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Research Article

Hypothesis Testing in Generalized Linear Models with Functional Coefficient Autoregressive Processes

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The paper studies the hypothesis testing in generalized linear models with functional coefficient autoregressive (FCA) processes. The quasi-maximum likelihood (QML) estimators are given, which extend those estimators of Hu (2010) and Maller (2003). Asymptotic chi-squares distributions of pseudo likelihood ratio (LR) statistics are investigated.

1. Introduction

Consider the following generalized linear model:

$$y_t = g\left(x_t^T \beta\right) + \varepsilon_t, \quad t = 1, 2, \dots, n, \tag{1.1}$$

where β is d-dimensional unknown parameter, $\{\varepsilon_t, t = 1, 2, \dots, n\}$ are functional coefficient autoregressive processes given by

$$\varepsilon_1 = \eta_1, \qquad \varepsilon_t = f_t(\theta)\varepsilon_{t-1} + \eta_t, \qquad t = 2, 3, \dots, n, \tag{1.2}$$

where $\{\eta_t, t = 1, 2, \dots, n\}$ are independent and identically distributed random variable errors with zero mean and finite variance σ^2, θ is a one-dimensional unknown parameter, and $f_t(\theta)$ is a real valued function defined on a compact set Θ which contains the true value θ_0 as an inner point and is a subset of R^1. The values of θ_0 and σ^2 are unknown. $g(\cdot)$ is a known continuous differentiable function.

Model (1.1) includes many special cases, such as an ordinary regression model (when $f_t(\theta) \equiv 0$, $g(\tau) = \tau$; see [1–7]), an ordinary generalized regression model (when $f_t(\theta) \equiv 0$; see [8–13]), a linear regression model with constant coefficient autoregressive processes (when $f_t(\theta) = \theta$, $g(\tau) = \tau$; see [14–16]), time-dependent and function coefficient autoregressive processes (when $g(\tau) = 0$; see [17]), constant coefficient autoregressive processes (when $f_t(\theta) = \theta$, $g(\tau) = 0$; see [18–20]), time-dependent or time-varying autoregressive processes (when $f_t(\theta) = a_t$, $g(\tau) = 0$; see [21–23]), and a linear regression model with functional coefficient autoregressive processes (when $g(\tau) = \tau$; see [24]). Many authors have discussed some special cases of models (1.1) and (1.2) (see [1–24]). However, few people investigate the model (1.1) with (1.2). This paper studies the model (1.1) with (1.2). The organization of this paper is as follows. In Section 2, some estimators are given by the quasi-maximum likelihood method. In Section 3, the main results are investigated. The proofs of the main results are presented in Section 4, with the conclusions and some open problems in Section 5.

2. The Quasi-Maximum Likelihood Estimate

Write the "true" model as

$$y_t = g\left(x_t^T \beta_0\right) + e_t, \quad t = 1, 2, \ldots, n, \tag{2.1}$$

$$e_1 = \eta_1, \qquad e_t = f_t(\theta_0) e_{t-1} + \eta_t, \quad t = 2, 3, \ldots, n, \tag{2.2}$$

where $g'(\tau) = (dg(\tau)/d\tau) \neq 0$, $f_t'(\theta) = (df_t(\theta)/d\theta) \neq 0$. Define $\prod_{i=0}^{-1} f_{t-i}(\theta_0) = 1$, and by (2.2), we have

$$e_t = \sum_{j=0}^{t-1} \left(\prod_{i=0}^{j-1} f_{t-i}(\theta_0) \right) \eta_{t-j}. \tag{2.3}$$

Thus e_t is measurable with respect to the σ–field H generated by $\eta_1, \eta_2, \ldots, \eta_t$, and

$$Ee_t = 0, \qquad \operatorname{Var}(e_t) = \sigma_0^2 \sum_{j=0}^{t-1} \left(\prod_{i=0}^{j-1} f_{t-i}^2(\theta_0) \right). \tag{2.4}$$

Assume at first that the η_t are i.i.d. $N(0, \sigma^2)$, we get the log-likelihood of $y_2, \ldots, y_n$ conditional on y_1 given by

$$\Phi_n = \ln L_n = -\frac{(n-1)\ln \sigma^2}{2} - \frac{\sum_{t=2}^{n} \left(\varepsilon_t - f_t(\theta)\varepsilon_{t-1}\right)^2}{2\sigma^2} - \frac{(n-1)\ln 2\pi}{2}. \tag{2.5}$$

At this stage we drop the normality assumption, but still maximize (2.5) to obtain QML estimators, denoted by $\widehat{\sigma}_n^2, \widehat{\beta}_n, \widehat{\theta}_n$. The estimating equations for unknown parameters in (2.5) may be written as

$$\frac{\partial \Phi_n}{\partial \sigma^2} = -\frac{n-1}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{t=2}^{n}\left(\varepsilon_t - f_t(\theta)\varepsilon_{t-1}\right)^2, \tag{2.6}$$

$$\begin{aligned} \frac{\partial \Phi_n}{\partial \theta} &= \frac{1}{\sigma^2}\sum_{t=2}^{n} f_t'(\theta)\left(\varepsilon_t - f_t(\theta)\varepsilon_{t-1}\right)\varepsilon_{t-1}, \\ \frac{\partial \Phi_n}{\partial \beta_{d\times 1}} &= \frac{1}{\sigma^2}\sum_{t=2}^{n}\left(\varepsilon_t - f_t(\theta)\varepsilon_{t-1}\right)\cdot\left(g'\left(x_t^T\beta\right)x_t - f_t(\theta)g'\left(x_{t-1}^T\beta\right)x_{t-1}\right). \end{aligned} \tag{2.7}$$

Thus, $\widehat{\sigma}_n^2, \widehat{\beta}_n, \widehat{\theta}_n$ satisfy the following estimation equations

$$\widehat{\sigma}_n^2 = \frac{1}{n-1}\sum_{t=2}^{n}\left(\widehat{\varepsilon}_t - f_t\left(\widehat{\theta}_n\right)\widehat{\varepsilon}_{t-1}\right)^2, \tag{2.8}$$

$$\sum_{t=2}^{n}\left(\widehat{\varepsilon}_t - f_t\left(\widehat{\theta}_n\right)\widehat{\varepsilon}_{t-1}\right)f_t'\left(\widehat{\theta}_n\right)\widehat{\varepsilon}_{t-1} = 0, \tag{2.9}$$

$$\sum_{t=2}^{n}\left(\widehat{\varepsilon}_t - f_t\left(\widehat{\theta}_n\right)\widehat{\varepsilon}_{t-1}\right)\left(g'\left(x_t^T\widehat{\beta}_n\right)x_t - f_t\left(\widehat{\theta}_n\right)g'\left(x_{t-1}^T\widehat{\beta}_n\right)x_{t-1}\right) = 0, \tag{2.10}$$

where

$$\widehat{\varepsilon}_t = y_t - g\left(x_t^T\widehat{\beta}_n\right). \tag{2.11}$$

Remark 2.1. If $g(x_t^T\beta) = x_t^T\beta$, then the above equations become the same as Hu's (see [24]). If $f_t(\theta) = \theta$, $g(x_t^T\beta) = x_t^T\beta$, then the above equations become the same as Maller's (see [15]). Thus we extend those QML estimators of Hu [24] and Maller [15].

