g]\right]d\sigma(x) = 0, \tag{4.18}$$

for all $v \in H^{1/2}(\Gamma) \cong H_{-\Delta+\kappa^2}(\Omega)$. By property of the integral, we obtain (3.12). □

5. Examples on the Unitary Disk

5.1. The Green Function for the Helmholtz Equation Dirichlet Problem in Circular Domains

In this section we will consider the Green function determination when the domain $\Omega \subset \mathbb{R}^2$ in problem (1.1) is circular with respect to the polar coordinate system.

A Green function to problem (1.1) is a solution of

$$\begin{gathered} -\Delta G(\zeta, x) - \kappa^2 G(\zeta, x) = \delta(\zeta - x), \quad \zeta \in \Omega, \\ G(\zeta, x) = 0, \quad \zeta \in \partial\Omega, \end{gathered} \tag{5.1}$$

where $x \in \Omega$ is the localization of the delta Dirac source. We may use the linearity of the problem for decomposing the solution in two additive parts

$$G(\zeta, x) = F(\zeta, x) + \frac{1}{2}\left(G_F(\zeta, x) + G_F(x, \zeta)\right). \tag{5.2}$$

Here F is the fundamental solution for the free space Helmholtz equation

$$-\Delta F(\zeta, x) - \kappa^2 F(\zeta, x) = \delta(\zeta - x), \tag{5.3}$$

and G_F is a homogeneous source regular solution of the Helmholtz equation

$$\begin{gathered} -\Delta G_F(\zeta, x) - \kappa^2 G_F(\zeta, x) = 0, \quad \zeta \in \Omega, \\ G(\zeta, x) = -F(\zeta, x), \quad \zeta \in \partial\Omega, \end{gathered} \tag{5.4}$$

and $x \in \overline{\Omega}$. In polar coordinates with $r = |\zeta - x|$

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F(r)}{\partial r}\right) - \kappa^2 F(r) = \delta(r) \quad \text{for } r \geq 0, \tag{5.5}$$

has at $r = 0$ singular behavior solution

$$F(\kappa r) = -\frac{1}{4}Y_0(\kappa r), \tag{5.6}$$

which is a Bessel function of second kind.

Remark 5.1. When κ is small, this solution has a singularity that has the same behavior of the logarithmic function, that is,

$$\begin{gathered} Y_0(\kappa r) \approx -\frac{2}{\pi}\log r, \\ \lim_{\kappa \to 0^+}\left(\Delta + \kappa^2\right) = \Delta \Longrightarrow \lim_{\kappa \to 0^+}\frac{1}{4}Y_0(\kappa r) = -\frac{1}{2}\log r. \end{gathered} \tag{5.7}$$

Remark 5.2 (addition theorem). Let $x = (\rho\cos(\theta), \rho\sin(\theta))$ and $\zeta = (\sigma\cos(\beta), \sigma\sin(\beta))$. Then

$$r = \sqrt{\sigma^2 + \rho^2 - 2\,\sigma\rho\cos(\beta-\theta)}, \tag{5.8}$$

$$\begin{aligned} Y_0(\kappa r) &= \sum_{n=-\infty}^{+\infty} Y_n(\kappa\sigma) J_n(\kappa\rho)\cos(n(\beta-\theta)), \\ 0 &= \sum_{n=-\infty}^{+\infty} Y_n(\kappa\sigma) J_n(\kappa\rho)\sin(n(\beta-\theta)), \\ Y_0(\kappa r) &= \sum_{n=-\infty}^{+\infty} Y_n(\kappa\sigma) J_n(\kappa\rho) e^{in(\beta-\theta)}. \end{aligned} \tag{5.9}$$

The nonhomogeneous Dirichlet boundary condition in (5.4) is

$$G(\zeta, x) = \frac{1}{4} Y_0\left(\kappa\sqrt{1+\rho^2-2\rho\cos(\beta-\theta)}\right), \quad \zeta \in \partial\Omega. \tag{5.10}$$

