Mathematical Problems in Engineering
Volume 4, pp. 165-185
Reprints available directly from the publisher
Photocopying permitted by license only

Robust Stability and Stabilization of a Class of Uncertain Nonlinear Systems with Delays*

MAGDI S. MAHMOUD

Dept. of Electrical and Computer Engineering, Kuwait University, PO Box 5969, Safat 13060, KUWAIT, e-mail: magdi@eng.kuniv.edu.kw

(Received 4 March, 1996; Revised 29 May, 1996; In final form 18 October, 1996)

A class of nonlinear systems with norm-bounded uncertainties and state-delay is considered. Two criteria are developed for the robust stability analysis: one is delay-independent and the other is delay-dependent. Methods for robust feedback synthesis are then examined. It is established that linear memoryless controllers are capable of guaranteeing the delay-dependent and delay-independent stabilizability of the closed-loop systems. All the results are expressed in the form of linear matrix inequalities which can be solved by efficient and numerically-stable routines. The developed theory is applied to the stability robustness problem of an industrial jacketed continuous stirred tank reactor.

1. INTRODUCTION

It has been increasingly apparent [4] that the presence of delayed information in physical and engineering systems may have a destabilizing effect and may lead to poor performance of control systems. Examples of systems with time delay include transport processes, population models, remote control problems, urban traffic; to name a few. Stability problems of time-delay systems have therefore been the subject of numerous studies [5,6,8,10–13,15,16]. Almost all the available results are however restricted to linear systems with norm bounded or matched parametric uncertainty. The objective of this paper is to examine the problems of robust stability and robust feedback synthesis for a class of nonlinear systems with

* Based on research supported by Kuwait University Research Administration under project code EE-063.

norm-bounded uncertainties and state-delay. The approach adopted in this work is based on the constructive use Lyapunov stability theory. Two criteria are developed for the robust stability analysis: one is delay-independent and the other is delay-dependent. Methods for robust feedback synthesis are then considered It is established that linear memoryless controllers are capable of guaranteeing both the delay-dependent at delay-independent stabilizability of the closed-loop systems. All the results are expressed in the form of linear matrix inequalities (LMI's), which can be solved by efficient and numerically-stable routines [1,3]. The developed theory is then illustrated by a simulation of an industrial jacketed continuous stirred tank reactor (JCSTR).

Notations. We use W^t, W^{-1}, $\lambda(W)$ to denote, respectively, the transpose of, the inverse of, and the eigenvalues of any square matrix W. The vector norm is taken to be Euclidian and the matrix norm is the corresponding induced one; that is $\|W\| = \lambda_M^{1/2}(W^tW)$, where $\lambda_{M(m)}(W)$ stands for the operation of taking the maximum (minimum) eigenvalue of W. We use $W > 0$ ($W < 0$) to denote a positive- (negative-) definite matrix W. Let C_0^- denote the proper left-half of the complex plane. Sometimes, the arguments of function will be omitted in the analysis when no confusion can arise.

2. PROBLEM STATEMENT AND DEFINITIONS

Consider a class of nonlinear dynamical systems with state-delay of the form

$$\begin{aligned}\dot{x}(t) &= [A_o + \Delta A(t)]x(t) + [B_o + \Delta B(t)]u(t) \\ &+ [D_o + \Delta D(t)]x(t-\tau) + [G_o + \Delta G(t)]g[x(t)]\end{aligned} \tag{1a}$$

$$x(t) = \phi(t) \quad \forall t \in [-\tau, 0], \quad 0 \leq \tau \leq \tau^* < \infty \tag{1b}$$

where $t \in \Re$ is the time, x E $\Re^n$ is the instantaneous state; $u \in \Re^m$ is the control input; and τ representing the delay of the system with the bound τ^* is known. The function g(.): $\Re^n \to \Re^n$, is unknown-but-bounded. The matrices $A_o \in \Re^{n\times n}$, $B_o \in \Re^{n\times m}$, represent the nominal system without delay and uncertainties where the pair (A_o, B_o) being controllable and $D_o \in \Re^{n\times n}$, $G_o \in \Re^{n \times n_g}$, are constant matrices. The uncertainty within the system are represented by the real-valued matrix functions $\Delta A(t)$,

$\triangle B(t)$, $\triangle D(t)$, $\triangle G(t)$, which we consider to satisfy the following assumptions:

A1. There exists real constant matrices M, N, L, E_g, T_g, E_d, T_d of appropriate dimensions such that $\forall\, t \in \Re$

$$\begin{aligned} &[\Delta A(t) \quad \Delta B(t)] = MF(t)[N\, L], \quad F^t(t)F(t) \le I \\ &\Delta G(t) = E_g F_g(t) T_g, \quad F_g^t(t)F_g(t) \le I, \\ &\Delta D(t) = E_d F_d(t) T_d, \quad F_d^t(t)F_d(t) \le I \end{aligned} \qquad (2)$$

where the elements $F_{ij}(t)$, $(F_d(t))_{ij}$ are Lebesgue measurable $\forall i$, j and $F(t) \in \Re^{\alpha\times\beta}$, $M \in \Re^{n\times\alpha}$, $N \in \Re^{\beta\times n}$, $L \in \Re^{\beta\times m}$, $F_g(t) \in \Re^{\alpha_g\times\beta_g}$, $F_d(t) \in \Re^{\alpha_d\times\beta_d}$, $T_g(t) \in \Re^{\beta_g\times n}$, $T_d(t) \in \Re^{\beta_d\times n}$.

A2. There exist a known scalar $\varepsilon_0 > 0$ and a matrix $R_g \in \Re^{n\times n}$ such that

$$||g(x)|| \le \varepsilon_o ||R_g x|| \qquad \forall x \in \Re^n \qquad (3)$$

Distinct from (1) is the **free** system described by:

$$\dot{x}(t) = [A_o + \Delta A(t)]x(t) + [D_o + \Delta D(t)]x(t-\tau) + [G_o + \Delta G(t)]g[x(t)] \quad (4a)$$

$$x(t) = \phi(t) \quad \forall t \in [-\tau, 0], \quad 0 \le \tau \le \tau^* < \infty \qquad (4b)$$

for which we invoke the following assumption:

A3. Suppose that $\lambda(A_o) \in C_n$ and there exist matrices $0 < P_o = P_o^t \in \Re^{n\times n}$ and $0 < Q_o = Q_o^t \in \Re^{n\times n}$ such that

$$P_o A_o + A_o' P_o = -Q_o \qquad (5)$$

Models of dynamical systems of the type (1) can be found in several engineering applications including river pollution control [7]; turbogenerator control [14] and recycled continuous stirred-tank reactors . In section section, a typical model of an industrial jacketed continuous stirred tank reactor with a delayed recycle stream will be studied for purpose of simulation.

In the sequel, we adopt the following concepts of robust stability and robust stabilization:

DEFINITION 1 The uncertain state-delay system (1) is said to be **robustly stable** if the null solution x(t) = 0 of system (4) is uniformly asymptotically stable for all admissible realizations of the uncertainties $\triangle A(t)$, $\triangle D(t)$, $\triangle G(t)$ satisfying (2) and (3).

DEFINITION 2 The uncertain state-delay system (1) is said to be **robustly stabilizable** if there exists a memoryless feedback control u(t) = K [x(t)] such that the resulting closed-loop system is robusty stable in the sense of Definition 1.

The objectives of this work are four-fold:

1. to develop delay-independent conditions for robust stability for the uncertain free system (4),
2. to develop delay-dependent conditions for robust stability for the uncertain free system (4),
3. to provide a feedback control synthesis that guarantee the robust stabilization of the uncertain system (1) and
4. to cast the solutions in all cases in the form of linear matrix inequalities that can be solved by efficient and numerically-stable routines.

For simplicity in exposition, let

$$\begin{aligned} \hat{A}(t) &= [A_o + MF(t)N] \quad , \quad \hat{D}(t) = [D_o + E_d F_d(t) T_d] \\ \hat{G}(t) &= [G_o + E_g F_g(t) T_g] \end{aligned} \tag{6}$$

Before proceeding further, some basic results that will be used in the sequel are provided.

LEMMA 1 Let $\Sigma_1, \Sigma_2, \Sigma_3$ be real constant matrices of compatible dimensions and H(t) be a real matrix function satisfying $H^t(t)\, H(t) \leq I$. Then the following inequalities hold:

1. $\Sigma_1 H(t)\Sigma_2 + \Sigma_2^t H^t(t)\Sigma_1^t \leq \rho^{-2}\Sigma_1\Sigma_1^t + \rho^2\Sigma_2^t\Sigma_2, \rho > 0$ (7a)
2. $\forall\rho > 0$ such that $\rho^2\Sigma_2^t\Sigma_2 < I$, (7b)
$(\Sigma_3^t + \Sigma_2^t H^t(t)\Sigma_1^t)(\Sigma_3 + \Sigma_1 H(t)\Sigma_2) \leq \rho^{-2}\Sigma_1^t\Sigma_1 + \Sigma_3^t[I - \rho^2\Sigma_2\Sigma_2^t]^{-1}\Sigma_3$

Proof See the appendix.