For ease of exposition, we will introduce the following notations, which will be used later in the paper. Let $(d+1)\times 1-$ vector $\varphi = (\beta^T, \theta)^T$. Define

$$S_n(\varphi) = \sigma^2\frac{\partial \Phi_n}{\partial \varphi} = \sigma^2\left(\frac{\partial \Phi_n}{\partial \beta}, \frac{\partial \Phi_n}{\partial \theta}\right), \qquad F_n(\varphi) = -\sigma^2\frac{\partial^2 \Phi_n}{\partial \varphi \partial \varphi^T}. \tag{2.12}$$

By (2.7), we have

$$F_n(\varphi) = \begin{pmatrix} X_n(\varphi, \omega) & U \\ * & \sum_{t=2}^{n}\left(\left(f_t'^2(\theta) + f_t(\theta)f_t''(\theta)\right)\varepsilon_{t-1}^2 - f_t''(\theta)\varepsilon_t\varepsilon_{t-1}\right) \end{pmatrix}, \tag{2.13}$$

where the * indicates that the elements are filled in by symmetry,

$$X_n(\varphi,\omega) = -\sigma^2\left(\frac{\partial^2\Phi_n}{\partial\beta\partial\beta^T}\right),$$

$$U = \sum_{t=2}^{n}\left(f_t'(\theta)\varepsilon_{t-1}g'\left(x_t^T\beta\right)x_t + f_t'(\theta)\varepsilon_t g'\left(x_{t-1}^T\beta\right)x_{t-1} - 2f_t(\theta)f_t'(\theta)\varepsilon_{t-1}g'\left(x_{t-1}^T\beta\right)x_{t-1}\right),$$

$$\frac{\partial^2\Phi_n}{\partial\beta\partial\beta^T} = -\frac{1}{\sigma^2}\sum_{t=2}^{n}\left(g'\left(x_t^T\beta\right)x_t - f_t(\theta)g'\left(x_{t-1}^T\beta\right)x_{t-1}\right)\left(g'\left(x_t^T\beta\right)x_t - f_t(\theta)g'\left(x_{t-1}^T\beta\right)x_{t-1}\right)^T$$
$$+\frac{1}{\sigma^2}\sum_{t=2}^{n}(\varepsilon_t - f_t(\theta)\varepsilon_{t-1})\left(g''\left(x_t^T\beta\right)x_t x_t^T - f_t(\theta)g''\left(x_{t-1}^T\beta\right)x_{t-1}x_{t-1}^T\right). \tag{2.14}$$

Because $\{e_{t-1}\}$ and $\{\eta_t\}$ are mutually independent, we have

$$D_n = E(F_n(\varphi_0)) = \begin{pmatrix} X_n(\varphi_0) & 0 \\ 0 & \sum_{t=2}^{n} f_t'^2(\theta_0)Ee_{t-1}^2 \end{pmatrix} = \begin{pmatrix} X_n(\varphi_0) & 0 \\ 0 & \Delta(\theta_0,\sigma_0) \end{pmatrix}, \tag{2.15}$$

where

$$X_n(\varphi_0) = \sum_{t=2}^{n}\left(g'\left(x_t^T\beta_0\right)x_t - f_t(\theta_0)g'\left(x_{t-1}^T\beta_0\right)x_{t-1}\right)\left(g'\left(x_t^T\beta_0\right)x_t - f_t(\theta_0)g'\left(x_{t-1}^T\beta_0\right)x_{t-1}\right)^T,$$

$$\Delta(\theta_0,\sigma_0) = \sum_{t=2}^{n} f_t'^2(\theta_0)Ee_{t-1}^2 = \sigma_0^2\sum_{t=2}^{n} f_t'^2(\theta_0)\sum_{j=0}^{t-2}\left(\prod_{i=0}^{j-1}f_{t-i}^2(\theta)\right) = O(n). \tag{2.16}$$

By (2.8) (2.7) and $E\eta_t = 0$, we have

$$\sigma_0^2 E\left(\left.\frac{\partial\Phi_n}{\partial\beta}\right|_{\beta=\beta_0}\right) = \sum_{t=2}^{n}E\eta_t\left(g'\left(x_t^T\beta_0\right)x_t - f_t(\theta_0)g'\left(x_{t-1}^T\beta_0\right)x_{t-1}\right) = 0,$$
$$\sigma_0^2 E\left(\left.\frac{\partial\Phi_n}{\partial\theta}\right|_{\theta=\theta_0}\right) = \sum_{t=2}^{n}f_t'(\theta_0)E(\eta_t e_{t-1}) = 0. \tag{2.17}$$

3. Statement of Main Results

In the section pseudo likelihood ratio (LR) statistics for various hypothesis tests of interest are derived. We consider the following hypothesis:

$$H_1 : g(\cdot), f(\cdot) \text{ are continuous functions, and } f^{'(\cdot)} \neq 0,\ \sigma_0^2 > 0. \tag{3.1}$$

When the parameter space is restricted by a hypothesis H_{0j}, $j = 1, 2, \ldots$, let $\widehat{\beta}_{jn}, \widehat{\theta}_{jn}, \widehat{\sigma}^2_{jn}$ be the corresponding QML estimators of β, θ, σ^2, and let

$$\widehat{L}_{jn} = -2\Phi_n\left(\widehat{\beta}_{jn}, \widehat{\theta}_{jn}, \widehat{\sigma}^2_{jn}\right) \tag{3.2}$$

be minus twice the log-likelihood, evaluated at the fitted parameters. Also let

$$\begin{gathered} \widehat{L}_n = -2\Phi_n\left(\widehat{\beta}_n, \widehat{\theta}_n, \widehat{\sigma}^2_n\right), \\ d_{jn} = \widehat{L}_{jn} - \widehat{L}_n \end{gathered} \tag{3.3}$$

be the "deviance" statistic for testing H_{0j} against H_1. From (2.5) and (2.8),

$$\widehat{L}_n = (n-1)\ln\widehat{\sigma}^2_n + (n-1)(1 + \ln 2\pi) \tag{3.4}$$

and similarly

$$\widehat{L}_{jn} = (n-1)\ln\widehat{\sigma}^2_{jn} + (n-1)(1 + \ln 2\pi). \tag{3.5}$$

In order to obtain our results, we give some sufficient conditions as follows.

(A1) $X_n = \sum_{t=2}^{n} x_t x_t^T$ is positive definite for sufficiently large n and

$$\lim_{n\to\infty}\max_{1\le t\le n} x_t^T X_n^{-1} x_t = O\left(n^{-\alpha}\right), \quad \forall \alpha \in \left(\frac{1}{2}, 1\right], \qquad \lim_{n\to\infty}\sup |\lambda|_{\max}\left(X_n^{-1/2} Z_n X_n^{-T/2}\right) < 1, \tag{3.6}$$

where $Z_n = (1/2)\sum_{t=2}^{n}(x_t x_{t-1}^T + x_{t-1} x_t^T)$ and $|\lambda|_{\max}(\cdot)$ denotes the maximum in absolute value of the eigenvalues of a symmetric matrix.

(A2) There is a constant $\alpha > 0$ such that

$$\sum_{j=1}^{t}\left(\prod_{i=0}^{j-1} f_{t-i}^2(\theta)\right) \le \alpha, \qquad \max_{1\le j\le n}\left|\sum_{t=j+1}^{n}\left(\prod_{i=0}^{t-j-1} f_{t-i}(\theta_0)\right)\right| \le \gamma. \tag{3.7}$$

(A3) $f_t'(\theta) = df_t(\theta)/d\theta \neq 0$ and $f_t''(\theta) = df_t'(\theta)/d\theta$ exist and are bounded, and $g(\cdot)$ is twice continuously differentiable, $0 < m \le \max_u |g'(u)| \le M < \infty$, $0 < \widetilde{m} \le \max_u |g''(u)| \le \widetilde{M} < \infty$.

Theorem 3.1. *Assume* (2.1), (2.2) *and (A1)–(A3).*

(1) *Suppose $H_{01} : f_t(\theta) = \theta$ and $g(u)$ is a continuous function, $\sigma_0^2 > 0$ holds. Then*

$$d_{1n} \xrightarrow{D} \chi_1^2, \quad n \longrightarrow \infty. \tag{3.8}$$

(2) *Suppose* $H_{02}: f_t(\theta) = \theta, g(u) = u,\ \sigma_0^2 > 0$ *holds. Then*

$$d_{2n} \xrightarrow{D} \chi_1^2, \quad n \longrightarrow \infty. \tag{3.9}$$

(3) *Suppose* $H_{03}: f_t(\theta) = \theta, g(u) = e^u/(1+e^u),\ \sigma_0^2 > 0$ *holds. Then*

$$d_{3n} \xrightarrow{D} \chi_1^2, \quad n \longrightarrow \infty. \tag{3.10}$$

4. Proof of Theorem

To prove Theorem 3.1, we first introduce the following lemmas.

Lemma 4.1. *Suppose that (A1)–(A3) hold. Then, for all* $A > 0$,

$$\sup_{\varphi \in N_n(A)} \left\| D_n^{-1/2} F_n(\varphi) D_n^{-T/2} - \Phi_n \right\| \xrightarrow{P} 0, \quad n \to \infty, \tag{4.1}$$

where

$$\Phi_n = \operatorname{diag}\left(I_d, \frac{\sum_{t=2}^n f_t'^2(\theta_0) e_{t-1}^2}{\Delta_n(\theta_0, \sigma_0)} \right), \tag{4.2}$$

$$N_n(A) = \left\{ \varphi \in R^{d+1} : (\varphi - \varphi_0)^T D_n (\varphi - \varphi_0) \leq A^2 \right\}. \tag{4.3}$$

Proof. Similar to proof of Lemma 4.1 in Hu [24], here we omit. □

Lemma 4.2. *Suppose that (A1)–(A3) hold. Then* $\widehat{\varphi}_n \to \varphi_0$, $\widehat{\sigma}_n^2 \to \sigma_0^2$ *and*

$$X_n\left(\beta^*, \beta^{**}, \widehat{\theta}_n\right) \longrightarrow X_n(\varphi_0), \tag{4.4}$$

where β^*, β^{**} *are on the line of* β_0 *and* $\widehat{\beta}_n$.