The regular solution of (5.4) is

$$G_F(\sigma,\beta;\rho,\theta) = \sum_{n=-\infty}^{+\infty} c_n(\rho,\theta) J_n(\kappa\sigma) e^{in\beta}, \tag{5.11}$$

where the coefficients $\overline{c}_n = -c_{-n}$ are determined by the Dirichlet boundary condition:

$$\begin{aligned} G_F(1,\beta;\rho,\theta) &= \frac{1}{4} Y_0\left(\kappa\sqrt{1+\rho^2-2\rho\cos(\beta-\theta)}\right) = \frac{1}{4}\sum_{n=-\infty}^{+\infty} Y_n(\kappa) J_n(\kappa\rho) e^{in(\beta-\theta)} \\ &= \frac{1}{4}\sum_{n=-\infty}^{+\infty} Y_n(\kappa\rho) J_n(\kappa) e^{in(\beta-\theta)} = \sum_{n=-\infty}^{+\infty} c_n(\rho,\theta) J_n(\kappa) e^{in\beta}. \end{aligned} \tag{5.12}$$

Since the solutions $\{e^{in\beta},\ n = -\infty, +\infty\}$ are linearly independent, we obtain

$$J_n(\kappa) c_n = \frac{1}{4} Y_n(\kappa) J_n(\kappa\rho) e^{-in\theta} = \frac{1}{4} Y_n(\kappa\rho) J_n(\kappa) e^{-in\theta}, \tag{5.13}$$

and when κ is not a root of the Bessel function J_n,

$$c_n(\rho,\theta) = \frac{1}{4}\frac{Y_n(\kappa)J_n(\kappa\rho)}{J_n(\kappa)}e^{-in\theta} = \frac{1}{4}\frac{Y_n(\kappa\rho)J_n(\kappa)}{J_n(\kappa)}e^{-in\theta}, \tag{5.14}$$

$$G_F(\sigma,\beta;\rho,\theta) = \frac{1}{4}\sum_{n=-\infty}^{+\infty}\frac{Y_n(\kappa)J_n(\kappa\rho)}{J_n(\kappa)}J_n(\kappa\sigma)e^{in(\beta-\theta)} = \frac{1}{4}\sum_{n=-\infty}^{+\infty}\frac{Y_n(\kappa\rho)J_n(\kappa)}{J_n(\kappa)}J_n(\kappa\sigma)e^{in(\beta-\theta)}, \tag{5.15}$$

$$G_F(\rho,\theta;\sigma,\beta) = \frac{1}{4}\sum_{n=-\infty}^{+\infty}\frac{Y_n(\kappa)J_n(\kappa\sigma)}{J_n(\kappa)}J_n(\kappa\rho)e^{in(\beta-\theta)} = \frac{1}{4}\sum_{n=-\infty}^{+\infty}\frac{Y_n(\kappa\sigma)J_n(\kappa)}{J_n(\kappa)}J_n(\kappa\rho)e^{in(\beta-\theta)}, \tag{5.16}$$

by noting that by addition theorem

$$F(\zeta,x) = -\frac{1}{4}\sum_{n=-\infty}^{+\infty}Y_n(\kappa\sigma)J_n(\kappa\rho)e^{in(\beta-\theta)} = -\frac{1}{4}\sum_{n=-\infty}^{+\infty}Y_n(\kappa\rho)J_n(\kappa\sigma)e^{in(\beta-\theta)}. \tag{5.17}$$

By substituting (5.6) and (5.15) and using the addition theorem (5.9) in (5.4), we finally obtain the Green function for the circular domain Helmholtz equation problem (5.1)

$$\begin{aligned} G(\sigma,\beta;\rho,\theta) = &-\frac{1}{8}\sum_{n=-\infty}^{+\infty}\frac{J_n(\kappa)Y_n(\kappa\sigma) - Y_n(\kappa)J_n(\kappa\sigma)}{J_n(\kappa)}J_n(\kappa\rho)e^{in(\beta-\theta)} \\ &-\frac{1}{8}\sum_{n=-\infty}^{+\infty}\frac{J_n(\kappa)Y_n(\kappa\rho) - Y_n(\kappa)J_n(\kappa\rho)}{J_n(\kappa)}J_n(\kappa\sigma)e^{in(\theta-\beta)}, \end{aligned} \tag{5.18}$$

if we define $r >= \max\sigma,\rho$ and $r <= \min\sigma,\rho$, which can also be rewritten as

$$G(\sigma,\beta;\rho,\theta) = -\frac{1}{4}\sum_{n=-\infty}^{+\infty}\frac{J_n(\kappa)Y_n(\kappa r >) - Y_n(\kappa)J_n(\kappa r >)}{J_n(\kappa)}J_n(\kappa r <)e^{in(\beta-\theta)}. \tag{5.19}$$