3. ROBUST STABILITY: DELAY-INDEPENDENT CASE

The objective here is to derive delay-independent conditions for robust stability of system (4). For this purpose, define the matrix functions

$$\Im(P,\varepsilon) = PA_o + A_o^t P + P\left[\varepsilon_2^1 MM^t + (\varepsilon_3^{-1} + \varepsilon_5^{-1})I\right]P + \left[\varepsilon_0^2 R_g^t R_g + \varepsilon_2 N^t N + \varepsilon_1 I\right] \tag{8}$$

$$\begin{aligned} \Omega_1(\varepsilon) &= I - \varepsilon_4 E_d E_d^t,\ \Omega_3(\varepsilon) = \varepsilon_1 I - \varepsilon_3\varepsilon_4^{-1} T_d^t T_d - \varepsilon_3 D_o^t (I - \varepsilon_4 E_d E_d^t)^{-1} D_o \\ \Omega_2(\varepsilon) &= I - \varepsilon_6 E_g E_g^t,\ \Omega_4(\varepsilon) = I - \varepsilon_5\varepsilon_6^{-1} T_g^t T_g - \varepsilon_5 G_o^t (I - \varepsilon_6 E_g E_g^t)^{-1} G_o \end{aligned} \tag{9}$$

where $0 < P = P^t \in \Re^{n\times n}$, $\varepsilon_0, \ldots, \varepsilon_6$ are positive scalars and $\varepsilon \in \Re^7 = [\varepsilon_0, \ldots, \varepsilon_6]$.

THEOREM 1 Suppose that **A1, A2** hold. Then system (4) is **robustly stable** if any of the following equivalent conditions holds:

a. There exist matrix $0 < P = P^t \in \Re^{n\times n}$ and $\varepsilon > 0$, satisfying the inequalities:

$$\Im(P, \varepsilon) < 0 \tag{10a}$$

$$\Omega_1(\varepsilon) > 0, \quad \Omega_2(\varepsilon) > 0, \quad \Omega_3(\varepsilon) > 0, \quad \Omega_4(\varepsilon) > 0 \tag{10b}$$

b. There exist matrix $0 < Z = Z^t \in \Re^{n\times n}$ and $\varepsilon > 0$, solving the following LMI:

$$\begin{bmatrix} ZA_o^t + A_o Z + (\varepsilon_3^{-1} + \varepsilon_5^{-1})I & . & Z\Sigma^t & M \\ \cdots\cdots\cdots & . & \cdots & \cdot\cdot \\ \Sigma Z & . & -\Delta & 0 \\ M^t & . & 0 & -\varepsilon_2 I \end{bmatrix} < 0 \tag{11a}$$

$$\Delta = \mathrm{diag}[\varepsilon_1^{-1}I, \varepsilon_2^{-1}I, \varepsilon_0^{-2}I], \quad \Sigma^t = [I\; N^t\; R_g^t]. \tag{11b}$$

Proof

a. Define the quadratic Lyapunov function candidate $V(.)$: $\Re^n \times \Re_+ \to \Re_+$ as

$$\begin{aligned} V(x,t) &= x^t(t)Px(t) + \varepsilon_1 \int_{t-\tau}^{t} x^t(\vartheta)x(\vartheta)d\vartheta \\ &\quad + \int_o^t \varepsilon_0^2 [R_g x]^t [R_g x] d\vartheta - \int_0^1 g^t[x(\vartheta)]g[x(\vartheta)]d\vartheta \end{aligned} \tag{12}$$

where $\varepsilon_1 > 0$ is a design parameter and observe that $V(x, t) \geq 0$ in view of (3). The derivative of the Lyapunov function (12) evaluated on the solutions of system (4) using (5) is given by

$$\begin{aligned} L(x,t) &= x^t(t)P\dot{x}(t) + \dot{x}^t(t)Px(t) + \varepsilon_1 x^t(t)x(t) \\ &\quad + \varepsilon_0^2[R_g x]^t[R_g x] - g^t[x(t)]g[x(t)] - \varepsilon_1 x^t(t-\tau)x(t-\tau) \\ &= x^t(t)P\hat{A}(t)x(t) + x^t(t)\hat{A}^t(t)Px(t) \\ &\quad + x^t(t)P\hat{D}(t)x(t-\tau) + x^t(t-\tau)\hat{D}^t(t)Px(t) \\ &\quad + g^t[x(t)]\hat{G}^t(t)Px(t) + x^t(t)P\hat{G}(t)g[x(t)] \\ &\quad + \varepsilon_1 x^t(t)x(t) - \varepsilon_1 x^t(t-\tau)x(t-\tau) + \varepsilon_0^2 x^t(t)R_g^t R_g x(t) - g^t[x(t)]g[x(t)] \end{aligned} \tag{13}$$

Through repeated application of **Lemma 1** subject to **A1** and **A2** to the different terms in (13), and grouping similar quantities, we find for some $\varepsilon_0, \ldots, \varepsilon_6 > 0$ that

$$\begin{aligned} L(x,t) \le\ & x^t(t)\left\{PA_o + A_o^tP + [\varepsilon_0^2 R_g^t R_g + \varepsilon_2 N^tN + \varepsilon_1 I] \right. \\ & \left. + P[\varepsilon_2^{-1}MM^t + (\varepsilon_3^{-1} + \varepsilon_5^{-1})I]P\right\} x(t) \\ & - g^t[x(t)]\left(I - \varepsilon_5\varepsilon_6^{-1}T_g^tT_g - \varepsilon_5 G_o^t(I - \varepsilon_6 E_g E_g^t)^{-1}G_o\right) g[x(t)] \\ & x^t(t-\tau)\left(\varepsilon_1 I - \varepsilon_3\varepsilon_4^{-1}T_d^tT_d - \varepsilon_3 D_o^t(I - \varepsilon_4 E_d E_d^t)^{-1}D_o\right) x(t-\tau) \end{aligned} \tag{14}$$

when conditions (10b) are satisfied for some $(\varepsilon_0,\ldots,\varepsilon_6) > 0$, then inequality (14) via (10a) reduces to

$$L(x,t) < x^t(t)\Im(P,\varepsilon)x(t) \tag{15}$$

where $\varepsilon = [\varepsilon_0,\ldots, \varepsilon_6]$. Since $\Im(P, \varepsilon)$ in monotonic nondecreasing function with respect to ε, then by **A3** the result (10a) follows immediately.

b. By defining by $Z = P^{-1}$, we obtain the equivalent conditions

$$A_oZ + ZA_o^t + (\varepsilon_3^{-1} + \varepsilon_5^{-1})I + Z[\varepsilon_0^2 R_g^t R_g + \varepsilon_2 N^tN + \varepsilon_1 I]Z + [\varepsilon_2^{-1}MM^t] < 0 \tag{16}$$

along with (9) and (10b). Then simple rearrangement of inequality (16) yields form (11) and the proof is completed.

COROLLARY 1 In the case of linear uncerain delay systems ($G_o = 0$, $\triangle G(t) = 0$), the conditions of delay-independent robust stability reduce for $\varepsilon > 0$ to $\Omega_1(\varepsilon) > 0$, $\Omega_3(\varepsilon) > 0$ and:

$$\begin{bmatrix} ZA_o^t + A_oZ + (\varepsilon_3^{-1} + \varepsilon_5^{-1}I) & . & Z[I, N^t] & M \\ \cdots\cdots\cdots & . & \cdots\cdots & \cdot\cdot \\ \begin{bmatrix} I \\ N \end{bmatrix} Z & . & -\begin{bmatrix} \varepsilon_1^{-1}I & 0 \\ 0 & \varepsilon_2^{-1}I \end{bmatrix} & 0 \\ M^t & . & 0 & -\varepsilon_2 I \end{bmatrix} < 0 \tag{17}$$

COROLLARY 2 In the absence of uncertainties ($\triangle A(t) = 0$, $\triangle D(t) = 0$, $\triangle G(t) = 0$), the stability robustness of the nonlinear delay systems corresponds to the solvability of

$$\begin{bmatrix} ZA_o^t + A_oZ + (\varepsilon_3^{-1} + \varepsilon_5^{-1})I & . & Z[I, R_g^t] \\ \cdots\cdots\cdots\cdots & . & \cdots\cdots \\ \begin{bmatrix} I \\ R_g \end{bmatrix} Z & . & -\begin{bmatrix} \varepsilon_1^{-1}I & 0 \\ 0 & \varepsilon_0^{-2}I \end{bmatrix} \end{bmatrix} < 0 \tag{18}$$

for $\varepsilon > 0$, and $(\varepsilon_1 I - \varepsilon_3 D_o^t D_o) > 0$, $(I - \varepsilon_5 G_o^t G_o) > 0$.

COROLLARY 3 By dropping out all the uncertainties and state-delay, it is easy to deduce that inequality (11) becomes

$$\begin{bmatrix} ZA_o^t + A_o Z + \varepsilon_5^{-1} I & . & ZR_g^t \\ \dots\dots\dots\dots & . & \dots\dots \\ R_g Z & . & -\varepsilon_0^{-2} I \end{bmatrix} < 0 \tag{19}$$

for $\varepsilon_0, \varepsilon_5 > 0$ and $(I - \varepsilon_5 G_o^t G_o) > 0$.

COROLLARY 4 On considering the case of linear systems without uncertainties and without state-delay, it turns out that inequality (11) converges to condition (5) with $Z = P_o^{-1}$, which corresponds to the standard Lyapunov stability requirement.

REMARK 1 The results of **Theorem 1** and **corollaries 1-4** provide tractable tools for control design of several engineering systems in which the dynamic relations exhibit time-delay with norm-bounded uncertainties and cone-bounded nonlinearities. It should be emphasized that the class of systems (1) encompasses almost all previous uncertain systems considered so far in the literature. Admittedly, the developed conditions of **Theorem 1** are only sufficient since we are dealing with nonlinear dynamical systems. However in the special cases described in **corollaries 1-4** and by reversing the order of proof of **Theorem 1,** it can be shown, following the results of [17], that conditions (17), (18) or (19) are both necessary and sufficient. The basic difference lies in the uncertainty structure.