Proof. Similar to proof of Theorem 3.1 in Hu [24], we easily prove that $\widehat{\varphi}_n \to \varphi_0$, and $\widehat{\sigma}_n^2 \to \sigma_0^2$. Since (4.4) is easily proved, here we omit the proof (4.4). □

Proof of Theorem 3.1. Note that $S_n(\widehat{\varphi}_n) = 0$ and $F_n(\widehat{\varphi}_n)$ are nonsingular. By Taylor's expansion, we have

$$0 = S_n(\widehat{\varphi}_n) = S_n(\varphi_0) - F_n(\widetilde{\varphi}_n)(\widehat{\varphi}_n - \varphi_0), \tag{4.5}$$

where $\widetilde{\varphi}_n = a\widehat{\varphi}_n + (1-a)\varphi_0$ for some $0 \le a \le 1$. Since $\widehat{\varphi}_n \in N_n(A)$, also $\widetilde{\varphi}_n \in N_n(A)$. By (4.1), we have

$$F_n(\widetilde{\varphi}_n) = D_n^{1/2}\left(\Phi_n + \widetilde{A}_n\right)D_n^{T/2}. \tag{4.6}$$

Thus $\widetilde{A}_n$ is a symmetric matrix with $\widetilde{A}_n \xrightarrow{P} 0$. By (4.5) and (4.6), we have

$$D_n^{T/2}(\widehat{\varphi}_n - \varphi_0) = D_n^{T/2}F_n^{-1}(\widetilde{\varphi}_n)S_n(\varphi_0) = \left(\Phi_n + \widetilde{A}_n\right)^{-1}D_n^{-1/2}S_n(\varphi_0). \tag{4.7}$$

Let $S_n(\varphi), F_n(\varphi)$ denote $S_n^{(\beta)}(\varphi), S_n^{(\theta)}(\varphi)$, and $F_n^{(\beta)}(\varphi), F_n^{(\theta)}(\varphi)$, respectively. By (4.7), we have

$$\Phi_n D_n^{T/2}\left(\widehat{\beta}_n - \beta_0, \widehat{\theta}_n - \theta_0\right) = D_n^{-1/2}\left(S_n^{(\beta)}(\varphi_0), S_n^{(\theta)}(\varphi_0)\right) + o_P(1). \tag{4.8}$$

Note that

$$\Phi_n D_n^{T/2} = \begin{pmatrix} X_n^{T/2}(\varphi_0) & 0 \\ 0 & \dfrac{\left(\sum_{t=2}^n f_t'^2(\theta_0)e_{t-1}^2\right)}{\sqrt{\Delta_n(\theta_0,\sigma_0)}} \end{pmatrix},$$
$$D_n^{-1/2} = \begin{pmatrix} X_n^{-1/2}(\varphi_0) & 0 \\ 0 & \dfrac{1}{\sqrt{\Delta_n(\theta_0,\sigma_0)}} \end{pmatrix}. \tag{4.9}$$

By (2.15), (4.2) and (4.8), we get

$$\begin{aligned} X_n^{T/2}(\varphi_0)\left(\widehat{\beta}_n - \beta_0\right) &= X_n^{-1/2}(\varphi_0)S_n^{(\beta)}(\varphi_0) + o_P(1) \\ &= X_n^{-1/2}(\varphi_0)\sum_{t=2}^n \eta_t\left(g'\left(x_t^T\beta_0\right)x_t - f_t(\theta_0)g'\left(x_{t-1}^T\beta_0\right)x_{t-1}\right) + o_P(1), \end{aligned} \tag{4.10}$$

$$\begin{aligned} \sum_{t=2}^n f_t'^2(\theta_0)e_{t-1}^2\left(\widehat{\theta}_n - \theta_0\right) &= S_n^{(\theta)}(\varphi_0) + o_P\left(\sqrt{\Delta_n(\theta_0,\sigma_0)}\right) \\ &= \sum_{t=2}^n f_t'(\theta_0)\eta_t e_{t-1} + o_P\left(\sqrt{\Delta_n(\theta_0,\sigma_0)}\right). \end{aligned} \tag{4.11}$$

Note that

$$\varepsilon_t = y_t - g\left(x_t^T\beta\right) = g'\left(x_t^T\beta^*\right)x_t^T(\beta_0 - \beta) + e_t \tag{4.12}$$

By (2.1), (2.11) and (4.12), we have

$$\hat{\varepsilon}_t - f_t\left(\hat{\theta}_n\right)\hat{\varepsilon}_{t-1} = \left(g'\left(x_t^T\beta^*\right)x_t^T - f_t\left(\hat{\theta}_n\right)g'\left(x_{t-1}^T\beta^{**}\right)x_{t-1}^T\right)\left(\beta_0 - \hat{\beta}_n\right) + \left(e_t - f_t\left(\hat{\theta}_n\right)e_{t-1}\right). \tag{4.13}$$

By (4.13) and (2.10), we have

$$\begin{aligned}\sum_{t=2}^{n}\left(\hat{\varepsilon}_t - f_t\left(\hat{\theta}_n\right)\hat{\varepsilon}_{t-1}\right)^2 &= \sum_{t=2}^{n}\left(\hat{\varepsilon}_t - f_t\left(\hat{\theta}_n\right)\hat{\varepsilon}_{t-1}\right)\left(\left(g'\left(x_t^T\beta^*\right)x_t^T - f_t\left(\hat{\theta}_n\right)g'\left(x_{t-1}^T\beta^{**}\right)x_{t-1}^T\right)\right.\\ &\quad\left.\times\left(\beta_0 - \hat{\beta}_n\right) + \left(e_t - f_t\left(\hat{\theta}_n\right)e_{t-1}\right)\right)\\ &= \sum_{t=2}^{n}\left(\hat{\varepsilon}_t - f_t\left(\hat{\theta}_n\right)\hat{\varepsilon}_{t-1}\right)\left(g'\left(x_t^T\beta^*\right)x_t^T - f_t\left(\hat{\theta}_n\right)g'\left(x_{t-1}^T\beta^{**}\right)x_{t-1}^T\right)\left(\beta_0 - \hat{\beta}_n\right)\\ &\quad+ \sum_{t=2}^{n}\left(\hat{\varepsilon}_t - f_t\left(\hat{\theta}_n\right)\hat{\varepsilon}_{t-1}\right)\left(e_t - f_t\left(\hat{\theta}_n\right)e_{t-1}\right)\\ &= \sum_{t=2}^{n}\left(\hat{\varepsilon}_t - f_t\left(\hat{\theta}_n\right)\hat{\varepsilon}_{t-1}\right)\left(e_t - f_t\left(\hat{\theta}_n\right)e_{t-1}\right).\end{aligned} \tag{4.14}$$

By (4.13), we have

$$\left(g'\left(x_t^T\beta^*\right)x_t^T - f_t\left(\hat{\theta}_n\right)g'\left(x_{t-1}^T\beta^{**}\right)x_{t-1}^T\right)\left(\beta_0 - \hat{\beta}_n\right) = \left(\hat{\varepsilon}_t - f_t\left(\hat{\theta}_n\right)\hat{\varepsilon}_{t-1}\right) - \left(e_t - f_t\left(\hat{\theta}_n\right)e_{t-1}\right). \tag{4.15}$$