5.2. The Integral Equation Kernel for the Unitary Disk

The kernel of the integral equation is obtain by taking the normal derivative trace of the Green function (5.18). By using the Wronskian identity for the Bessel functions

$$J_n(z)Y_n'(z) - J_n'(z)Y_n(z) = \frac{2}{\pi z}, \tag{5.20}$$

we obtain

$$P_\kappa(\rho,\beta-\theta) = \frac{\partial G(x,\zeta)}{\partial \nu_x} = \frac{\partial G(\sigma,\beta;\rho,\theta)}{\partial\sigma}\Big|_{\sigma=1} = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\frac{J_n(\kappa\rho)}{J_n(\kappa)}e^{in(\beta-\theta)}, \tag{5.21}$$

which is the modified Poisson kernel for Dirichlet Helmholtz equation problem in the disk. Note that, when $\kappa \to 0$, it tends to the classical Poisson kernel for the disk.

Note that, when $\kappa \equiv i\kappa$ is substitute in (1.1), it becomes a modified Helmholtz equation for which extensions of the Maximum Modulo Principle and the Strong Maximum Principle for metaharmonics functions are applicable. In this case the Poisson Kernel may be rewrit with modified Bessel function of first kind

$$I_n(r) = e^{-in(\pi/2)} J_n(ir), \tag{5.22}$$

$$P_{i\kappa}(\rho,\beta-\theta) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\frac{I_{|n|}(\kappa\rho)}{I_{|n|}(\kappa)}e^{in(\beta-\theta)}. \tag{5.23}$$

Remark 5.3. Substituting the kernel (5.23) in (3.12), we obtain the strong inverse source equation (3.12) for unitary disk

$$\int_0^{2\pi}\int_0^1\left(\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\frac{I_{|n|}(\kappa\rho)}{I_{|n|}(\kappa)}e^{in(\beta-\theta)}\right)f(\rho,\theta)\rho\, d\rho\, d\theta = \Lambda(\beta), \quad \beta\in[0,2\pi). \tag{5.24}$$

Remark 5.4. Another important class of sources to be reconstructed is the characteristic source with star-shaped boundary with parametrization given by $(x,y) = (r(\theta)\cos(\theta), r(\theta)\sin(\theta))$, with $r \in H^1(0,2\pi)$. The strong inverse star-shaped characteristic source equation (3.12) for unitary disk is

$$\int_0^{2\pi}\int_0^{r(\theta)}\left(\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\frac{I_{|n|}(\kappa\rho)}{I_{|n|}(\kappa)}e^{in(\beta-\theta)}\right)\rho\, d\rho\, d\theta = \Lambda(\beta), \quad \beta\in[0,2\pi). \tag{5.25}$$

Remark 5.5. Note that the kernel with finite sum

$$P_{i\kappa}^N(\rho,\beta-\theta) = \frac{1}{2\pi}\sum_{n=-N}^{+N}\frac{I_{|n|}(\kappa\rho)}{I_{|n|}(\kappa)}e^{in(\beta-\theta)}, \tag{5.26}$$

converges with respect to (ρ,θ) pointwise to $P_{i\kappa}(\rho,\beta-\theta)$. Since

$$\begin{aligned}\left|P_{i\kappa}(\rho,\beta-\theta)\right| &= \frac{1}{2\pi}\sum_{n=-N}^{+N}\frac{I_{|n|}(\kappa\rho)}{I_{|n|}(\kappa)} = \frac{1}{2\pi}\sum_{n=-N}^{+N}\rho^{|n|}\frac{1+\sum_{m=1}^{\infty}\left(\left(\kappa^2/4\right)^m|n|!\rho^{2m}/m!(|n|+m)!\right)}{1+\sum_{m=1}^{\infty}\left(\left(\kappa^2/4\right)^m|n|!/m!(|n|+m)!\right)}\\ &\le \frac{1}{2\pi}\sum_{n=-N}^{+N}\rho^{|n|},\end{aligned} \tag{5.27}$$

it is bounded by the function $\rho d\rho$ integrable $2/(1-\rho)$, in such a way that the limit and the integral can be transposed.