4. ROBUST STABILITY: DELAY-DEPENDENT CASE

The problem of interest is to determine the upper bound τ^* for the time-delay τ such that system (4) is robustly stable $\forall\, \tau \in [0, \tau^*]$. Let $x(t)$, $t \geq 0$ be the solution of system (4). Since

$$x(t-\tau) = x(t) - \int_{\tau}^{0} \dot{x}(t+\vartheta)d\vartheta = x(t) - \xi(x,t) \tag{20a}$$

where

$$\begin{aligned} \xi(x,t) \;&= \textstyle\int_{-\tau}^{0}[\hat{A}(t+\vartheta)x(t+\vartheta)] + \hat{D}(t+\vartheta)x(t-\tau+\vartheta)]d\vartheta \\ &+ \textstyle\int_{-\tau}^{0}\hat{G}(t+\vartheta)g[x(t+\vartheta)]d\vartheta \end{aligned} \tag{20b}$$

then an equivalent form of the free system (4) would be

$$\begin{aligned} \dot{x}(t) &= [\hat{A}(t) + \hat{D}(t)]x(t) - \hat{D}(t)\xi(x,t) + \hat{G}(t)g[x(t)] \\ x(\vartheta) &= \phi(\vartheta), \quad \forall \vartheta \in [-2\tau, 0] \end{aligned} \tag{21}$$

Define the matrix function

$$\begin{aligned} \aleph(P,\sigma,\tau) = {} & P(A_o + D_o) + (A_o + D_o)^t P + P[\sigma_4 MM^t + \sigma_5 E_d E_d^t]P \\ & + [\varepsilon_0^2 \sigma_3^{-1} R_g^t R_g + \sigma_4^{-1} N^t N + \sigma_5^{-1} T_d^t T_d] \\ & + \tau\sigma_1^{-1} P[D_o(I - \sigma_6 E_d^t E_d)^{-1} D_o^t + \sigma_6^{-1} T_d T_d^t]P \\ & + \sigma_3 P[G_o(I - \sigma_7 E_g^t E_g)^{-1} G_o^t + \sigma_7^{-1} T_g T_g^t]P \\ & + \tau\sigma_1(1 + \sigma_2^{-1})(1 + \sigma_o)[A_o^t(I - \sigma_8 NN^t)^{-1} A_o + \sigma_8^{-1} M^t M] \\ & + \tau\varepsilon_0^2 \sigma_1 (1 + \sigma_2^{-1})(1 + \sigma_o^{-1}) R_g^t [G_o^t (I - \sigma_9 T_g T_g^t)^{-1} G_o \\ & + \sigma_9^{-1} E_g^t E_g] R_g + \tau\sigma_1(1 + \sigma_2)[D_o^t(I - \sigma_{10} T_d T_d^t)^{-1} D_o + \sigma_{10}^{-1} E_d^t E_d] \end{aligned} \tag{22}$$

where $0 < P = P^t \in \Re^{n \times n}$, σ_o, ..., σ_{10} are positive scalars and $(I - \sigma_6 E_d^t E_d) > 0$, $(I - \sigma_7 E_g^t E_g) > 0$, $(I - \sigma_8 NN^t) > 0$, $(I - \sigma_9 T_g T_g^t) > 0$, $(I - \sigma_{10} T_d T_d^t) > 0$, with $\sigma \in \Re^{11} = [\sigma_0, ..., \sigma_{10}]$. Instead of **A3,** we invoke the following assumption

A4. Suppose that $\lambda(A_o + D_o) \in C_o^-$ and there exist matrices $0 < P_+ = P_+^t \in \Re^{n \times n}$ and $0 < Q_+ = Q_+^t \in \Re^{n \times n}$ such that

$$P_+(A_o + D_o) + (A_o + D_o)^t P_+ = -Q_+ \tag{23}$$

REMARK 2 Observe that **A4** is equivalent to the asymptotic stability of the dynamical system (1) without uncertainties and time-delay and is indeed necessary for the uniform asymptotic stability of system (1) in the presence of uncertaintity matrices $\triangle A(t)$, $\triangle D(t)$, $\triangle G(t)$ satisfying the mismatched conditions (2) and (3).

THEOREM 2 Suppose that **A1, A2 and A4** hold. Then given a scalar τ^*, system (21) is **robustly stable** for any constant time-delay $\tau \in [0, \tau^*]$ provided that one of the two equivalent conditions is met:

a. There exist matrix $0 < P = P^t \in \mathbb{R}^{n \times n}$ and $\sigma > 0$ satisfying the inequalities:

$$\aleph(P,\sigma,\tau^*) < 0 \tag{24}$$

$$\begin{aligned} &(I - \sigma_6 E_d^t E_d) > 0, \quad (I - \sigma_7 E_g^t E_g) > 0, \quad (I - \sigma_8 NN^t) > 0 \\ &(I - \sigma_9 T_g T_g^t) > 0, \quad (I - \sigma_{10} T_d T_d^t) > 0 \end{aligned} \tag{25}$$

b. There exist matrix $0 < Z = Z^t \in \Re^{n \times n}$ and $\varepsilon \in \Re^{11} = [\varepsilon_1, ..., \varepsilon_{11}] > 0$, $\eta = 1/\tau^*$ solving the following LM

$$\begin{bmatrix} Z(A_o+D_o)^t+(A_o+D_o)Z+\Omega J\Omega^t & . & Z\Sigma^t & Z\Theta^t & \Psi \\ \cdots\cdots\cdots & . & \cdots & \cdots & .. \\ \Sigma Z & . & -\Pi & 0 & 0 \\ \Theta Z & . & 0 & -\eta\Phi & 0 \\ \Psi^t & . & 0 & 0 & -\eta\Gamma \end{bmatrix} < 0 \qquad (26a)$$

where

$$\Omega = [M\ T_d\ T_g\ G_o],\ J = \mathrm{diag}[\varepsilon_1 I,\ \varepsilon_2 I,\ \varepsilon_8\varepsilon_7^{-1} I,\ \varepsilon_8(I-\varepsilon_7 E_g^t E_g)^{-1}] \qquad (26b)$$

$$\Psi = [E_d\ D_o],\ \Sigma^t = [N^t\ T_d^t\ R_g^t],\ \Pi = \mathrm{diag}[\varepsilon_1 I,\ \varepsilon_2 I,\ \varepsilon_o^{-2}\varepsilon_8 I] \qquad (26c)$$

$$\Theta = [A_o^t\ M^t\ D_o^t\ E_d^t\ R_g^t G_o^t\ E_g^t],\ \Gamma = \mathrm{diag}[\varepsilon_6 I, (I-\varepsilon_6 T_d^t T_d)] \qquad (26d)$$

$$\Phi = \mathrm{diag}[(\varepsilon_3 I-\varepsilon_4 NN^t),\ \varepsilon_4 I,\ (\varepsilon_{10} I-\varepsilon_5 T_d T_d^t),\ \varepsilon_5 I,\ (\varepsilon_3 I-\varepsilon_9 T_g T_g^t),\ \varepsilon_9 I] \qquad (26e)$$

Proof

a. Introduce the Lyapunov function candidate $V_d(.)$: $\Re^n \times \Re_+ \rightarrow \Re_+$ as

$$\begin{aligned} V_d(x,t) &= \\ x^t(t)Px(t) &+ \int_{-\tau}^{0} \left(\sigma_1(1+\sigma_2^{-1})(1+\sigma_0^{-1}) \int_{t+\vartheta}^{t} \|\hat{G}(\alpha)g[x(\alpha)]\|^2 d\alpha\right) d\vartheta \\ &+ \int_{-\tau}^{0} \left(\sigma_1(1+\sigma_2^{-1})(1+\sigma_0) \int_{t+\vartheta}^{t} \|\hat{A}(\alpha)x(\alpha)\|^2 d\alpha \right. \\ &\left. + \sigma_1(1+\sigma_2) \int_{t-\tau+\vartheta}^{t} \|\hat{D}(\alpha+\tau)x(\alpha)\|^2 d\alpha\right) d\vartheta \end{aligned} \qquad (27)$$

where $\sigma_0,\sigma_1,\sigma_2$, are weighting factors to be selected. The Lyapunov derivative $L_d(.)$ evaluated along the solutions of system (21) using (20) is given by

$$\begin{aligned} L_d(x,t) &= x^t(t)\{P[\hat{A}(t)+\hat{D}(t)]+[\hat{A}(t)+\hat{D}(t)]^t P\}x(t) - 2x^t(t)P\hat{D}(t)\xi(x,t) \\ &+ 2x^t(t)P\hat{G}(t)g[(x,t)] \\ &- \left(\sigma_1(1+\sigma_2^{-1})(1+\sigma_0) \int_{-\tau}^{0} \|\hat{A}(t+\alpha)x(t+\alpha)\|^2 d\alpha \right. \\ &\left. + \sigma_1(1+\sigma_2) \int_{-\tau}^{0} \|\hat{D}(t+\alpha)x(t-\tau+\alpha)\|^2 d\alpha\right) \\ &- \left(\sigma_1(1+\sigma_2^{-1})(1+\sigma_0^{-1}) \int_{-\tau}^{0} \|\hat{G}(t+\alpha)g[x(t+\alpha)]\|^2 d\alpha\right) \\ &+ \tau\sigma_1(1+\sigma_2^{-1})(1+\sigma_0^{-1})\|\hat{G}(t)g[x(t)]\|^2 \\ &+ \left[\tau\sigma_1(1+\sigma_2^{-1})(1+\sigma_0)\|\hat{A}(t)x(t)\|^2 + \tau\sigma_1(1+\sigma_2)\|\hat{D}(t+\tau)x(t)\|^2\right] \end{aligned} \qquad (28)$$