By (4.15), we have

$$\begin{aligned}&\sum_{t=2}^{n}\left(\left(g'\left(x_t^T\beta^*\right)x_t^T - f_t\left(\hat{\theta}_n\right)g'\left(x_{t-1}^T\beta^{**}\right)x_{t-1}^T\right)\left(\beta_0 - \hat{\beta}_n\right)\right)^2\\ &\quad= \sum_{t=2}^{n}\left(\hat{\varepsilon}_t - f_t\left(\hat{\theta}_n\right)\hat{\varepsilon}_{t-1}\right)^2 + \sum_{t=2}^{n}\left(e_t - f_t\left(\hat{\theta}_n\right)e_{t-1}\right)^2\\ &\qquad- 2\sum_{t=2}^{n}\left(\hat{\varepsilon}_t - f_t\left(\hat{\theta}_n\right)\hat{\varepsilon}_{t-1}\right)\left(e_t - f_t\left(\hat{\theta}_n\right)e_{t-1}\right)\\ &\quad= \sum_{t=2}^{n}\left(e_t - f_t\left(\hat{\theta}_n\right)e_{t-1}\right)^2 - \sum_{t=2}^{n}\left(\hat{\varepsilon}_t - f_t\left(\hat{\theta}_n\right)\hat{\varepsilon}_{t-1}\right)^2.\end{aligned} \tag{4.16}$$

By (4.14) and (4.16), we have

$$\sum_{t=2}^{n}\left(\widehat{\varepsilon}_t - f_t\left(\widehat{\theta}_n\right)\widehat{\varepsilon}_{t-1}\right)^2 = \sum_{t=2}^{n}\left(e_t - f_t\left(\widehat{\theta}_n\right)e_{t-1}\right)^2 - \sum_{t=2}^{n}\left(\left(g'\left(x_t^T\beta^*\right)x_t^T - f_t\left(\widehat{\theta}_n\right)g'\left(x_{t-1}^T\beta^{**}\right)x_{t-1}^T\right)\left(\beta_0 - \widehat{\beta}_n\right)\right)^2. \tag{4.17}$$

By (4.15), we have

$$\begin{aligned}\sum_{t=2}^{n}&\left(\widehat{\varepsilon}_t - f_t\left(\widehat{\theta}_n\right)\widehat{\varepsilon}_{t-1}\right)^2\\ &= \sum_{t=2}^{n}\left(e_t - f_t\left(\widehat{\theta}_n\right)e_{t-1}\right)^2 + \sum_{t=2}^{n}\left(\left(g'\left(x_t^T\beta^*\right)x_t^T - f_t\left(\widehat{\theta}_n\right)g'\left(x_{t-1}^T\beta^{**}\right)x_{t-1}^T\right)\left(\beta_0 - \widehat{\beta}_n\right)\right)^2\\ &\quad + 2\sum_{t=2}^{n}\left(e_t - f_t\left(\widehat{\theta}_n\right)e_{t-1}\right)\left(\left(g'\left(x_t^T\beta^*\right)x_t^T - f_t\left(\widehat{\theta}_n\right)g'\left(x_{t-1}^T\beta^{**}\right)x_{t-1}^T\right)\left(\beta_0 - \widehat{\beta}_n\right)\right).\end{aligned} \tag{4.18}$$

Thus, by (4.17) and (4.18), we have

$$\begin{aligned}\sum_{t=2}^{n}&\left(\left(g'\left(x_t^T\beta^*\right)x_t^T - f_t\left(\widehat{\theta}_n\right)g'\left(x_{t-1}^T\beta^{**}\right)x_{t-1}^T\right)\left(\beta_0 - \widehat{\beta}_n\right)\right)^2\\ &+ \sum_{t=2}^{n}\left(e_t - f_t\left(\widehat{\theta}_n\right)e_{t-1}\right)\left(\left(g'\left(x_t^T\beta^*\right)x_t^T - f_t\left(\widehat{\theta}_n\right)g'\left(x_{t-1}^T\beta^{**}\right)x_{t-1}^T\right)\left(\beta_0 - \widehat{\beta}_n\right)\right) = 0.\end{aligned} \tag{4.19}$$

Since $\eta_t = e_t - f_t(\theta_0)e_{t-1}$, we have

$$\begin{aligned}\sum_{t=2}^{n}\left(e_t - f_t\left(\widehat{\theta}_n\right)e_{t-1}\right)^2 &= \sum_{t=2}^{n}\left(\eta_t + f_t(\theta_0)e_{t-1} - f_t\left(\widehat{\theta}_n\right)e_{t-1}\right)^2\\ &= \sum_{t=1}^{n}\eta_t^2 + \sum_{t=2}^{n}\left(f_t(\theta_0) - f_t\left(\widehat{\theta}_n\right)\right)^2 e_{t-1}^2 + 2\left(f_t(\theta_0) - f_t\left(\widehat{\theta}_n\right)\right)\eta_t e_{t-1}.\end{aligned} \tag{4.20}$$

Thus, by (4.17), (4.20) and mean value theorem, we have

$$\begin{aligned}(n-1)\widehat{\sigma}_n^2 &= \sum_{t=2}^{n}\left(\widehat{\varepsilon}_t - f_t\left(\widehat{\theta}_n\right)\widehat{\varepsilon}_{t-1}\right)^2\\ &= \sum_{t=1}^{n}\eta_t^2 + \sum_{t=2}^{n}\left(f_t(\theta_0) - f_t\left(\widehat{\theta}_n\right)\right)^2 e_{t-1}^2 + 2\left(f_t(\theta_0) - f_t\left(\widehat{\theta}_n\right)\right)\eta_t e_{t-1}\end{aligned}$$

$$
\begin{aligned}
&-\sum_{t=2}^{n}\left(\left(g'\left(x_t^T\beta^*\right)x_t^T - f_t\left(\widehat{\theta}_n\right)g'\left(x_{t-1}^T\beta^{**}\right)x_{t-1}^T\right)\left(\beta_0-\widehat{\beta}_n\right)\right)^2 \\
&= \sum_{t=1}^{n}\eta_t^2 + \left(\theta_0-\widehat{\theta}_n\right)^2\sum_{t=2}^{n}f_t'^2\left(\widetilde{\theta}\right)e_{t-1}^2 + 2\left(\theta_0-\widehat{\theta}_n\right)\sum_{t=2}^{n}f_t'\left(\widetilde{\theta}\right)e_{t-1}\eta_t \\
&-\sum_{t=2}^{n}\left(\left(g'\left(x_t^T\beta^*\right)x_t^T - f_t\left(\widehat{\theta}_n\right)g'\left(x_{t-1}^T\beta^{**}\right)x_{t-1}^T\right)\left(\beta_0-\widehat{\beta}_n\right)\right)^2,
\end{aligned}
\tag{4.21}
$$

where $\widetilde{\theta} = a\theta_0 + (1-a)\widehat{\theta}_n$ for some $0 \le a \le 1$.

It is easy to know that

$$
\begin{aligned}
&\left(\widehat{\beta}_n-\beta_0\right)^T X_n(\varphi_0)\left(\widehat{\beta}_n-\beta_0\right) \\
&\quad= \left(\sum_{t=2}^{n}\eta_t X_n^{-1/2}(\varphi_0)\left(g'\left(x_t^T\beta_0\right)x_t - f_t(\theta_0)g'\left(x_{t-1}^T\beta_0\right)x_{t-1}\right)\right)^2 + o_p(1).
\end{aligned}
\tag{4.22}
$$

By Lemma 4.2 and (4.22), we have

$$
\begin{aligned}
(n-1)\widehat{\sigma}_n^2 &= \sum_{t=1}^{n}\eta_t^2 + \left(\theta_0-\widehat{\theta}_n\right)^2\sum_{t=2}^{n}f_t'^2\left(\widetilde{\theta}\right)e_{t-1}^2 + 2\left(\theta_0-\widehat{\theta}_n\right)\sum_{t=2}^{n}f_t'\left(\widetilde{\theta}\right)e_{t-1}\eta_t \\
&\quad-\left(\sum_{t=2}^{n}\eta_t X_n^{-1/2}\left(\beta^*,\beta^{**},\widehat{\theta}_n\right)\left(g'\left(x_t^T\beta_0\right)x_t - f_t(\theta_0)g'\left(x_{t-1}^T\beta_0\right)x_{t-1}\right)\right)^2 + o_P(1) \\
&= \sum_{t=1}^{n}\eta_t^2 + \left(\theta_0-\widehat{\theta}_n\right)^2\sum_{t=2}^{n}f_t'^2\left(\widetilde{\theta}\right)e_{t-1}^2 + 2\left(\theta_0-\widehat{\theta}_n\right)\sum_{t=2}^{n}f_t'\left(\widetilde{\theta}\right)e_{t-1}\eta_t \\
&\quad-\left(\sum_{t=2}^{n}\eta_t X_n^{-1/2}(\varphi_0)\left(g'\left(x_t^T\beta_0\right)x_t - f_t(\theta_0)g'\left(x_{t-1}^T\beta_0\right)x_{t-1}\right)\right)^2 + o_P(1).
\end{aligned}
\tag{4.23}
$$

