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Effect of Rotation on Wave Propagation in a Transversely Isotropic Medium

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Wave propagation in a transversely isotropic unbounded medium rotating about its axis of symmetry is studied. For propagation at high frequencies, effects of rotation are negligible but for a frequency which is much smaller than the frequency of rotation, there is a fast wave and two very slow waves. When the two frequencies are equal, the speed of a wave becomes unbounded.

Keywords: Frequency of rotation; Longitudinal; Transverse; Elastic wave; Isotropic; Transversely isotropic

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1. INTRODUCTION

Schoenberg and Censor [1] have considered the effect of rotation on plane wave propagation in an isotropic medium. Their results showed that a rotating isotropic medium behaves like a transversely isotropic medium to the extent that, in any given direction, there propagate three waves having different polarizations and phase speeds. They also showed that such waves are, in general, neither dilatational nor transverse. Such waves exist only if the axis of rotation and the direction of propagation are either parallel or

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perpendicular to each other. Chandrasekharaiah and Srikantiah [2] discussed thermoelastic waves in a rotating medium and Chandrasekharaiah and Srinath [3] have discussed thermoelastic plane waves without energy dissipation. However it has been recently shown by the present authors [4, 5] that the results of [2] and [3] are erroneous since the dilatational and transverse waves cannot propagate in an arbitrary direction.

Chadwick [6] has discussed, in great detail, wave propagation in a transversely isotropic medium. He gives expressions for the three wave speeds and the corresponding polarization vectors for propagation in an arbitrary direction. One of the three waves is always transverse and the other two, in general, are neither longitudinal nor transverse save for some special directions.

In this paper we shall discuss wave propagation in a rotating transversely isotropic medium. We find that for a frequency much smaller than the frequency of rotation, there is a fast wave and two very slow waves. Also for frequencies approaching the rotation frequency a very fast wave will propagate in an arbitrary direction. We present numerical results for cadmium for four directions of propagation.

2. BASIC EQUATIONS

The constitutive equation of an anisotropic elastic solid is expressed by the generalized Hooke's law, which can be written as

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} \quad i, j, k, l = 1, 2, 3,$$

where σ_{ij} are the Cartesian components of the stress and ε_{kl} is the strain tensor which is related with the displacement vector, u_i, as

$$\varepsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}),$$

C_{ijkl} are the components of a fourth-order tensor and are called the stiffnesses of the medium. The Einstein convention for repeated indices is used. For a homogeneous elastic body

$$\sigma_{ij,j} = C_{ijkl}\, u_{k,jl}, \tag{1}$$

where the comma denotes differentiation with respect to the appropriate component of x. If a body is rotating about an axis with a constant angular velocity Ω then the equation of motion in the absence of body forces can be written as

$$\sigma_{ij,j} = \rho\{\ddot{u}_i + \Omega_j u_j \Omega_i - \Omega^2 u_i + 2\varepsilon_{ijk}\Omega_j \dot{u}_k\}, \tag{2}$$

[1], where the dots indicate differentiation with respect to time. The ε_{ijk} is Levi–Civita tensor and ρ is the mass density of the material. Thus the equation of motion (2), by using the relation (1), becomes

$$C_{ijkl} u_{k,jl} = \rho\{\ddot{u}_i + \Omega_j u_j \Omega_i - \Omega^2 u_i + 2\varepsilon_{ijk}\Omega_j \dot{u}_k\} \tag{3}$$

We assume a plane wave solution of the form

$$u_i = A[\exp\{i(k\ n_j\ x_j - \omega\ t)\}]p_i.$$

Here ω is the frequency and k is assumed to be a complex wave number. The amplitude A is also assumed to be complex whereas n_i and p_i are unit vectors specifying the direction of propagation and the polarization of the displacement wave. The speed of the wave is given by

$$\frac{\omega}{\mathrm{Re}(k)}. \tag{4}$$

Substitution of u_i in Eq. (3), after cancellation of A and the exponential factor, gives

$$\begin{aligned} -k^2 C_{ijkl}\ n_j\ n_l\ p_k = \rho\{&-\omega^2\ p_i + \Omega_j p_j \Omega_i \\ &- \Omega^2\ p_i - 2\ i\ \omega\varepsilon_{ijk}\ \Omega_j\ p_k\} \end{aligned}$$

which can be written as

$$\begin{aligned} -k^2 Q_{ik}\ p_k = \rho\{&-\omega^2\ p_i + \Omega_j p_j \Omega_i - \Omega^2\ p_i \\ &- 2\ i\ \omega\varepsilon_{ijk}\ \Omega_j\ p_k\} \end{aligned} \tag{5}$$