Recalling **Lemma 1** and (20), it follows that

$$2x^t(t)P\hat{D}(t)\xi(x,t) \le \tau\sigma_1^{-1}[x^t(t)P\hat{D}(t)\hat{D}^t(t)Px(t)] + \sigma_1\tau^{-1}[\xi^t(x,t)\xi(x,t)] \qquad (29a)$$

$$2x^t(t)P\hat{G}(t)g[(x,t)] \le \sigma_3[x^t(t)P\hat{G}(t)\hat{G}^t(t)Px(t)] + \sigma_3^{-1}g^t[(x,t)]g[(x,t)] \qquad (29b)$$

By the algebraic inequality $||p + q||^2 \leq (1 + \upsilon^{-1})\, ||p||^2 + (1 + \upsilon)\, ||q||^2$, $\upsilon > 0$ it follows that

$$\begin{aligned}\xi^t(x,t)\xi(x,t) \quad & \leq (1+\sigma_2^{-1})(1+\sigma_0)||\textstyle\int_{-\tau}^{0}\hat{A}(t+\alpha)x(t+\alpha)d\alpha||^2 \\ & +(1+\sigma_2^{-1})(1+\sigma_0^{-1})||\textstyle\int_{-\tau}^{0}\hat{G}(t+\alpha)g[x(t+\alpha)]d\alpha||^2 \quad (30) \\ & +(1+\sigma_2)||\textstyle\int_{-\tau}^{0}\hat{D}(t+\alpha)x(t-\tau+\alpha)d\alpha||^2\end{aligned}$$

Using the Schwartz inquality in (30), it reduces to

$$\begin{aligned}\xi^t(x,t)\xi(x,t) \quad & \leq \tau(1+\sigma_2^{-1})(1+\sigma_0)\textstyle\int_{-\tau}^{0}||\hat{A}(t+\alpha)x(t+\alpha||^2 d\alpha \\ & +\tau(1+\sigma_2^{-1})(1+\sigma_0^{-1})\textstyle\int_{-\tau}^{0}\hat{G}(t+\alpha)g[x(t+\alpha)]||^2 d\alpha \quad (31) \\ & +\tau(1+\sigma_2)\textstyle\int_{-\tau}^{0}||\hat{D}(t+\alpha)x(t-\tau+\alpha)||^2 d\alpha\end{aligned}$$

By substituting (3), (29) and (31) into (28), we get

$$\begin{aligned}L_d(x,t) \leq\ & x^t(t)\left(P[\hat{A}(t)+\hat{D}(t)]+[\hat{A}(t)+\hat{D}(t)^t]P\right)x(t) \qquad (32) \\ & +x^t(t)\left(\tau\sigma_1^{-1}P\hat{D}(t)\hat{D}^t(t)P\right)x(t)+x^t(t)\left(\sigma_3 P\hat{G}(t)\hat{G}^t(t)P\right)x(t) \\ & +\sigma_3^{-1}g^t[x(t)]g[x(t)]+g^t[x(t)]\left(\tau\sigma_1(1+\sigma_2^{-1})(1+\sigma_0^{-1})\hat{G}^t(t)\hat{G}(t)\right)g[x(t)] \\ & +x^t(t)\left(\tau\sigma_1(1+\sigma_2^{-1})(1+\sigma_0)\hat{A}^t(t)\hat{A}(t)+\tau\sigma_1(1+\sigma_2)\hat{D}^t(t+\tau)\hat{D}(t+\tau)\right)x(t)\end{aligned}$$

Algebraic manipulation of (32) using **Lemma 1** subject to **A1, A2** yields:

$$\begin{aligned}L_d(x,t) \leq\ & x^t(t)\left\{P(A_o+D_o)+(A_o+D_o)^t P++P[\sigma_4 MM^t+\sigma_5 E_d E_d^t]P\right. \\ & +[\varepsilon_0^2\sigma_3^{-1}R_g^t R_g+\sigma_4^{-1}N^t N+\sigma_5^{-1}T_d^t T_d] \\ & +\tau\sigma_1^{-1}P[D_o(I-\sigma_6 E_d^t E_d)^{-1}D_o^t+\sigma_6^{-1}T_d T_d^t]P \\ & +\sigma_3 P[G_o(I-\sigma_7 E_g^t E_g)^{-1}G_o^t+\sigma_7^{-1}T_g T_g^t]P \qquad (33) \\ & +\tau\sigma_1(1+\sigma_2^{-1})(1+\sigma_0)[A_o^t(I-\sigma_8 NN^t)^{-1}A_o+\sigma_8^{-1}M^t M] \\ & +\tau\varepsilon_0^2\sigma_1(1+\sigma_2^{-1})(1+\sigma_0^{-1})R_g^t[G_o^t(I-\sigma_9 T_g T_g^t)^{-1}G_o+\sigma_9^{-1}E_g^t E_g]R_g \\ & \left.+\tau\sigma_1(1+\sigma_2)[D_o^t(I-\sigma_{10}T_d T_d^t)^{-1}D_o+\sigma_{10}^{-1}E_d^t E_d]\right\}x(t)\end{aligned}$$

for any scalars $\sigma_o,\ldots,\sigma_{10} > 0$ along with conditions (25). In view of (22), inequality (33) can be expressed

$$L_d(x,t) < x^t(t)\aleph(P,\sigma,\tau)x(t) \qquad (34)$$

Since $\Re(P, \sigma, \tau)$ is monotonic nondecreasing with respect to τ, the rersult follows immediately.

b. By making the following changes of variables

$$\begin{aligned}& \tilde{P}=\sigma_1^{-1}P,\ \varepsilon_1=\sigma_1\sigma_4,\ \varepsilon_2=\sigma_1\sigma_5,\ \varepsilon_3=(1+\sigma_2^{-1})^{-1}(1+\sigma_0)^{-1},\ \varepsilon_6=\sigma_6,\ \varepsilon_7=\sigma_7 \\ & \varepsilon_4=\sigma_8(1+\sigma_2^{-1})^{-1}(1+\sigma_0)^{-1},\ \varepsilon_5=\sigma_{10}(1+\sigma_2)^{-1},\ \varepsilon_8=\sigma_1\sigma_3,\ \varepsilon_{10}=(1+\sigma_2)^{-1} \\ & \varepsilon_9=\varepsilon_0^2\sigma_9(1+\sigma_2^{-1})^{-1}(1+\sigma_0^{-1})^{-1},\ \varepsilon_{11}=\varepsilon_0^2(1+\sigma_2^{-1})^{-1}(1+\sigma_0^{-1})^{-1} \quad (35)\end{aligned}$$

it is easy to see that conditions (24) and (25) are met if and only if

$$I - \varepsilon_7 E_g^t E_g > 0, \quad \varepsilon_3 I - \varepsilon_4 NN^t > 0, \quad I - \varepsilon_6 E_d^t E_d > 0 \tag{36a}$$

$$\varepsilon_{11} I - \varepsilon_5 T_d T_d^t > 0, \quad \varepsilon_{11} I - \varepsilon_9 T_g T_g^t > 0 \tag{36b}$$

$$\begin{aligned}
&\tilde{P}(A_o + D_o) + (A_o + D_o)^t \tilde{P} + \tilde{P}(\varepsilon_1 MM^t + \varepsilon_2 E_d E_d^t)\tilde{P} \\
&+\varepsilon_1^{-1} N^t N + \varepsilon_2^{-1} T_d^t T_d + \tau^* \tilde{P}(\varepsilon_6^{-1} T_d T_d^t + D_o[I - \varepsilon_6 E_d^t E_d]^{-1} D_o^t)\tilde{P} \\
&+\varepsilon_0^2 \varepsilon_8^{-1} R_g^t R_g + \varepsilon_8 \tilde{P}(\varepsilon_7^{-1} T_g T_g^t + G_o[I - \varepsilon_7 E_g^t E_g]^{-1} G_o^t)\tilde{P} \\
&+\tau^*(\varepsilon_4^{-1} M^t M + A_o^t[\varepsilon_3 I - \varepsilon_4 NN^t]^{-1} A_o) \\
&+\tau^* R_g^t (\varepsilon_9^{-1} E_g^t E_g + G_o^t[\varepsilon_{11} I - \varepsilon_9 T_g T_g^t]^{-1} G_o) R_g \\
&+\tau^* (\varepsilon_5^{-1} E_d^t E_d + D_o^t[\varepsilon_{10} I - \varepsilon_5 T_d T_d^t]^{-1} D_o) \qquad < 0
\end{aligned} \tag{37}$$

By letting $\eta = 1/\tau^*$, and $Z = \tilde{P}^{-1}$ we obtain the following equivalent conditions:

$$\begin{aligned}
&(A_o + D_o)^t + (A_o + D_o)Z + \Omega J \Omega^t + Z\Sigma^t \Pi^{-1} \Sigma Z \\
&\eta^{-1} Z \Theta^t \Phi^{-1} \Theta Z + \eta^{-1} \Psi \Gamma^{-1} \Psi^t \qquad < 0
\end{aligned} \tag{38}$$

where Ω, Θ, Ψ, Γ, Π, J, Φ and Σ are given by (26b) through (26e). Since inequalities (36) and (37) are equivalent to (26), the proof is completed.

REMARK 3 **Theorem 2** establishes a systematic method for delay-dependent robust control synthesis base on LMI formalism, which should prove very useful in engineering applications. By similarity to recent existing results [12,13,16], the condition (26) is only sufficient. However, it is considered to yield less conservative results than those corresponding to the case of delay-independent of **Theorem 1** since it incorporates the size of the delay into effect .