Hence, by (4.11), we have

$$
\begin{aligned}
\widehat{\theta}_n-\theta_0 &= \frac{\sum_{t=2}^{n}f_t'(\theta_0)\eta_t e_{t-1}}{\sum_{t=2}^{n}f_t'^2(\theta_0)e_{t-1}^2} + o_P\left(\frac{\sqrt{\Delta_n(\theta_0,\sigma_0)}}{\sum_{t=2}^{n}f_t'^2(\theta_0)e_{t-1}^2}\right) \\
&= \frac{\sum_{t=2}^{n}f_t'(\theta_0)\eta_t e_{t-1}}{\sum_{t=2}^{n}f_t'^2(\theta_0)e_{t-1}^2} + o_P\left(\frac{1}{\sqrt{\sum_{t=2}^{n}f_t'^2(\theta_0)e_{t-1}^2}}\right).
\end{aligned}
\tag{4.24}
$$

By (4.24), we have

$$\begin{aligned}
&\left(\theta_0-\widehat{\theta}_n\right)^2\sum_{t=2}^{n} f_t'^2\left(\widetilde{\theta}\right)e_{t-1}^2+2\left(\theta_0-\widehat{\theta}_n\right)\sum_{t=2}^{n} f_t'\left(\widetilde{\theta}\right)e_{t-1}\eta_t\\
&\quad=\left(\frac{\sum_{t=2}^{n} f_t'(\theta_0)\eta_t e_{t-1}}{\sum_{t=2}^{n} f_t'^2(\theta_0)e_{t-1}^2}+o_P\left(\frac{1}{\sqrt{\sum_{t=2}^{n} f_t'^2(\theta_0)e_{t-1}^2}}\right)\right)^2\sum_{t=2}^{n} f_t'^2\left(\widetilde{\theta}\right)e_{t-1}^2\\
&\qquad+2\left(\frac{\sum_{t=2}^{n} f_t'(\theta_0)\eta_t e_{t-1}}{\sum_{t=2}^{n} f_t'^2(\theta_0)e_{t-1}^2}+o_P\left(\frac{1}{\sqrt{\sum_{t=2}^{n} f_t'^2(\theta_0)e_{t-1}^2}}\right)\right)\sum_{t=2}^{n} f_t'\left(\widetilde{\theta}\right)e_{t-1}\eta_t+o_P(1)\\
&\quad=\left(\frac{\sum_{t=2}^{n} f_t'(\theta_0)\eta_t e_{t-1}}{\sum_{t=2}^{n} f_t'^2(\theta_0)e_{t-1}^2}+o_P\left(\frac{1}{\sqrt{\sum_{t=2}^{n} f_t'^2(\theta_0)e_{t-1}^2}}\right)\right)^2\sum_{t=2}^{n}\left(f_t'(\theta_0)+o(1)\right)^2e_{t-1}^2\\
&\qquad+2\left(\frac{\sum_{t=2}^{n} f_t'(\theta_0)\eta_t e_{t-1}}{\sum_{t=2}^{n} f_t'^2(\theta_0)e_{t-1}^2}+o_P\left(\frac{1}{\sqrt{\sum_{t=2}^{n} f_t'^2(\theta_0)e_{t-1}^2}}\right)\right)\\
&\qquad\quad\cdot\sum_{t=2}^{n}\left(f_t''(\theta_0)+o(1)\right)e_{t-1}\eta_t+o_P(1)
\end{aligned}\tag{4.25}$$

$$\begin{aligned}
&\quad=\frac{\left(\sum_{t=2}^{n} f_t'(\theta_0)\eta_t e_{t-1}\right)^2}{\sum_{t=2}^{n} f_t'^2(\theta_0)e_{t-1}^2}-\frac{2\left(\sum_{t=2}^{n} f_t'(\theta_0)\eta_t e_{t-1}\right)^2}{\sum_{t=2}^{n} f_t'^2(\theta_0)e_{t-1}^2}+o_P(1)\\
&\quad=-\frac{\left(\sum_{t=2}^{n} f_t'(\theta_0)\eta_t e_{t-1}\right)^2}{\sum_{t=2}^{n} f_t'^2(\theta_0)e_{t-1}^2}+o_P(1).
\end{aligned}$$

By Lemma 4.2, we have

$$\begin{aligned}
(n-1)\widehat{\sigma}_n^2&=\sum_{t=1}^{n}\eta_t^2-\frac{\left(\sum_{t=2}^{n} f_t'(\theta_0)\eta_t e_{t-1}\right)^2}{\sum_{t=2}^{n} f_t'^2(\theta_0)e_{t-1}^2}\\
&\quad-\left(\sum_{t=2}^{n}\eta_t X_n^{-1/2}\left(\beta^*,\beta^{**},\widehat{\theta}_n\right)\left(g'\left(x_t^T\beta_0\right)x_t-f_t(\theta_0)g'\left(x_{t-1}^T\beta_0\right)x_{t-1}\right)\right)^2+o_P(1)\\
&=\sum_{t=1}^{n}\eta_t^2-\frac{\left(\sum_{t=2}^{n} f_t'(\theta_0)\eta_t e_{t-1}\right)^2}{\sum_{t=2}^{n} f_t'^2(\theta_0)e_{t-1}^2}\\
&\quad-\left(\sum_{t=2}^{n}\eta_t X_n^{-1/2}(\varphi_0)\left(g'\left(x_t^T\beta_0\right)x_t-f_t(\theta_0)g'\left(x_{t-1}^T\beta_0\right)x_{t-1}\right)\right)^2+o_P(1).
\end{aligned}\tag{4.26}$$

Now, we prove (3.8). By (4.12), we have

$$\widehat{\varepsilon}_t(1) = y_t - g\left(x_t^T\widehat{\beta}_{1n}\right) = g'\left(x_t^T\widehat{\beta}_{1n}^*\right)x_t^T\left(\beta_0 - \widehat{\beta}_{1n}\right) + e_t. \tag{4.27}$$

Note that

$$\varepsilon_t - f_t(\theta_0)\varepsilon_{t-1} = \left(g'\left(x_t^T\beta^*\right)x_t^T - f_t(\theta_0)g'\left(x_{t-1}^T\beta^{**}\right)x_{t-1}^T\right)(\beta_0 - \beta) + \eta_t. \tag{4.28}$$

From (4.28), we have

$$\widehat{\varepsilon}_t(1) - \widehat{\theta}_{1n}\widehat{\varepsilon}_{t-1}(1) = \left(g'\left(x_t^T\widehat{\beta}_{1n}^*\right)x_t^T - \widehat{\theta}_{1n}g'\left(x_{t-1}^T\widehat{\beta}_{1n}^{**}\right)x_{t-1}^T\right)\left(\beta_0 - \widehat{\beta}_{1n}\right) + \eta_t. \tag{4.29}$$

By (2.8) and (2.10), we have

$$\begin{aligned}
0 &= \sum_{t=2}^{n}\left(\widehat{\varepsilon}_t(1) - \widehat{\theta}_{1n}\widehat{\varepsilon}_{t-1}(1)\right)\left(g'\left(x_t^T\widehat{\beta}_{1n}^*\right)x_t - \widehat{\theta}_{1n}g'\left(x_{t-1}^T\widehat{\beta}_{1n}^{**}\right)x_{t-1}\right)\\
&= \sum_{t=2}^{n}\left(g'\left(x_t^T\widehat{\beta}_{1n}^*\right)x_t^T - \widehat{\theta}_{1n}g'\left(x_{t-1}^T\widehat{\beta}_{1n}^{**}\right)x_{t-1}^T\right)\left(\beta_0 - \widehat{\beta}_{1n}\right)\left(g'\left(x_t^T\widehat{\beta}_{1n}^*\right)x_t - \widehat{\theta}_{1n}g'\left(x_{t-1}^T\widehat{\beta}_{1n}^{**}\right)x_{t-1}\right)\\
&\quad + \sum_{t=2}^{n}\eta_t\left(g'\left(x_t^T\widehat{\beta}_{1n}^*\right)x_t - \widehat{\theta}_{1n}g'\left(x_{t-1}^T\widehat{\beta}_{1n}^{**}\right)x_{t-1}\right)\\
&= \left(\beta_0 - \widehat{\beta}_{1n}\right)^T \mathrm{X}_{1n}\left(\widehat{\beta}_{1n}^*, \widehat{\beta}_{1n}^{**}, \widehat{\theta}_{1n}\right) + \sum_{t=2}^{n}\eta_t\left(g'\left(x_t^T\widehat{\beta}_{1n}^*\right)x_t - \widehat{\theta}_{1n}g'\left(x_{t-1}^T\widehat{\beta}_{1n}^{**}\right)x_{t-1}\right).
\end{aligned} \tag{4.30}$$