Remark 5.6. The transposed equation

$$\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\left(\int_0^{2\pi}\int_0^{r(\theta)}\frac{I_{|n|}(\kappa\rho)}{I_{|n|}(\kappa)}e^{-in\theta}\rho\,d\rho\,d\theta\right)e^{in\beta}=\Lambda(\beta),\quad \beta\in[0,2\pi), \tag{5.28}$$

can be Fourier transformed giving

$$\int_0^{2\pi}\int_0^{r(\theta)}\frac{I_{|n|}(\kappa\rho)}{I_{|n|}(\kappa)}e^{-in\theta}\rho\,d\rho\,d\theta=\widehat{\Lambda}(n),\quad n\in\mathcal{Z}. \tag{5.29}$$

5.3. The Variational Formulation for the Unitary Disk

Take $\Omega = D$ as the unitary disk. The weak equation (4.4) can also be specialized for the characteristic star-shaped source $\chi_\omega = \chi_r$ inside the unitary disk Modified Helmholtz problem

$$\begin{aligned}\int_D v(\rho,\theta)\chi_\omega(\rho,\theta)\rho\,d\rho\,d\theta&=\int_0^{2\pi}\int_0^{r(\theta)}v(\rho,\theta)d\rho\,d\theta\\&=\int_0^{2\pi}\left(g_{\nu_x}(\theta)v(1,\theta)-g(\theta)\frac{\partial v}{\partial\nu_x}(\theta)\right)d\theta,\end{aligned} \tag{5.30}$$

for all $v\in H_{-\Delta+\kappa^2}(D)$. As we have proved in Lemma 3.5, without loss of generality we can consider the data from the homogeneous Dirichlet problem $g=0$. Note that the Modified Poisson Dirichlet kernel

$$P_{i\kappa}(\rho,\beta-\theta)=\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\frac{I_{|n|}(\kappa\rho)}{I_{|n|}(\kappa)}e^{in(\beta-\theta)}, \tag{5.31}$$

is a function inside $H_{-\Delta+\kappa^2}(D)$ and can be substituted in this equation giving

$$\int_0^{2\pi}\int_0^{r(\theta)}\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\frac{I_{|n|}(\kappa\rho)}{I_{|n|}(\kappa)}e^{in(\beta-\theta)}\rho\,d\rho\,d\theta=\int_0^{2\pi}g_{\nu_x}(\theta)\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}e^{in(\beta-\theta)}d\theta. \tag{5.32}$$

Proposition 5.7. *If $g_{\nu_x}\in H^l(0,2\pi)$ for $l\geq 1/2$, then*

$$\int_0^{2\pi}g_{\nu_x}(\theta)\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}e^{in(\beta-\theta)}d\theta=g_{\nu_x}(\beta). \tag{5.33}$$

Proof. This is consequence from the fact that $D_N=\sum_{n=-N}^{+N}e^{in(\beta-\theta)}$ is the Dirichlet kernel of order N. When it acts on a continuous function, the fast oscillations of D_N far from $\theta=\beta$ do not contribute to N-truncated Fourier series

$$g_{\nu_x}^N(\beta)=\frac{1}{2\pi}\int_0^{2\pi}g_{\nu_x}(\theta)D_N(\theta-\beta)d\theta. \tag{5.34}$$

As N grows, the dominant contribution to this integral comes from an arbitrarily small neighborhood of $\theta - \beta$. This behavior gives the Dirac delta distribution character to the series

$$\delta(\theta - \beta) = \lim N \longrightarrow \infty D_N(\theta - \beta), \tag{5.35}$$

and the integral is well defined only when g_{ν_x} is a continuous function. By Sobolev embedding, the minimal admissible index is $1/2$. □

Remark 5.8. Note that this result is in agreement with the sufficient condition for (ii) ⇒ (i) in the equivalence theorem Section 4.3.