REMARK 4 The problem of determining the largest upper bound of $\tau^* = \hat{\tau}^*$ which ensures the robust stability of system (4) for any delay $\tau \in [0, \hat{\tau}^*]$ amounts to the solution of the generalized eigenvalue problem in Z, $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_{10})$ and $\eta^* = 1/\tau^*$

$$\begin{aligned}
&\textit{Minimize}\ \eta^* \\
&\textit{subject to}\ Z > 0,\ \varepsilon > 0,\ \eta^* > 0
\end{aligned} \tag{39}$$

and using (26). The minimum value will be $\hat{\eta}^* = 1/\hat{\tau}^*$.

COROLLARY 5 In the case of linear uncerain delay systems ($G_o = 0$, $\triangle G(t) = 0$), the conditions of delay-dependent robust stability reduce to:

$$\begin{bmatrix} Z(A_o + D_o)^t + (A_o + D_o)Z + \Omega_o J \Omega_o^t & . & Z\Sigma^t & Z\Theta_o^t & \Psi \\ \cdots\cdots\cdots\cdots & . & \cdots\cdots & \cdots\cdots & \cdot\cdot \\ \Sigma Z & . & -\Pi & 0 & 0 \\ \Theta_o Z & . & 0 & -\eta\Phi_o & 0 \\ \Psi^t & . & 0 & 0 & -\eta\Gamma \end{bmatrix} < 0 \quad (40)$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{10}) > 0$, $\Omega_0 = [M, T_d]$, $J_o = \mathrm{diag}[\varepsilon_1 I, \varepsilon_2 I]$, $\Theta_o = [A_o^t, M^t, D_o^t, E_d^t]$, and $\Phi_o = \mathrm{diag}[(\varepsilon_3 I - \varepsilon_4 NN^t), \varepsilon_4 I, (\varepsilon_{10} I - \varepsilon_5 T_d T_d^t), \varepsilon_5 I]$.

COROLLARY 6 In the absence of uncertainties ($\triangle A(t) = 0$, $\triangle D(t) = 0$, $\triangle G(t) = 0$), the stability robustness of the nonlinear delay systems corresponds to the solvability of

$$\begin{bmatrix} Z(A_o+D_o)^t+(A_o+D_o)Z+G_o(\varepsilon_8^{-1}I)G_o^t & . & ZR_g^t & Z[A_o^t\, D_o^t\, G_o^t] & D_o \\ \cdots\cdots\cdots & . & \cdots\cdots & \cdots\cdots & \cdot\cdot \\ R_g Z & . & -\varepsilon_{10} I & 0 & 0 \\ \begin{bmatrix} A_o \\ D_o \\ G_o \end{bmatrix} Z & . & 0 & -\eta\Phi_o & 0 \\ D_o^t & . & 0 & 0 & -\eta I \end{bmatrix} < 0 \quad (41)$$

COROLLARY 7 By dropping out all the uncertainties and state-delay, it is easy to deduce that inequality (11) becomes

$$\begin{bmatrix} Z(A_o + D_o)^t + (A_o + D_o)Z + G_o(\varepsilon_8^{-1}I)G_o^t & . & ZR_g^t & Z[A_o^t\, G_o^t] \\ \cdots\cdots\cdots & . & \cdots & \cdots\cdots\cdots \\ R_g Z & . & -\varepsilon_{10} I & 0 \\ \begin{bmatrix} A_o \\ G_o \end{bmatrix} Z & . & 0 & -\eta\Phi_o \end{bmatrix} < 0 \quad (42)$$

REMARK 5 Once again, the developed conditions of **Theorem 2** are only sufficient since we are dealing with nonlinear dynamical systems. However in the special cases described in **corollaries 5-7** and by reversing the order of proof of **Theorem 2,** it can be shown, following the results of [17], that conditions (40), (41) or (42) are both necessary and sufficient. The basic difference lies in the uncertainty structure where Zhou et al. [17] have considered real time-varying parameter uncertainties on compact intervals and we employ a structure with real-valued mismatched matrix functions.

5. ROBUST FEEDBACK STABILIZATION

Now, the problem is to determine the upper bound τ_c for the time-delay τ such that the nonlinear system (1) is robustly stabilizable by a linear memoryless feedback control $u(t) = K\ x(t)\ \forall\ \tau \in [0, \tau_c]$. For this purpose, the closed-loop system has the form

$$\begin{aligned}\dot{x}(t) &= [(A_o + B_oK) + (\Delta A(t) + \Delta B(t)K)]x(t) \\ &+[D_o + \Delta D(t)]x(t-\tau) + [G_o + \Delta G(t)]g[x(t)]\end{aligned} \tag{43a}$$

$$x(t) = \phi_o(t) \quad \forall t \in [-\tau, 0], \quad 0 \le \tau \le \tau_o < \infty \tag{43b}$$

for which we invoke the following assumption:

A4. The pair ($A_o + D_o$, B_o) is stabilizable

REMARK 6 In line of **A3,** we note that **A4** is equivalent to the stabilizability of system (1) without time-delay and uncertainty and it is therefore necessary for the existence of a stabilizing linear state feedback controller for system (1).

Before proceeding further, we introduce

$$\tilde{A}(t) = [(A_o + B_oK) + MF(t)(N + LK)] \tag{44}$$

and in line of section 4, two cases will be considered

5.1 Delay-Independent Case

Using (2), (6) and (44), we rewrite the closed-loop system (44) as:

$$\dot{x}(t) = \tilde{A}(t)x(t) + \hat{D}(t)x(t-\tau) + \hat{G}(t)g[x(t)] \tag{45}$$

THEOREM 3 Suppose that **A1-A4** hold. Given a scalar $\tau_c > 0$, then system (1) is **robustly stablizable** by the linear memoryless controller $u(t) = \varepsilon_2^{-1} B_o^{\ t}\ Z^{-1}\ x(t)$ for any constant time-delay $\tau \in [0, \tau_c]$ if condition (10b) is met and there exist $\varepsilon > 0$ such that $\Omega_1(\varepsilon) > 0$, $\Omega_2(\varepsilon) > 0$, $\Omega_3(\varepsilon) > 0$, $\Omega_4(\varepsilon) > 0$ and a matrix $0 < Z = Z^1 \in \Re^{n\times n}$ solving the LMI

$$\begin{bmatrix} ZA_o^t + A_oZ + (\varepsilon_3^{-1} + \varepsilon_5^{-1})I & . & Z\Sigma^t & H \\ \cdots\cdots\cdots\cdots & . & \cdots\cdots & \cdot\cdot \\ \Sigma Z & . & -\Delta & 0 \\ H^t & . & 0 & -\Lambda \end{bmatrix} < 0 \tag{46a}$$

where

$$\begin{aligned}\Delta = \mathrm{diag}[\varepsilon_1^{-1}I,\ \varepsilon_2^{-1}I,\ \varepsilon_0^{-2}I],\ \Sigma^t = [I,\ N^t,\ R_g^t],\\ H = [M,\ B_o],\ \Lambda = [\varepsilon_2 I,\ (1-\varepsilon_2^{-1})^{-2}I]\end{aligned} \tag{46b}$$

Proof By similarity to **Theorem 1**, we use V(x, t) of (12). By evaluating its derivative L(x, t) along the solutions of system (45) followed by expansion of some terms using (2), (6), (7) and (44), we obtain

$$\begin{aligned}L(x,t) = {} & x^t(t)\,(P(A_o+B_oK)+(A_o+B_oK)^tP)\,x(t)\\ & + x^t(t)\left[\varepsilon_2^{-1}PMM^tP+\varepsilon_2(N+LK)^t(N+LK)^t\right]x(t)\\ & + x^t(t)P\hat{D}(t)x(t-\tau)+x^t(t-\tau)\hat{D}^t(t)Px(t)\\ & + g^t[x(t)]\hat{G}^t(t)Px(t)+x^t(t)P\hat{G}(t)g[x(t)]\\ & + \varepsilon_1x^t(t)x(t)-\varepsilon_1x^t(t-\tau)x(t-\tau)\\ & + \varepsilon_0^2x^t(t)R_g^tR_gx(t)-g^t[x(t)]g[x(t)]\end{aligned} \tag{47}$$

Applying the feedback gain $K = -\varepsilon_2^{-1}\ B_o^t\,P$ and by standard grouping of terms, we get

$$\begin{aligned}L(x,t) \le {} & x^t(t)\left\{PA_o+A_o^tP+[\varepsilon_0^2R_g^tR_g+\varepsilon_2N^tN+\varepsilon_1I]\right.\\ & \left.+P[\varepsilon_2^{-1}MM^t+(\varepsilon_2^{-1}+\varepsilon_5^{-1}I+B_o(I-\varepsilon_2^{-1})^2B_o^t]P\right\}x(t)\\ & - g^t[x(t)]\left(I-\varepsilon_5\varepsilon_6^{-1}T_g^tT_g-\varepsilon_5G_o^t(I-\varepsilon_6E_gE_g^t)^{-1}G_o\right)g[x(t)]\\ & - x^t(t-\tau)\left(\varepsilon_1I-\varepsilon_3\varepsilon_4^{-1}T_d^tT_d-\varepsilon_3D_o^t(I-\varepsilon_4E_dE_d^t)^{-1}D_o\right)x(t-\tau)\end{aligned} \tag{48}$$

When conditions (10b) are satisfied and by letting $Z = P^{-1}$, then inequality (48) reduces to

$$\begin{aligned}L(x,t) \quad & \le x^t(t)\left\{ZA_o^t+A_oZ+Z[\varepsilon_0^2R_g^tR_g+\varepsilon_2N^tN+\varepsilon_1I]Z\right.\\ & \left.+\ [\varepsilon_2^{-1}MM^t+(\varepsilon_3^{-1}+\varepsilon_5^{-1})I+B_o(I-\varepsilon_2^{-1})^2B_o^t]\right\}x(t)\end{aligned} \tag{49}$$

Now, it is easy to infer that the robust stability requirement corresponds to (46) as desired.