From (4.30), we obtain that

$$\widehat{\beta}_{1n} - \beta_0 = \mathrm{X}_{1n}^{-1}\left(\widehat{\beta}_{1n}^*, \widehat{\beta}_{1n}^{**}, \widehat{\theta}_{1n}\right)\sum_{t=2}^{n}\eta_t\left(g'\left(x_t^T\widehat{\beta}_{1n}\right)x_t - \widehat{\theta}_{1n}g'\left(x_{t-1}^T\widehat{\beta}_{1n}\right)x_{t-1}\right). \tag{4.31}$$

By (4.29), (4.31) and Lemma 4.2, we have

$$\begin{aligned}
(n-1)\widehat{\sigma}_{1n}^2 &= \sum_{t=2}^{n}\left(\widehat{\varepsilon}_t(1) - \widehat{\theta}_{1n}\widehat{\varepsilon}_{t-1}(1)\right)^2\\
&= \sum_{t=1}^{n}\eta_t^2 + \left(\beta_0 - \widehat{\beta}_{1n}\right)^T \mathrm{X}_{1n}\left(\widehat{\beta}_{1n}^*, \widehat{\beta}_{1n}^{**}, \widehat{\theta}_{1n}\right)\left(\beta_0 - \widehat{\beta}_{1n}\right)\\
&\quad + 2\left(\beta_0 - \widehat{\beta}_{1n}\right)^T\sum_{t=2}^{n}\eta_t\left(g'\left(x_t^T\widehat{\beta}_{1n}^*\right)x_t - \widehat{\theta}_{1n}g'\left(x_{t-1}^T\widehat{\beta}_{1n}^{**}\right)x_{t-1}\right)
\end{aligned}$$

$$= \sum_{t=1}^{n} \eta_t^2 - \left(\sum_{t=2}^{n} \eta_t X_{1n}^{-1/2} \left(\widehat{\beta}_{1n}^{*}, \widehat{\beta}_{1n}^{**}, \widehat{\theta}_{1n}\right) \left(g'\left(x_t^T \widehat{\beta}_{1n}^{*}\right) x_t - \widehat{\theta}_{1n} g'\left(x_{t-1}^T \widehat{\beta}_{1n}^{**}\right) x_{t-1} \right) \right)^2 + o_p(1)$$

$$= \sum_{t=1}^{n} \eta_t^2 - \left(\sum_{t=2}^{n} \eta_t X_{1n}^{-1/2} (\varphi_0) \left(g'\left(x_t^T \beta_0\right) x_t - \theta_0 g'\left(x_{t-1}^T \beta_0\right) x_{t-1} \right) \right)^2 + o_p. \tag{4.32}$$

By (3.3)–(3.5), we have

$$d_{1n} = \widehat{L}_{1n} - \widehat{L}_n = (n-1) \ln\left(\frac{\widehat{\sigma}_{1n}^2}{\widehat{\sigma}_n^2} \right) = (n-1)\left(\left(\frac{\widehat{\sigma}_{1n}^2}{\widehat{\sigma}_n^2} \right) - 1 \right) + o_P(1). \tag{4.33}$$

Under the H_{01}, and by (4.26), (4.32) and (4.33), we have

$$\begin{aligned} \frac{(n-1)\left(\widehat{\sigma}_{1n}^2 - \widehat{\sigma}_n^2\right)}{\widehat{\sigma}_n^2} &= \frac{\left(\sum_{t=2}^{n} \eta_t e_{t-1}\right)^2}{\widehat{\sigma}_n^2 \sum_{t=2}^{n} e_{t-1}^2} + o_P(1) \\ &= \frac{\left(\sum_{t=2}^{n} \eta_t e_{t-1}\right)^2}{\sigma_0^2 \sum_{t=2}^{n} e_{t-1}^2} + o_P(1). \end{aligned} \tag{4.34}$$

It is easily proven that

$$\frac{\sum_{t=2}^{n} \eta_t e_{t-1}}{\sigma_0 \sqrt{\sum_{t=2}^{n} e_{t-1}^2}} \longrightarrow N(0,1). \tag{4.35}$$

Thus, by (4.33)–(4.35), we finish the proof of (3.8).

Next we prove (3.9). Under $\mathrm{H}_{02} : f_t(\theta) = \theta$, $g(u) = u$, and $y_t = x_t^T \beta_0 + e_t$, we have

$$\widehat{\varepsilon}_t(2) = y_t - x_t^T \widehat{\beta}_{2n} = x_t^T \beta_0 - x_t^T \widehat{\beta}_{2n} + e_t = x_t^T \left(\beta_0 - \widehat{\beta}_{2n}\right) + e_t. \tag{4.36}$$

Hence

$$\begin{aligned} \widehat{\varepsilon}_t(2) - \widehat{\theta}_{2n} \widehat{\varepsilon}_{t-1}(2) &= x_t^T \left(\beta_0 - \widehat{\beta}_{2n}\right) + e_t - \widehat{\theta}_{2n} \left(x_{t-1}^T \left(\beta_0 - \widehat{\beta}_{2n}\right) + e_{t-1} \right) \\ &= \left(x_t^T - \widehat{\theta}_{2n} x_{t-1}^T \right) \left(\beta_0 - \widehat{\beta}_{2n}\right) + \eta_t. \end{aligned} \tag{4.37}$$

By (2.8), (2.10), we have

$$\begin{aligned} 0 &= \sum_{t=2}^{n} \left(\widehat{\varepsilon}_t(2) - \widehat{\theta}_{2n} \widehat{\varepsilon}_{t-1}(2) \right) \left(x_t - \widehat{\theta}_{2n} x_{t-1} \right) \\ &= \sum_{t=2}^{n} \left(x_t^T - \widehat{\theta}_{2n} x_{t-1}^T \right) \left(\beta_0 - \widehat{\beta}_{2n}\right) \left(x_t - \widehat{\theta}_{2n} x_{t-1} \right) + \sum_{t=2}^{n} \eta_t \left(x_t - \widehat{\theta}_{2n} x_{t-1} \right). \end{aligned} \tag{4.38}$$

From (4.38), we obtain,

$$\widehat{\beta}_{2n} - \beta_0 = X_{2n}^{-1}\left(\widehat{\theta}_{2n}\right)\sum_{t=2}^{n}\eta_t\left(x_t - \widehat{\theta}_{2n}x_{t-1}\right). \tag{4.39}$$

Thus, by (4.37), (4.39) and Lemma 4.2, we have

$$\begin{aligned}(n-1)\widehat{\sigma}_{2n}^2 &= \sum_{t=2}^{n}\left(\widehat{\varepsilon}_t(2) - \widehat{\theta}_{2n}\widehat{\varepsilon}_{t-1}(2)\right)^2 \\ &= \sum_{t=1}^{n}\eta_t^2 + \left(\beta_0 - \widehat{\beta}_{2n}\right)^T X_{2n}\left(\widehat{\theta}_{2n}\right)\left(\beta_0 - \widehat{\beta}_{2n}\right) + 2\left(\beta_0 - \widehat{\beta}_{2n}\right)^T\sum_{t=2}^{n}\eta_t\left(x_t - \widehat{\theta}_{2n}x_{t-1}\right) \\ &= \sum_{t=1}^{n}\eta_t^2 - \left(\sum_{t=2}^{n}\eta_t\left(x_t - \widehat{\theta}_{2n}x_{t-1}\right)^T\right)X_{2n}^{-1}\left(\widehat{\theta}_{2n}\right)\left(\sum_{t=2}^{n}\eta_t\left(x_t - \widehat{\theta}_{2n}x_{t-1}\right)\right) \\ &= \sum_{t=1}^{n}\eta_t^2 - \left(\sum_{t=2}^{n}\eta_t X_{2n}^{-1/2}(\theta_0)(x_t - \theta_0 x_{t-1})\right)^2 + o_p(1).\end{aligned} \tag{4.40}$$

By (3.3)–(3.5), we have

$$d_{2n} = \widehat{L}_{2n} - \widehat{L}_n = (n-1)\ln\left(\frac{\widehat{\sigma}_{2n}^2}{\widehat{\sigma}_n^2}\right) = (n-1)\left(\left(\frac{\widehat{\sigma}_{2n}^2}{\widehat{\sigma}_n^2}\right) - 1\right) + o_P(1). \tag{4.41}$$

Under the H_{02}, by (4.26), (4.40), and (4.41), we obtain

$$\begin{aligned}\frac{(n-1)(\widehat{\sigma}_{2n}^2 - \widehat{\sigma}_n^2)}{\widehat{\sigma}_n^2} &= \frac{\left(\sum_{t=2}^{n}\eta_t e_{t-1}\right)^2}{\widehat{\sigma}_n^2\sum_{t=2}^{n}e_{t-1}^2} + o_P(1) \\ &= \frac{\left(\sum_{t=2}^{n}\eta_t e_{t-1}\right)^2}{\sigma_0^2\sum_{t=2}^{n}e_{t-1}^2} + o_P(1).\end{aligned} \tag{4.42}$$

Thus, by (4.35), (4.42), (3.9) holds.