where

$$Q_{ik} = C_{ijkl}\ n_j\ n_l,$$

is the acoustical tensor for the problem. It is shown in [6] that

$$\begin{aligned}\rho^{-1}Q_{ik} = \{a_2 - (a_2 - a_4)(\mathbf{e}.\mathbf{n})^2\}\delta_{ik} + (a_1 - a_2)n_i n_k \\ - (a_1 - a_2 - a_5)(\mathbf{e}.\mathbf{n})(n_k\delta_{i3} + n_i\delta_{k3}) \\ - \{a_2 - a_4 - (a_1 + a_3 - 2a_4 - 2a_5)(\mathbf{e}.\mathbf{n})^2\}\delta_{i3}\delta_{k3}. \quad (6)\end{aligned}$$

a_1, a_2 *etc*., are related to the stiffnesses of the material by the following definitions

$$\begin{aligned}a_1 = \frac{1}{\rho}c_{11},\ a_2 = \frac{1}{\rho}(c_{11} - c_{12}),\ a_3 = \frac{1}{\rho}c_{33}, \\ a_4 = \frac{1}{\rho}c_{44},\ a_5 = \frac{1}{\rho}(c_{13} + c_{44})\end{aligned}$$

ρ being the density of the material. These parameters have the dimension of the square of velocity and e is the axis of symmetry *i.e.*, $e_i = \delta_{i3}$. Equation (6) gives

$$\begin{aligned}Q_{ik}\ p_k = \rho[\{a_2 - (a_2 - a_4)n_3^2\}p_i + (a_1 - a_2)n_k\ p_k\ n_i \\ - (a_1 - a_2 - a_5)n_3(n_k\ p_k\delta_{i3} + n_i\ p_3) \\ - \{a_2 - a_4 - (a_1 + a_3 - 2a_4 - 2a_5)n_3^2\}p_3\ \delta_{i3}]. \quad (7)\end{aligned}$$

From Eqs. (5) and (6), it follows

$$\begin{aligned}- k^2[\{a_2 - (a_2 - a_4)n_3^2\}p_i + (a_1 - a_2)n_k\ p_k\ n_i \\ - (a_1 - a_2 - a_5)n_3(n_k\ p_k\delta_{i3} + n_i\ p_3) \\ - \{a_2 - a_4 - (a_1 + a_3 - 2a_4 - 2a_5)n_3^2\}p_3\ \delta_{i3}] \\ = \{-\omega^2\ p_i + \Omega_j\ p_j\Omega_i - \Omega^2\ p_i - 2\ i\ \omega\varepsilon_{ijk}\Omega_j\ p_k\}\end{aligned}$$

or

$$\begin{aligned}[(\omega^2 + \Omega^2) - k^2\{a_2 - (a_2 - a_4)n_3^2\}]p_i - k^2[(a_1 - a_2)n_k\ p_k\ n_i \\ - (a_1 - a_2 - a_5)n_3(n_k\ p_k\delta_{i3} + n_i\ p_3) \\ - \{a_2 - a_4 - (a_1 + a_3 - 2a_4 - 2a_5)n_3^2\}p_3\ \delta_{i3}] \\ = \{\Omega_j\ p_j\ \Omega_i - 2\ i\ \omega\varepsilon_{ijk}\Omega_j\ p_k\}.\end{aligned}$$

We assume, for simplicity, that the body is rotating about its axis of symmetry, thus $\mathbf{\Omega} = \Omega\ (0, 0, 1)$. We choose a coordinate system such that the propagation vector $\mathbf{n}$ is in the yz-plane *i.e.*, $\mathbf{n}=(0, n_2, n_3)$. For $i=1,2,3$, the following system of equations is obtained

$$[(1+\Gamma^2) - x^2\{a_2\, n_2^2 + a_4\, n_3^2\}]p_1 - 2i\Gamma\, p_2 = 0, \tag{8a}$$

$$2i\,\Gamma\, p_1 + [(1+\Gamma^2) - x^2(a_1\, n_2^2 + a_4\, n_3^2)]p_2 - x^2\, a_5\, n_2\, n_3\, p_3 = 0, \tag{8b}$$

$$-x^2\, a_5\, n_2\, n_3\, p_2 + [1 - x^2(a_3\, n_3^2 + a_4\, n_2^2)]p_3 = 0. \tag{8c}$$

where we have defined $\Gamma=(\Omega/\omega)$ and $x=(k/\omega)$.