5.2 Delay-Dependent Case

Let x(t), t ≥ 0 be the solution of system (46) . In terms of

$$x(t-\tau) = x(t) - \int_{-\tau}^{0}\dot{x}(t+\vartheta)d\vartheta = x(t)-\zeta(x,t) \tag{50a}$$

where

$$\begin{aligned}\zeta(x,t) \quad & = \textstyle\int_{-\tau}^{0}[\tilde{A}(t+\vartheta)x(t+\vartheta)]+[\hat{D}(t+\vartheta)x(t-\tau+\vartheta)]d\vartheta\\ & + \textstyle\int_{-\tau}^{0}\hat{G}(t+\vartheta)g[x(t+\vartheta)]d\vartheta\end{aligned} \tag{50b}$$

the closed-loop system (45) can be cast into the form

$$\begin{aligned}\dot{x}(t) &= [\tilde{A}(t) + \hat{D}(t)]x(t) - \hat{D}(t)\zeta(t) + \hat{G}(t)g[x(t)] \\ x(\vartheta) &= \phi_o(\vartheta) \quad \forall\vartheta \in [-2\tau, 0]\end{aligned} \tag{51}$$

THEOREM 4 Suppose that **A1-A4** hold. Given a scalar $\tau_c > 0$, system (46) is **robustly stabilizable** for any constant time-delay $\tau \in [0, \tau_c]$ by the controller $u(t) = Y Z^{-1} x(t)$ if there exist a matrix $Y \in \Re^{m\times n}$, $\varepsilon = (\varepsilon_1,...,\varepsilon_{11}) > 0$ and a matrix $0 < Z = Z^t \in \Re^{n\times n}$ solving the LMI

$$\begin{bmatrix} \begin{matrix} Z(A_o+D_o)^t+(A_o+D_o)Z \\ +\Omega J\Omega^t + B_oY + Y^tB_o^t \end{matrix} & \cdot & Z\Sigma_c^t & (ZN^t+Y^tL^t) & Z\Theta_c^t & \Psi & \Xi(Z,Y) \\ \cdots\cdots & \cdot & \cdots & \cdots & \cdots & \cdot\cdot & \cdots \\ \Sigma_c Z & \cdot & -\Pi & 0 & 0 & 0 & 0 \\ (NZ+LY) & \cdot & 0 & -\varepsilon_1 I & 0 & 0 & 0 \\ \Theta_c Z & \cdot & 0 & 0 & -\eta_c\Phi_c & 0 & 0 \\ \Psi^t & \cdot & 0 & 0 & 0 & -\eta_c\Gamma & 0 \\ \Xi^t(Z,Y) & \cdot & 0 & 0 & 0 & 0 & -\eta_c\Lambda \end{bmatrix} < 0 \tag{52}$$

where $\Sigma_c^t = [T_d^t, R_g^t]$, $\Pi_c = \text{diag}[\varepsilon_2 I, \varepsilon_{10} I]$, $\Theta_c = [D_o^t, T_d^t, G_o^t, T_g^t]$, $O^t = (ZN^t + Y^tL^t)$, $\Xi(Z, Y) = [(ZN^t + Y^tL^t), (ZA_o^t + Y^tB_o^t)]$, $\Lambda = \text{diag}[\varepsilon 4I, (\varepsilon_3 I - \varepsilon_4 MM^t)]$, $\eta_c = 1/\tau_c$, and $\Phi c = \text{diag}[(I(1 - \varepsilon_3) - \varepsilon_5 T_d T_d^t), \varepsilon_5 I, (\varepsilon_3 I - \varepsilon_9 E_g E_g^t), \varepsilon_9 I]$ with the remaining terms as in (26).

Proof Following **Theorem 2,** we use $V_d(x, t)$ of (27) whose derivative $L_d(x, t)$ is given by (28) but will $\tilde{A}(t)$ and $\zeta(x, t)$ replacing $\hat{A}(t)$ and $\xi(x, t)$, respectively. Proceeding in line with (29) through (32), we get

$$\begin{aligned} L_d(x,t) \le\; & x^t(t)\left(P[\tilde{A}(t) + \hat{D}(t)] + [\tilde{A}(t) + \hat{D}(t)]^tP\right)x(t) \\ & + x^t(t)\left(\tau\sigma_1^{-1}P\hat{D}(t)\hat{D}^t(t)P\right)x(t) + x^t(t)\left(\sigma_3 P\hat{G}(t)\hat{G}^t(t)P\right)x(t) \\ & + \sigma_3^{-1}g^t[x(t)]g[x(t)] + x^t(t)\left(\tau\sigma_1(1+\sigma_2^{-1})\hat{G}^t(t)\hat{G}(t)\right)x(t) \\ & + x^t(t)\left(\tau\sigma_1(1+\sigma_2^{-1})\tilde{A}^t(t)\tilde{A}(t) + \tau\sigma_1(1+\sigma_2)\hat{D}^t(t+\tau)\hat{D}(t+\tau)\right)x(t) \end{aligned} \tag{53}$$

By repeated applications of **Lemma 1** to (53) and using (2), (3) and (44) we obtain

$$\begin{aligned} L_d(x,t,) \le\; & x^t(t)\{P(A_o + D_o + B_oK + (A_o + D_o + B_oK)^tP \\ & + P[\sigma_4 MM^t + \sigma_5 E_dE_d^t]P + [\varepsilon_0^2\sigma_3^{-1}R_g^tR_g + \sigma_5^{-1}T_d^tT_d] \\ & + \sigma_4^{-1}(N+LK)^t(N+LK) + \tau_c\sigma_1(1+\sigma_2^{-1})\sigma_8^{-1}(N+LK)^t(N+LK)] \\ & + \tau_c\sigma_1^{-1}P[D_o(I - \sigma_6T_d^tT_d)^{-1}D_o^t + \sigma_6^{-1}E_dE_d^t]P \\ & + \sigma_3 P[G_o(I - \sigma_7T_g^tT_g)^{-1}G_o^t + \sigma_7^{-1}E_gE_g^t]P \\ & + \tau_c\sigma_1(1+\sigma_2^{-1})[(A_o + B_oK)^t(I - \sigma_8MM^t)^{-1}(A_o + B_oK)] \\ & + \tau_c\sigma_1(1+\sigma_2^{-1})[G_o^t(I - \sigma_9E_gE_g^t)^{-1}G_o + \sigma_9^{-1}T_g^tT_g] \\ & + \tau_c\sigma_1(1+\sigma_2)[D_o^t(I - \sigma_{10}E_dE_d^t)_{-1}D_o + \sigma_{10}^{-1}T_d^tT_d]\}\,x(t) \end{aligned} \tag{54}$$

Using (35) and defining $Z = \tilde{P}^{-1}$, $Y = K\tilde{P}^{-1}$, then inequality (54) can be rewritten as

$$Z(A_o+D_o)^t+(A_o+D_o)Z+\Omega J\Omega^t+B_oY+Y^tB_o^t+O^t(\varepsilon_1^{-1}I)O+Z\Sigma_c^t\Pi_c^{-1}\Sigma_cZ \\ +\eta_c^{-1}Z\Theta_c^t\Phi_c^{-1}\Theta_cZ+\eta_c^{-1}\Psi\Gamma^{-1}\Psi^t+\eta_c^{-1}\Xi(Z,Y)\Lambda^{-1}\Xi^t(Z,Y)<0 \qquad (55)$$

It is readily evident that (55) is equivalent to (52) as desired.

REMARK 7 It can be immediately seen that the largest upper bound of $\tau_c = \hat{\tau}_c$ which ensures the robust stabilizability of system (1) can be determined from the solution of the following generalized eigenvalue problem in Y, W $\varepsilon = (\varepsilon_1,...,\varepsilon_{11})$ and $\eta_c = 1/\hat{\tau}_c$

$$\begin{array}{l} Minimize\ \eta_c \\ subject\ to\ Y > 0,\ W,\ \varepsilon > 0,\ \eta_c > 0 \end{array} \qquad (56)$$

and using (52). The bound $\hat{\tau}_c = 1/\hat{\eta}_c$, where $\hat{\eta}_c$ is the solution of (56).

COROLLARY 8 Consider the nonlinear system (1) without uncertainties and suppose that **A4** holds. Given scalar τ^o, system (4) is **robustly stabilizable** for any constant time-delay $\tau \in [0, \tau^o]$ if there exist matrices $0 < Z = Z^t \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{m\times n}$ and scalars $\beta_o > 0$, $\gamma_o > 0$, $0 < \mu_o < 1$ solving the LMI

$$\begin{bmatrix} Z(A_o+D_o)^t+(A_o+D_o)Z \\ +G_o(\beta_oI)G_o^t+B_oY+Y^tB_o^t & \cdot & ZR_g^t & Z[D_o^t,G_o^t] & D_o & ZA_o^t+Y^tB_o^t \\ \cdots & \cdot & \cdots & \cdots & \cdots & \cdots \\ R_gZ & \cdot & -(\gamma_oI) & 0 & 0 & 0 \\ \begin{bmatrix} D_o \\ G_o \end{bmatrix} Z & \cdot & 0 & -\eta^o\begin{bmatrix}(1-\mu_o)I & 0 \\ 0 & (\mu_oI)\end{bmatrix} & 0 & 0 \\ D_o^t & \cdot & 0 & 0 & -\eta^oI & 0 \\ A_oZ+B_oY & \cdot & 0 & 0 & 0 & -\eta^o(\mu_oI) \end{bmatrix} < 0 \qquad (57)$$

where $\eta^o = 1/\tau^o$. Th e stabilizing control is given by $u(t) = YZ^{-1}x(t)$.