5.4. The Integral Operator for Star-Shaped Characteristic Source

Let us consider the nonlinear mapping

$$\mathcal{F}[r] = F[\chi_r], \tag{5.36}$$

where F is the function in the formulation of problem given by (3.12). Note that the strong integral equation (5.25) can be formally stated as the nonlinear problem: to find $r(\theta)$ such that

$$\mathcal{F}[r] = \Lambda. \tag{5.37}$$

We can investigate the operator $\mathcal{F} : H^1(0, 2\pi) \to H^l(0, 2\pi)$ with respect to a possible set of values $l \in \mathbb{R}$ for which the regularity of the source boundary r influences the regularity in the range of the functional.

Proposition 5.9. *One considers two possible cases for the star-shaped source inside the unitary disk in which the source boundary can do or do not touch the disk border*

(i) *if* $r(\theta) \leq 1$, *then* $\mathcal{F}[r] \in H^l(0, 2\pi)$ *with* $l < 1/2$;

(ii) *when* $r(\theta) < 1$, $\mathcal{F}[r] \in C^\infty(0, 2\pi)$.

Proof. We will estimate the functional norm in $H^l(0, 2\pi)$ by using its Fourier transform (5.29). Note that

$$\begin{gathered}\frac{I_{|n|}(\kappa\rho)}{I_{|n|}(\kappa)} = \rho^{|n|} \frac{1 + \sum_{m=1}^{\infty}\left(\left(\kappa^2/4\right)^m |n|! \rho^{2m} / m!(|n| + m)!\right)}{1 + \sum_{m=1}^{\infty}\left(\left(\kappa^2/4\right)^m |n|! / m!(|n| + m)!\right)} \leq \rho^{|n|}, \\ \int_0^{r(\theta)} \frac{I_{|n|}(\kappa\rho)}{I_{|n|}(\kappa)} \rho \, d\rho \leq \frac{r(\theta)^{|n|+2}}{|n| + 2}.\end{gathered} \tag{5.38}$$

Since, for $r \leq 1$, $\int_0^{r(\theta)} (I_{|n|}(\kappa\rho)/I_{|n|}(\kappa))\rho\, d\rho \leq 1/(|n|+2)$ and $(1+n^2)^l/(|n|+2)^2 \leq 2^l |n|^{2l-2}$, we obtain that

$$\begin{aligned}\|\mathcal{F}[r]\|_{H^l(0,2\pi)} &= \frac{1}{2\pi}\sum_{-\infty}^{\infty}\left(1+n^2\right)^l \left| \int_0^{2\pi} \left(\int_0^{r(\theta)} \frac{I_{|n|}(\kappa\rho)}{I_{|n|}(\kappa)} \rho\, d\rho \right) e^{-in\theta} d\theta \right|^2 \\ &\leq \sum_{-\infty}^{\infty}\left(1+n^2\right)^l \left| \int_0^{r(\theta)} \frac{I_{|n|}(\kappa\rho)}{I_{|n|}(\kappa)} \rho\, d\rho \right|^2 \leq \sum_{-\infty}^{\infty} \frac{(1+n^2)^l}{(|n|+2)^2} \leq \sum_{-\infty}^{\infty} 2^l |n|^{2l-2},\end{aligned} \tag{5.39}$$

converges when $2l-2<-1$, which proves (i). If $2l-2=-1$, that is, $l=1/2$, the upper bound series results in the divergent harmonic series. Note that the result (i) in the proposition gives only a sufficient condition, since