Finally, we prove (3.10). Under H_{03}, we have

$$\widehat{\varepsilon}_t(3) = y_t - \frac{e^{x_t^T\widehat{\beta}_{3n}^*}}{1 + e^{x_t^T\widehat{\beta}_{3n}^*}} = \frac{e^{x_t^T\widehat{\beta}_{3n}^*}}{\left(1 + e^{x_t^T\widehat{\beta}_{3n}^*}\right)^2}x_t^T\left(\beta_0 - \widehat{\beta}_{3n}\right) + e_t. \tag{4.43}$$

Thus

$$\begin{aligned}
\widehat{\varepsilon}_t(3) - \widehat{\theta}_{3n}\widehat{\varepsilon}_{t-1}(3) &= \frac{e^{x_t^T\widehat{\beta}_{3n}^*}}{\left(1+e^{x_t^T\widehat{\beta}_{3n}^*}\right)^2} x_t^T\left(\beta_0 - \widehat{\beta}_{3n}\right) + e_t \\
&\quad - \widehat{\theta}_{3n}\frac{e^{x_{t-1}^T\widehat{\beta}_{3n}^{**}}}{\left(1+e^{x_{t-1}^T\widehat{\beta}_{3n}^{**}}\right)^2} x_{t-1}^T\left(\beta_0 - \widehat{\beta}_{3n}\right) - \widehat{\theta}_{3n}e_{t-1} \\
&= \left(\frac{e^{x_t^T\widehat{\beta}_{3n}^*}}{\left(1+e^{x_t^T\widehat{\beta}_{3n}^*}\right)^2} x_t^T - \widehat{\theta}_{3n}\frac{e^{x_{t-1}^T\widehat{\beta}_{3n}^{**}}}{\left(1+e^{x_{t-1}^T\widehat{\beta}_{3n}^{**}}\right)^2} x_{t-1}^T\right)\left(\beta_0 - \widehat{\beta}_{3n}\right) + \eta_t.
\end{aligned} \tag{4.44}$$

By (2.8) and (2.10), we have

$$\begin{aligned}
0 &= \sum_{t=2}^{n}\left(\widehat{\varepsilon}_t(3) - \widehat{\theta}_{3n}\widehat{\varepsilon}_{t-1}(3)\right)\left(\frac{e^{x_t^T\widehat{\beta}_{3n}^*}}{\left(1+e^{x_t^T\widehat{\beta}_{3n}^*}\right)^2} x_t - \widehat{\theta}_{3n}\frac{e^{x_{t-1}^T\widehat{\beta}_{3n}^{**}}}{\left(1+e^{x_{t-1}^T\widehat{\beta}_{3n}^{**}}\right)^2} x_{t-1}\right) \\
&= \sum_{t=2}^{n}\left(\frac{e^{x_t^T\widehat{\beta}_{3n}^*}}{\left(1+e^{x_t^T\widehat{\beta}_{3n}^*}\right)^2} x_t^T - \widehat{\theta}_{3n}\frac{e^{x_{t-1}^T\widehat{\beta}_{3n}^{**}}}{\left(1+e^{x_{t-1}^T\widehat{\beta}_{3n}^{**}}\right)^2} x_{t-1}^T\right)\left(\beta_0 - \widehat{\beta}_{3n}\right) \\
&\quad \times \left(\frac{e^{x_t^T\widehat{\beta}_{3n}^*}}{\left(1+e^{x_t^T\widehat{\beta}_{3n}^*}\right)^2} x_t - \widehat{\theta}_{3n}\frac{e^{x_{t-1}^T\widehat{\beta}_{3n}^{**}}}{\left(1+e^{x_{t-1}^T\widehat{\beta}_{3n}^{**}}\right)^2} x_{t-1}\right) \\
&\quad + \sum_{t=2}^{n}\eta_t\left(\frac{e^{x_t^T\widehat{\beta}_{3n}^*}}{\left(1+e^{x_t^T\widehat{\beta}_{3n}^*}\right)^2} x_t - \widehat{\theta}_{3n}\frac{e^{x_{t-1}^T\widehat{\beta}_{3n}^{**}}}{\left(1+e^{x_{t-1}^T\widehat{\beta}_{3n}^{**}}\right)^2} x_{t-1}\right) \\
&= \left(\beta_0 - \widehat{\beta}_{3n}\right)^T \mathrm{X}_{3n}\left(\widehat{\beta}_{3n}^*, \widehat{\beta}_{3n}^{**}, \widehat{\theta}_{3n}\right) + \sum_{t=2}^{n}\eta_t\left(\frac{e^{x_t^T\widehat{\beta}_{3n}^*}}{\left(1+e^{x_t^T\widehat{\beta}_{3n}^*}\right)^2} x_t - \widehat{\theta}_{3n}\frac{e^{x_{t-1}^T\widehat{\beta}_{3n}^{**}}}{\left(1+e^{x_{t-1}^T\widehat{\beta}_{3n}^{**}}\right)^2} x_{t-1}\right).
\end{aligned} \tag{4.45}$$

From (4.45), we obtain

$$\widehat{\beta}_{3n} - \beta_0 = \mathrm{X}_{3n}^{-1}\left(\widehat{\beta}_{3n}^*, \widehat{\beta}_{3n}^{**}, \widehat{\theta}_{3n}\right)\sum_{t=2}^{n}\eta_t\left(\frac{e^{x_t^T\widehat{\beta}_{3n}^*}}{\left(1+e^{x_t^T\widehat{\beta}_{3n}^*}\right)^2} x_t - \widehat{\theta}_{3n}\frac{e^{x_{t-1}^T\widehat{\beta}_{3n}^{**}}}{\left(1+e^{x_{t-1}^T\widehat{\beta}_{3n}^{**}}\right)^2} x_{t-1}\right). \tag{4.46}$$

By (4.44), (4.46) and Lemma 4.2, we have

$$\begin{aligned}
(n-1)\widehat{\sigma}_{3n}^{2} &= \sum_{t=2}^{n}\left(\widehat{\varepsilon}_t(3)-\widehat{\theta}_{3n}\widehat{\varepsilon}_{t-1}(3)\right)^2 \\
&= \sum_{t=1}^{n}\eta_t^2+\left(\beta_0-\widehat{\beta}_{3n}\right)^T X_{3n}\left(\widehat{\beta}_{3n}^{*},\widehat{\beta}_{3n}^{**},\widehat{\theta}_{3n}\right)\left(\beta_0-\widehat{\beta}_{3n}\right) \\
&\quad +2\left(\beta_0-\widehat{\beta}_{3n}\right)^T\sum_{t=2}^{n}\eta_t\left(\frac{e^{x_t^T\widehat{\beta}_{3n}^{*}}}{\left(1+e^{x_t^T\widehat{\beta}_{3n}^{*}}\right)^2}x_t-\widehat{\theta}_{3n}\frac{e^{x_{t-1}^T\widehat{\beta}_{3n}^{**}}}{\left(1+e^{x_{t-1}^T\widehat{\beta}_{3n}^{**}}\right)^2}x_{t-1}\right) \\
&= \sum_{t=1}^{n}\eta_t^2-\left(\sum_{t=2}^{n}\eta_t X_{3n}^{-1/2}\left(\widehat{\beta}_{3n}^{*},\widehat{\beta}_{3n}^{**},\widehat{\theta}_{3n}\right)\right. \\
&\qquad \left.\times\left(\frac{e^{x_t^T\widehat{\beta}_{3n}^{*}}}{\left(1+e^{x_t^T\widehat{\beta}_{3n}^{*}}\right)^2}x_t^T-\widehat{\theta}_{3n}\frac{e^{x_{t-1}^T\widehat{\beta}_{3n}^{**}}}{\left(1+e^{x_{t-1}^T\widehat{\beta}_{3n}^{**}}\right)^2}x_{t-1}^T\right)\right)^2 \\
&= \sum_{t=1}^{n}\eta_t^2-\left(\sum_{t=2}^{n}\eta_t X_{3n}^{-1/2}(\varphi_0)\left(\frac{e^{x_t^T\beta_0}}{\left(1+e^{x_t^T\beta_0}\right)^2}x_t-\theta_0\frac{e^{x_{t-1}^T\beta_0}}{\left(1+e^{x_{t-1}^T\beta_0}\right)^2}x_{t-1}\right)\right)^2+o_p(1).
\end{aligned} \tag{4.47}$$