REMARK 8 The developed conditions of **Theorems 3 , 4** are only sufficient since we are dealing with nonlinear dynamical systems. However in the special case of **corollary 9** with $R_g = 0$ and by reversing the order of proof of **Theorem 4,** it can be shown, following the results of [17], that condition (57) is both necessary and sufficient.

REMARK 9 It is important to note that alll the LMI's developed in this paper can be effectively solved using the software **LMI LAB** [3] or **LAAS** routines [9].

6. EXAMPLE

As an application of the developed theory and associated computational procedure, we consider an industrial jacketed continuous stirred tank reactor (JCSTR) of 5000 gallons volume with a delayed recycle stream and in which the reaction is a unimolecular and irreversible (exothermic). The reactor accepts a feed of reactant which contains a substance **A** in intital concentration C_{Ao}. The feed enters at a rate of F gallons per hour and at a temperature T_o. Perfect mixing is assumed and heat losses are neglected. The tank is cooled by a flow of water around the jacket of the tank and the water flow in the jacket F_J, is controled by actuating a valve . Suppose that fresh feed of pure (C_A) is to be mixed with a recycled stream of unreacted (C_A) with recycle flow rate (1-c) where $0 \le c \le I$ is the coefficient of recirculation. The amount of delay in the recycle stream is d. The change in the concentration arises from three terms: the amount of **A** that is added with feed under recycling, the amount of **A** that leaves with the product flow, and the amount of **A** that is used up in the reaction. The change in the temperature of the fluid arises from four terms: a term for the heat that enters with the feed flow under recycling, a term for the heat that leaves with the product flow. term for the heat created by the reaction and a term for the heat that is transferred to the cooling jacket. There are three term associated with the changes of the temperature of the fluid in the jacket: one term representing the heat entering the jacket with the cooling fluid flow, one term representing the heat leaving the jacket with the outflow of cooling liquid, and one term representing the heat transferred from the fluid: the reaction tank to the fluid in the jacket. Under the conditions of constant holdup, constant densities and perfect mixing, the energy and material balances can be expressed mathematically as:

$$\begin{aligned}
\dot{C}_A(t) &= (F/V)[cC_{A0} - cC_A(t) + (1-c)C_A(t-d)] - k_1 C_A e^{-k_2/T} \\
&= f_c(C_A, T) \\
\dot{T}(t) &= (F/V)[cT_0 - cT(t) + (1-c)T(t-d)] - k_1 k_3 C_A e^{-k_2/T} \\
&\quad - k_4[T(t) - T_J(t)] = f_T(C_A, T, T_J) \\
\dot{T}_J(t) &= (F_J/V_J)[T_{J0} - T_J(t)] - k_5[T(t) - T_J(t)] \\
&= f_J(T, T_J)
\end{aligned} \tag{58}$$

To cast this dynamic model into the form (1), we apply the following procedure:

1. Insert correct values for the constants
2. Choose proper initial values of the dynamic variables (C_{Ao}, T_o, T_{Jo}, F_{Jo}).

3. Expand the right-hand side of the dynamical equations using a second-order approximation of the form:

$$f_c(C_A,T)\approx f_c(\bar{C}_A,\bar{T})+\left.\frac{\partial f_c(C_A,T)}{\partial C_A}\right|_{\bar{C}_A,\bar{T}}\delta C_A+\left.\frac{\partial f_C(C_A,T)}{\partial T}\right|_{\bar{C}_A,\bar{T}}\delta T \quad (59)$$

$$+\left.\frac{\partial^2 f_c(C_A,T)}{\partial C_A^2}\right|_{\bar{C}_A,\bar{T}}\delta C_A^2+\left.\frac{\partial^2 f_c(C_A,T)}{\partial C_A\partial T}\right|_{\bar{C}_A,\bar{T}}\delta C_A\delta T+\left.\frac{\partial^2 f_c(C_A,T)}{\partial T^2}\right|_{\bar{C}_A,\bar{T}}\delta T^2$$

$$\begin{aligned}f_T(C_A,T,T_J)\approx\ & f_T(\bar{C}_A,\bar{T},\bar{T}_J)+\left.\frac{\partial f_T(\bar{C}_a,\bar{T},\bar{T}_J)}{\partial C_A}\right|_{\bar{C}_A,\bar{T},\bar{T}_J}\delta C_A\\ &+\left.\frac{\partial f_t(\bar{C}_A,\bar{T},\bar{T}_J)}{\partial T}\right|_{\bar{C}_A,\bar{T},\bar{T}_J}\delta T+\left.\frac{\partial f_T(\bar{C}_A,\bar{T},\bar{T}_J)}{\partial T_J}\right|_{\bar{C}_A,\bar{T},\bar{T}_J}\delta T_J\\ &+\left.\frac{\partial^2 f_T(\bar{C}_A,\bar{T},\bar{T}_J)}{\partial C_A^2}\right|_{\bar{C}_A,\bar{T},\bar{T}_J}\delta C_A^2+\left.\frac{\partial f_T(\bar{C}_A,\bar{T},\bar{T}_J)}{\partial T^2}\right|_{\bar{C}_A,\bar{T},\bar{T}_J}\delta T^2 \quad (60)\\ &+\left.\frac{\partial^2 f_T(\bar{C}_A,\bar{T},\bar{T}_J)}{\partial T_J^2}\right|_{\bar{C}_A,\bar{T},\bar{T}_J}\delta T_j^2+\left.\frac{\partial^2 f_T(\bar{C}_A,\bar{T},\bar{T}_J)}{\partial C_A\partial T}\right|_{\bar{C}_A,\bar{T},\bar{T}_J}\delta C_A\delta T\\ &+\left.\frac{\partial^2 f_T(\bar{C}_A,\bar{T},\bar{T}_J)}{\partial T_J\partial C_A}\right|_{\bar{C}_A,\bar{T},\bar{T}_J}\delta C_A\delta T_J+\left.\frac{\partial^2 f_T(\bar{C}_A,\bar{T},\bar{T}_J)}{\partial T_J\partial T}\right|_{\bar{C}_A,\bar{T},\bar{T}_J}\delta T_J\delta T\end{aligned}$$

$$\begin{aligned}f_J(T,T_J)\approx\ & f_J(\bar{T},\bar{T}_J)+\left.\frac{\partial f_J(\bar{T},\bar{T}_J)}{\partial T}\right|_{\bar{T},\bar{T}_J}\delta T+\left.\frac{\partial f_J(\bar{T},\bar{T}_J)}{\partial T}\right|_{\bar{T},\bar{T}_J}\delta T_J\\ &+\left.\frac{\partial^2 f_J(\bar{T},\bar{T}_J)}{\partial T^2}\right|_{\bar{T},\bar{T}_J}\delta T^2+\left.\frac{\partial^2 f_J(\bar{T},\bar{T}_J)}{\partial T_J\partial T}\right|_{\bar{T},\bar{T}_J}\delta T_J\delta T+\left.\frac{\partial^2 f_J(\bar{T},\bar{T}_J)}{\partial T_J^2}\right|_{\bar{T},\bar{T}_J}\delta T_J^2\end{aligned} \quad (61)$$

where $(\overline{C}_A, \overline{T}_J, \overline{T})$ represents the equilibrium state of the reactor.

4. Simplify the left-hand sides of the nonlinear differential equations (58) by

$$\dot{C}_A \approx \dot{\bar{C}}_A + \delta\dot{C}_A = \delta\dot{C}_A,\ \dot{T} \approx \dot{\bar{T}} + \delta\dot{T} = \delta\dot{T},\ T_J \approx \dot{\bar{T}}_J + \delta\dot{T}_J = \delta\dot{T}_J$$

Since $\dot{\bar{C}} = 0$, $\dot{\bar{T}} = 0$, $\dot{\bar{T}}_j = 0$

5. Compute the equilibuirm state $(\overline{C}_A, \overline{T}_J, \overline{T})$, the incremental changes $(\delta C_A, \delta T, \delta T_J)$ and their derivatives are set to zero to insure that the reactor system do not change at the operating point.

Using typical data values with standard units

$$\kappa_2 = 15098,\ \kappa_1 = 7.08 \times 10^{10},\ \kappa_3 = 800,\ \kappa_4 = 2.4,\ F/V = 0.83,$$
$$\kappa_5 = 16.5,\ F_J/V_J = 13,\ C_{A0} = 0.5,\ T_0 = T_{J0} = 70,$$

the operating point is computed to be $(\overline{C}_A = 0.245, \overline{T} = 601°, \overline{T}_J = 93.30°, \overline{F}_J = 1240)$.