For a nontrivial solution one must set the determinant of the above system to zero. This gives the characteristic equation which can be solved for (ω/k) to find the velocities of the three waves which propagate in the medium. The determinantal equation is

$$\Delta=\begin{vmatrix} 1+\Gamma^2-x^2(a_2n_2^2+a_4n_3^2) & -2i\Gamma & 0 \\ 2i\Gamma & 1+\Gamma^2-x^2(a_1n_2^2+a_4n_3^2) & -x^2a_5n_2n_3 \\ 0 & -x^2a_5n_2n_3 & 1-x^2(a_3n_3^2+a_4n_2^2) \end{vmatrix}=0.$$

3. WAVE SPEEDS AND POLARIZATIONS

3.1. Two Special Cases

First we consider two special cases. When, $\Gamma \gg 1$, we can approximately write Δ as the product of the diagonal terms and it will vanish if $x=x_1, x_2, x_3$ where

$$x_1 = \frac{1}{\sqrt{a_3n_3^2 + a_4n_2^2}}, \quad x_2 = \frac{\Gamma}{\sqrt{a_1n_2^2 + a_4n_3^2}},$$

$$x_3 = \frac{\Gamma}{\sqrt{a_2n_2^2 + a_4n_3^2}}.$$

If θ is the angle between the propagation vector and the axis of symmetry, $n_3=\cos\theta$, $n_2=\sin\theta$ and we conclude that for the three waves travelling in an arbitrary direction and having a frequency ω much smaller than the rotation frequency one will be a fast wave having the speed

$$\sqrt{a_3 \cos^2\theta + a_4 \sin^2\theta}$$

while the other two will travel at a much slower speed. In the limit $\Gamma \to \infty$, these latter speeds approach zero. From Eq. (8) it is obvious that for the fast wave $p_1 = p_2 = 0$ and $p_3 = 1$, thus this wave will be longitudinal. Another observation about the determinant Δ is that the choice $\Gamma = 1$, $x = 0$ makes it vanish. This implies that the speed of a wave, whose frequency equals the rotation frequency, will approach infinity. However various approximations involved in the theory, will set an upper bound on this speed.

3.2. Special Directions

Consider a wave propagating parallel to the axis of symmetry *i.e.*, $n_2 = 0$, $n_3 = 1$. In this case, Eq. (8c) gives us a longitudinal wave travelling with speed $\sqrt{a_3}$. For the other two waves we have, from Eqs. (8a) and (8b)

$$(1 + \Gamma^2 - a_4 x^2)^2 - 4\Gamma^2 = 0,$$

or

$$x = \frac{|1 \pm \Gamma|}{\sqrt{a_4}}.$$

Thus there are two waves with speeds respectively $(\sqrt{a_4}/|1 + \Gamma|)$ and $(\sqrt{a_4}/|1 - \Gamma|)$. The speed of the second wave will be unbounded when $\Gamma \to 1$, as has been noted above. Both of these waves will be transverse.

Now consider wave propagation in the basal plane *i.e.*, a plane perpendicular to the axis of symmetry. Here $n_3 = 0$ and $n_2 = 1$. Equation (8c) gives a transverse wave traveling with a velocity of $\sqrt{a_4}$. The speeds of the other two waves, which are neither longitudinal nor transverse, are determined from the equation

$$(1 - \Gamma^2)^2 \nu^4 - (1 + \Gamma^2)(a_1 + a_2)\nu^2 + a_1 a_2 = 0$$

where we have put $\nu = (1/x)$. The above equation is a quadratic in ν^2 and has two positive roots.

3.3. Arbitrary Directions

The characteristic equation for an arbitrary direction is a cubic and can be solved for any given material and a specified direction. For cadmium the material parameters are as follows, in units of giga pascals

Material	ρa_1	ρa_2	ρa_3	ρa_4	ρa_5	ρ (Density)
Cadmium	116	37	50.9	19.6	60.6	8642 kg/m^3

In Table I we list values of the three velocities for four directions of propagation *i.e.*, $\theta = 0°$, 30°, 45° and 90°, as a function of (Ω/ω). Table I exhibits the qualitative features discussed above.

TABLE I Velocities of propagation (m/sec.) in cadmium

Γ	ν_1	ν_2	ν_3
(a) $\theta = 0°$			
0	1505.99	1505.99	2426.9
0.1	1369.08	1673.32	2426.9
0.5	1003.99	3011.97	2426.9
0.99	756.77	150599	2426.9
10	136.98	167.332	2426.9
100	14.91	15.21	2426.9
(b) $\theta = 30°$			
0	1408.52	1664.74	2838.51
0.1	1375.18	1700	2875.78
0.5	1094.3	1943.92	4171.77
0.99	845.77	1998.81	197842
10	155.87	192.82	2236.87
100	16.62	17.93	2232.61
(c) $\theta = 45°$			
0	1407.93	1809.62	3153.1
0.1	1395.06	1811.48	3211.01
0.5	1198.71	1822.84	4902.11
0.99	952.02	179829	2358.1
10	174.16	229.8	2027.4
100	18.08	21.99	2019.71
(d) $\theta = 90°$			
0	2069.16	3663.72	1505.99
0.1	2040.71	3752.31	1505.99
0.5	1671.96	6045.43	1505.99
0.99	1280.38	297526	1505.99
10	204.07	275.23	1505.99
100	20.68	36.64	1505.99

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