$$\begin{aligned}&\|\mathcal{F}[r]\|_{H^l(0,2\pi)} \\ &= \frac{1}{2\pi}\sum_{-\infty}^{\infty}\left(1+n^2\right)^l \left| \int_0^{2\pi} \left(\int_0^{r(\theta)} \rho^{|n|} \frac{1+\sum_{m=1}^{\infty}\left(\left(\kappa^2/4\right)^m |n|!\rho^{2m}/m!(|n|+m)!\right)}{1+\sum_{m=1}^{\infty}((\kappa^2/4)^m |n|!/m!(|n|+m)!)} \rho\, d\rho \right) e^{-in\theta} d\theta \right|^2 \\ &= \sum_{-\infty}^{\infty}\left(1+n^2\right)^l \\ &\quad \times \left| \int_0^{2\pi} \frac{r(\theta)^{|n|+2}}{|n|+2} \frac{1+\sum_{m=1}^{\infty}((|n|+2)/(|n|+2m+2))\left(\left(\kappa^2/4\right)^m |n|! r(\theta)^{2m}\right)/m!(|n|+m)!)}{1+\sum_{m=1}^{\infty}((\kappa^2/4)^m |n|!/m!(|n|+m)!)} e^{-in\theta} d\theta \right|^2,\end{aligned} \tag{5.40}$$

if $r(\theta)=1$ for all $\theta \in [0,2\pi)$, then

$$\|\mathcal{F}[r]\|_{H^l(0,2\pi)} = 1 \quad \forall l \in \mathbb{R}. \tag{5.41}$$

For (ii) we note that $r \in H^1(0,2\pi) \subset C(0,2\pi)$ and that $\sup_\theta r(\theta) = \|r\|_\infty < 1$. In this case, for each $m \in \mathbb{N}$, we have

$$\lim_{n\to\infty}\left(|n|^m \left|\widehat{\mathcal{F}[r]}(n)\right|\right) = \lim_{n\to\infty}\left(\frac{|n|^m \mid \|r\|_\infty^{|n|+2}}{|n|+2}\right) = 0, \tag{5.42}$$

which proves that $\mathcal{F}[r] \in C^\infty(0,2\pi)$. □

Proposition 5.10. *Suppose that the source star-shaped boundaries do not touch the unitary circumference. Then*

$$\mathcal{F} : \mathcal{N}[r] \subset H^1(0,2\pi) \longrightarrow H^l(0,2\pi), \tag{5.43}$$

is locally Lipschitz continuous for every $l \in \mathbb{R}$; that is, for all r, there exists a neighborhood $\mathcal{N}(r)$ of r in $H^1(0,2\pi)$ such that $\mathcal{F}$ is Lipschitz.

Proof. Let r_1 and r_2 be two star-shaped boundaries in the same neighborhood. Then

$$\|\mathcal{F}[r_1] - \mathcal{F}[r_2]\|_{H^l(0,2\pi)} = \left(\frac{1}{2\pi} \sum_{-\infty}^{\infty} \left(1+n^2\right)^l \left| \int_0^{2\pi} (F[r_1] - F[r_2]) e^{-in\theta} d\theta \right|^2 \right)^{1/2}, \tag{5.44}$$

where

$$\begin{aligned} &\mathcal{F}[r_1] - \mathcal{F}[r_2] \\ &\quad = \frac{\left(r_1(\theta)^{|n|+2} - r_2(\theta)^{|n|+2}\right)/(|n|+2)}{1 + \sum_{m=1}^{\infty}\left((\kappa^2/4)^m |n|!/m!(|n|+m)!\right)} \\ &\qquad + \frac{\sum_{m=1}^{\infty}(1/(|n|+2m+2))\left((\kappa^2/4)^m |n|!\left(r_1(\theta)^{|n|+2+2m} - r_2(\theta)^{|n|+2+2m}\right)/m!(|n|+m)!\right)}{1 + \sum_{m=1}^{\infty}\left((\kappa^2/4)^m |n|!/m!(|n|+m)!\right)}, \end{aligned} \tag{5.45}$$

can be factored in $(r_1(\theta) - r_2(\theta))$

$$\mathcal{F}[r_1] - \mathcal{F}[r_2] = (r_1(\theta) - r_2(\theta)) \times f_{|n|}(\theta, r_1(\theta), r_2(\theta)), \tag{5.46}$$