By (3.3)–(3.5), we know that

$$d_{3n}=\widehat{L}_{3n}-\widehat{L}_n=(n-1)\ln\left(\frac{\widehat{\sigma}_{3n}^2}{\widehat{\sigma}_n^2}\right)=(n-1)\left(\left(\frac{\widehat{\sigma}_{3n}^2}{\widehat{\sigma}_n^2}\right)-1\right)+o_P(1). \tag{4.48}$$

Under the H_{03}, by (4.26), (4.47) and (4.48), we have

$$\frac{(n-1)\left(\widehat{\sigma}_{3n}^2-\widehat{\sigma}_n^2\right)}{\widehat{\sigma}_n^2}=\frac{\left(\sum_{t=2}^{n}\eta_t e_{t-1}\right)^2}{\sigma_0^2\sum_{t=2}^{n}e_{t-1}^2}+o_P(1). \tag{4.49}$$

Thus, (3.10) follows from (4.48), (4.49), and (4.35). Therefore, we complete the proof of Theorem 3.1. □

5. Conclusions and Open Problems

In the paper, we consider the generalized linear mode with FCA processes, which includes many special cases, such as an ordinary regression model, an ordinary generalized regression model, a linear regression model with constant coefficient autoregressive processes, time-dependent and function coefficient autoregressive processes, constant coefficient autoregressive processes, time-dependent or time-varying autoregressive processes, and a linear regression model with functional coefficient autoregressive processes. And then we obtain the QML estimators for some unknown parameters in the generalized linear mode model and extend some estimators. At last, we use pseudo LR method to investigate three hypothesis tests of interest and obtain the asymptotic chi-squares distributions of statistics.

However, several lines of future work remain open.

(1) It is well known that a conventional time series can be regarded as the solution to a differential equation of integer order with the excitation of white noise in mathematics, and a fractal time series can be regarded as the solution to a differential equation of fractional order with a white noise in the domain of stochastic processes (see [25]). In the paper, $\{\varepsilon_t\}$ is a conventional nonlinear time series. We may investigate some hypothesis tests by pseudo LR method when the $\{\varepsilon_t\}$ is a fractal time series (the idea is given by an anonymous reviewer). In particular, we assume that

$$\sum_{i=0}^{p} a_{p-i} D^{v_i} \varepsilon_t = \eta_t, \tag{5.1}$$

where $v_p, v_{p-1}, \ldots, v_0$ is strictly decreasing sequence of nonnegative numbers, a_i is a constant sequence, and D^v is the Riemann-Liouville integral operator of order $v > 0$ given by

$$D^v h(t) = \frac{1}{\Gamma(v)} \int_0^t (t-u)^{v-1} h(u)\, du, \tag{5.2}$$

where Γ is the Gamma function, and $h(\mathrm{t})$ is a piecewise continuous on $(0, \infty)$ and integrable on any finite subinterval of $[0, \infty)$ (See [25, 26]). Fractal time series may have a heavy-tailed probability distribution function and has been applied various fields of sciences and technologies (see [25, 27–32]). Thus it is very significant to investigate various regression models with fractal time series errors, including regression model (1.1) with (5.1).

(2) We maybe investigate the others hypothesis tests, for example:

H_{04}: $f_t(\theta) = 0, g(u) = \mathrm{u}, \sigma_0^2 > 0$;

H_{05}: $f_t(\theta) = \theta, g(u) = 0, \sigma_0^2 > 0$;

H_{06}: $f_t(\theta) = 0, g(u) = e^u/(1+e^u), \sigma_0^2 > 0$;

H_{07}: $f_t(\theta) = a_t$ and $g(u)$ is a continuous function, $\sigma_0^2 > 0$;

H_{08}: $f_t(\theta) = a_t, g(u) = u, \sigma_0^2 > 0$;

H_{09}: $f_t(\theta) = a_t, g(u) = e^u/(1+e^u), \sigma_0^2 > 0$.

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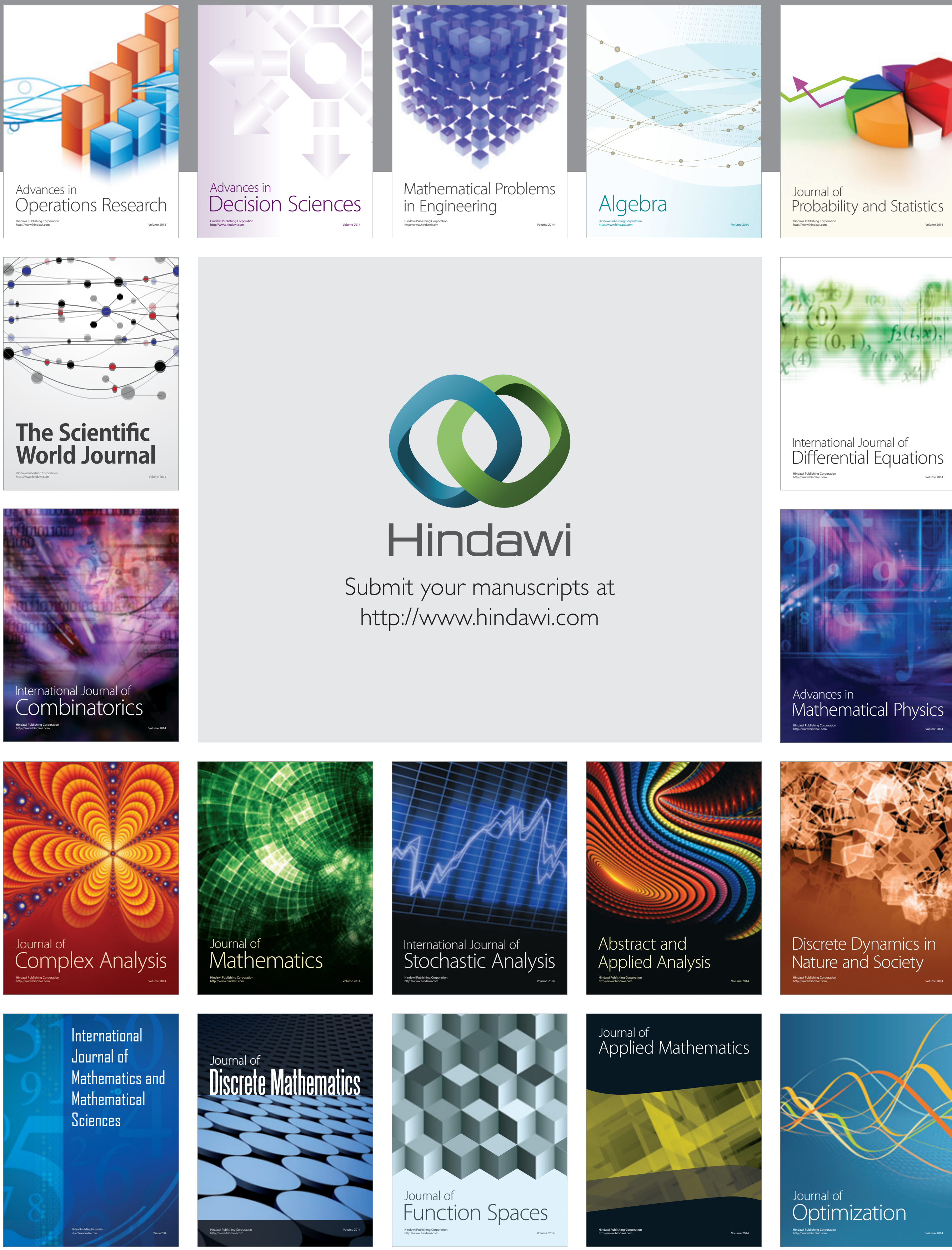
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