The state variables are taken as the values of the concentration of substance A in the tank (C_A), the temperature in the tank (T) and the temperature in the cooling jacket (T_J) and the control variable is the volumetric flow rate of water in the jacket (F_J), that is,

$$x_1 = \delta C_A,\ x_2 = \delta T,\ x_3 = \delta T_J,\ u = \delta F_J$$

In view of the above calculations and definitions, model (1) is obtained with the following matrices

$$A_0 = \begin{bmatrix} -1.7 & -0.0088 & 0 \\ 696 & -10.358 & 2.4 \\ 0 & 16.5 & -29.5 \end{bmatrix},\quad B_0 = \begin{bmatrix} 0 \\ 0 \\ -0.16 \end{bmatrix},$$

$$D_0 = \begin{bmatrix} -0.75 & 0 & 0 \\ 0 & -0.39 & 0 \\ 0 & 0 & -0.04 \end{bmatrix},\ G_0 = \begin{bmatrix} 0.15 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.05 \end{bmatrix}$$

The uncertainties within the JCSTR system arise from changes in model parameters and effect of recycling and are represented by

$$M = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.2 \\ 0.1 & -0.3 \end{bmatrix}, N = \begin{bmatrix} 0.2 & 0 & -0.2 \\ 0 & 0.5 & 0.1 \end{bmatrix}, L = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, F = \begin{bmatrix} 0.5\sin 3t & 0 \\ 0 & 0.8\cos 3t \end{bmatrix}$$

$$E_g = \begin{bmatrix} -0.06 \\ 0.22 \\ 0.02 \end{bmatrix},\ F_g = 0.4\sin 5t,\ T_g = [\ 0.12 \quad 0.03 \quad -0.2\ ]$$

$$E_d = \begin{bmatrix} -0.14 \\ 0.08 \\ -0.6 \end{bmatrix},\ F_d = 0.5\cos 6t,\ T_d = [\ -0.03 \quad -0.5 \quad 0.04]$$

Evaluation of (3) in the light of (59) to (61) shows that a good estimate of ε_0, R_g would be $\varepsilon_0 = 0.3$ and $R_g = \text{diag}[0.6\ 0.4\ 0.2]$

With focus on the issue of stability and using the soft ware **LMI LAB** for **Theorem 2**, it has been foun that the JCSTR model (58) is robustly stable for any constant delay $d \leq 0.4537$.

Next by considereing the feedback control synthesis, application of **Theorem 4** with the aid of the **LMI LAB** shows that the JCSTR system is robustly stabilizable for any constant delay $d \leq 0.3745$. Moreove the corresponding stabilizing control is given by

$$\delta F_J(t) = -12.809\,\delta\, C_A + 1.238\,\delta\, T + 1.733\,\delta\, T_J$$

which provides the expected extra amount of water flow into the jacket in response to changes in the concentration of substance A in the tank (C_A), the temperature in the tank (T) and the temperature in the cooling jacket (T_J). We note that if the cooling water flows slowly the system heats up. On the other hand if the cooling water flows too quickly, the reaction slows down and poor product yield results.

CONCLUSIONS

Using a linear matrix inequality formulation, this paper has established new results and provided insigh into the problems of robust stability analysis and robust feedback synthesis for a class of nonlinear system with norm-bounded and state-delay. Both the cases of delay-dependent and delay-independent have been considered. It has been further established that linear memoryless controllers are capable of guaranteeing tl closed-loop system stabilizability. Simulation on a linearized model of an industrial jacketed continuous stirred tank reactor with a delayed recycle stream has illustrated the potential of our methodology.

APPENDIX (Proof of Lemma 1)

1. Consider the matrix function

$$[\rho^{-1}\Sigma_1^t - \rho H(t)\Sigma_2]^t[\rho^{-1}\Sigma_1^t - \rho H(t)\Sigma_2] \geq 0 \tag{A1}$$

On expanding (A1), we get $\rho^{-2}\Sigma_1\Sigma_1^t - \Sigma_2^t H^t(t)\Sigma_1^t - \Sigma_1 H(t)\Sigma_2 + \rho^2\Sigma_2^t H^t(t)H(t)\Sigma_2 \geqslant 0$, and by rearranging the terms using $H^t(t)\, H(t) \leqslant I$, the desired form (7a) results.

2. Again, instead of (A1), consider the matrix function

$$\Sigma(t) = [\rho^{-2}I - \Sigma_2\Sigma_2^t]^{-1/2}\Sigma_2\Sigma_3^t - [\rho^{-2}I - \Sigma_2\Sigma_2^t]^{1/2}H^t(t)\Sigma_1^t \tag{A2}$$

Expanding $\Sigma^t(t)\Sigma(t) \geqslant 0$ using the fact that $\rho^2\Sigma_2^t(t)\Sigma_2(t) < I$ implies $\rho^2\Sigma_2(t)\Sigma_2^t(t) < I$ we get

$$\begin{aligned}
&\Sigma_3\Sigma_2^t H^t(t)\Sigma_1^t + \Sigma_1 H(t)\Sigma_2\Sigma_3^t + \Sigma_1 H(t)\Sigma_2\Sigma_2^t H^t(t)\Sigma_1^t \\
\leq\ &\Sigma_3\Sigma_2^t[\rho^{-2}I - \Sigma_2\Sigma_2^t]^{-1}\Sigma_2\Sigma_3^t + \rho^{-2}\Sigma_1 H(t)H^t(t)\Sigma_1^t \\
&(\Sigma_3 + \Sigma_1 H(t)\Sigma_2)(\Sigma_3^t + \Sigma_2^t H^t(t)\Sigma_1^t) - \Sigma_3\Sigma_3^t \\
\leq\ &\Sigma_3\Sigma_2^t[\rho^{-2}I - \Sigma_2\Sigma_2^t]^{-1}\Sigma_2\Sigma_3^t + \rho^{-2}\Sigma_1 H(t)H^t(t)\Sigma_1^t \\
&(\Sigma_3 + \Sigma_1 H(t)\Sigma_2)(\Sigma_3^t + \Sigma_2^t H^t(t)\Sigma_2^t) \\
\leq\ &\Sigma_3[I + \Sigma_2^t[\rho^{-2}I - \Sigma_2\Sigma_2^t]^{-1}\Sigma_2]\Sigma_3^t + \rho^{-2}\Sigma_1 H(t)H^t(t)\Sigma_1^t
\end{aligned} \tag{A3}$$

Since $[I - \rho^2\Sigma_2^t \Sigma_2]^{-1} = [I + \rho^2\Sigma_2^t [I - \rho^2\Sigma_2^t \Sigma_2]^{-1}\Sigma_2]$ and $H^t(t)\ H(t) \leqslant I$ implies $H(t)H^t(t) \leqslant I$, then (7 ?? follows directly by transposing (A3).

References

[1] S. P. Boyd and L. El-Ghaoui "Methods of Centers for Minimizing Generalized Eigenvalues", **Linear Algebra and Its Applications, vol. 15**, 1993, pp. 63–111.

[2] S. P. Boyd, L. El-Ghaoui, E. Feron and V. Balakrishnan "**Linear Matrix Inequalities in System and Control Theory**", SIAM, **vol. 15,** Philadelphia, 1994.

[3] P. Gahinet , A. Nemirovski and A. J. Laub "**LMI Control Toolbox User's Guide**", The MathWorks, Inc. Mass. ,1995.

[4] H. Gorecki, S. Fuska, P. Grabowski and A. Korytowski, "**Analysis and Synthesis of Time Delay Systems**", J. Wiley and Sons, New York , 1989.

[5] M. S. Mahmoud and N.F. Al-Muthairi, "Design of Robust Controllers for Time-Delay Systems", **IEEE Trans. Automatic Control, AC–39,** 1994, pp. 995–999.

[6] M. S. Mahmoud and N.F. Al-Muthairi, "Quadratic Stabilization of Continuous-Time Systems with State-Delay and Norm-Bounded Time-Varying Uncertainties", **IEEE Trans. Automatic Control, AC–39**, 1994, pp. 2135–2139.

[7] M. S. Mahmoud, M. F. Hassan and S. J. Saleh, "Decentralized Structures for Stream Water Quality Control Problems", **Optimal Control Applications and Methods, vol. 6,** 1985, pp. 167–186.

[8] M. S. Mahmoud, "Output Feedback Stabilization of Uncertain Systems with State Delay", in **Analysis and Synthesis Techniques in Complex Control and Dynamic Systems", vol. 63 of (Advances in Theory and Applications); C. T. Leondes (Ed.),** 1994, pp. 197–257.

[9] M. S. Mahmoud, "Synthesis of Delay-Dependent Dynamic Controllers for Uncertain Systems by LMI Approach", **Kuwait University Technical Report ECE–063–95-G01,** 1995.

[10] S. I. Niculesscu, C. E. De Souza, J. M. Dion and L. Dugard, "Robust Stabilization of Uncertain Linear Systems with State-Delay: Single delay case", **Proc. IFAC Symposium on Robust Control Design,** BRAZIL, 1994.

[11] J. C. Shen, B. S. Chen and F. C. Kung, "Memoryless Stabilization of Uncertain Dynamic Delay Systems: Riccati equation approach", **IEEE Trans. Automatic Control, AC–36,** 1991, pp. 638–640.

[12] T. H. Su, "Further Results on the Robust Stability of Linear Systems with a Single Time Delay", **Systems and Control Letters, vol. 23,**1994, pp. 375–379.

[13] T. J. Su and C. G. Huang, "Robust Stability of Delay Dependence for Linear Uncertain Systems", **IEEE Trans. Automatic Control, AC–37,** 1992, pp. 1656–1659.

[14] T. Taniguchi and H. Kaizu, "New Approach to the Synthesis of a Stabilizing Control Method for a Power System", **Int. Journal of Control, vol. 49,** 1989, pp. 691–712.

[15] C. L. Tseng , I. K. Fong and J. H. Su, "Robust Stability Analysis for Uncertain Delay Systems with Output Feedback Controller for Time-Delay", **Systems and Control Letters, vol. 23**, 1994, pp. 271–278.

[16] S. S. Wang, B. S. Chen and T. P. Lin, "Robust Stability of Uncertain Time-Delay Systems", **Int. Journal of Control, vol. 46,** 1987, pp. 963–976.

[17] K. P Zhou, P. P. Khargonekar, J. Stoustrup and H. H. Niemann, "Robust Performance of Systems with Structured Uncertainties in State Space", **Automatica, vol. 31,** 1995, pp. 249-255.