where the factor function

$$\begin{aligned} &f_{|n|}(\theta, r_1(\theta), r_2(\theta)) \\ &\quad := \frac{\sum_{t=0}^{|n|+1} r_1(\theta)^{|n|+1-t} r_2(\theta)^t}{(|n|+2)\left(1 + \sum_{m=1}^{\infty}\left((\kappa^2/4)^m |n|!/m!(|n|+m)!\right)\right)} \\ &\qquad + \frac{\sum_{m=1}^{\infty}((|n|+2)/(|n|+2m+2))\left((\kappa^2/4)^m |n|!/m!(|n|+m)!\right) \sum_{t=0}^{|n|+1+2m} r_1(\theta)^{|n|+1+2m-t} r_2(\theta)^t}{(|n|+2)\left(1 + \sum_{m=1}^{\infty}\left((\kappa^2/4)^m |n|!/m!(|n|+m)!\right)\right)} \\ &\quad \geq 0 \end{aligned} \tag{5.47}$$

is nonnegative. So

$$\|\mathcal{F}[r_1]-\mathcal{F}[r_2]\|_{H^l(0,2\pi)}=\left(\frac{1}{2\pi}\sum_{-\infty}^{\infty}\left(1+n^2\right)^l\left|\int_0^{2\pi}(r_1(\theta)-r_2(\theta))\times f_{|n|}(\theta,r_1(\theta),r_2(\theta))e^{-in\theta}d\theta\right|^2\right)^{1/2}$$

$$\|\mathcal{F}[r_1]-\mathcal{F}[r_2]\|_{H^l(0,2\pi)}\leq\left(\frac{1}{2\pi}\sum_{-\infty}^{\infty}\left(1+n^2\right)^l\int_0^{2\pi}|r_1(\theta)-r_2(\theta)|^2 f_{|n|}(\theta,r_1(\theta),r_2(\theta))^2d\theta\right)^{1/2}, \tag{5.48}$$

or

$$\|\mathcal{F}[r_1]-\mathcal{F}[r_2]\|_{H^l(0,2\pi)}\leq C_0\|r_1(\theta)-r_2(\theta)\|_{C(0,2\pi)}, \tag{5.49}$$

where

$$C_0\geq\left(\sum_{-\infty}^{\infty}\left(1+n^2\right)^l\int_0^{2\pi}\left(f_{|n|}(\theta,r_1(\theta),r_2(\theta))\right)^2d\theta\right)^{1/2} \tag{5.50}$$

is the Lipschitz constant. By denoting $r_0=\max_{\theta\in(0,2\pi)}\{r_1(\theta),r_2(\theta)\}$, we have the following estimate:

$$\begin{aligned}f_{|n|}(\theta,r_0,r_0)&=r_0^{|n|+1}\frac{1+\sum_{m=1}^{\infty}((|n|+2)/(|n|+2m+2))\left(\left(\kappa^2/4\right)^m|n|!/m!(|n|+m)!\right)r_0^{2m}}{1+\sum_{m=1}^{\infty}\left(\kappa^2/4\right)^m|n|!/m!(|n|+m)!}\\&\leq r_0^{|n|+1}\end{aligned} \tag{5.51}$$

which gives

$$C_0=\left(\sum_{-\infty}^{\infty}\left(1+n^2\right)^l r_0^{2(|n|+1)}\right)^{1/2}<\infty\quad\forall l\in\mathbb{R}. \tag{5.52}$$

Note that we have used the boundedness of the embedding of $H^1(0,2\pi)$ in $C(0,2\pi)$ to find a neighborhood $\mathcal{N}[r_1]$ of $r_1<1\in H^1(0,2\pi)$ such that $\sup\{\|r_2\|_\infty : r_2\in\mathcal{N}[r_1]\}<1$. This embedding says that there also exists a neighborhood in $H^1(0,2\pi)$ such that the nonlinear mapping $\mathcal{F}[r]$ is Lipschitz continuous, as enunciated in the proposition. □

6. Conclusions

The central question investigated in this paper is nonuniqueness of the inverse source problem, which is related with nonobservability of the source by using only boundary data. The Hilbert space framework constrains the class of functions that can be reconstructed and may be considered a kind of a priory information about the source. For more generic Banach spaces and other optimal formulations, different sources may be obtained. The demonstrated equivalence Theorem 4.2 can be used to investigate questions such as stability and regularization. Further numerical studies for the unitary disk based on the equations presented in Section 5 remain as future work.

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