Internat. J. Math. & Math. Sci.
VOL. 13 NO. 2 (1990) 349-352

ON ULTRACONNECTED SPACES

P.M. MATHEW

Department of Mathematics and Statistics
Cochin University of Science and Technology
Cochin 682 022, India

(Received March 17, 1988 and in revised form March 16, 1989)

ABSTRACT. In this paper, we study some properties of ultraconnected spaces and show that ultraconnected $T_{\frac{1}{2}}$ spaces are maximal ultraconnected and minimal $T_{\frac{1}{2}}$. We also introduce the notion of F-connected spaces, topological spaces which are both hyperconnected and ultraconnected and characterize compact maximal F-connected topologies on a set.

KEY WORDS AND PHRASES. Ultraconnected, hyperconnected, Semi-topological, generalized closed.
1980 AMS SUBJECT CLASSIFICATION CODES. 6A; 54D, 54G.

1. INTRODUCTION.

A topological space is ultraconnected if the intersection of any two nonempty closed sets is nonempty (Steen and Seebach [1]). Each topology τ on a set X may be associated with a pre-order relation $\rho(\tau)$ on X, defined by $(a,b) \in \rho(\tau)$ if every open set containing b contains a. In 1978 Andima and Thron [2] defined a topological space (X,τ) to be upward directed if any two elements in $(X,\rho(\tau))$ have an upper bound, and it can easily be seen that the notion of upward directed and that of ultraconnected are equivalent.

Let (X,R) be a pre-ordered set. Define $\overline{\{x\}} = \{y \in X \mid x R y\}$ and $\{x\} = \{y \in X \mid y R x\}$, for each $x \in X$. $\mu(R)$, the point closure topology of R, is the smallest topology in which all sets $\overline{\{x\}}$, $x \in X$, are closed and V(R), the kernel topology of R, is the topology with basis $\{\hat{\{x\}} \mid x \in X\}$. A topology τ on X induces a pre-order R as described above iff $\mu(R) \subset \tau \subset V(R)$ [2].

2. ULTRACONNECTED SPACES.

In [2], it is proved that a topological space (X,τ) is maximal upward directed iff $(X,\rho(\tau))$ is a partially ordered set of length 1, with a greatest element and

$\tau = V(\rho(\tau))$. If (X,R) is a partially ordered set of length 1, with a greatest element, say a, then $V(R) = P(X\setminus\{a\}) \cup \{X\}$. Thus the maximal ultraconnected topologies on a set X are precisely $P(X\setminus\{a\}) \cup \{X\}$, where $a \in X$.

DEFINITION 2.1. A topological space is $T_{\frac{1}{2}}$ if each singleton subset is either open or closed (Levine [3]).

REMARK 2.1. Any $T_{\frac{1}{2}}$ space is T_o and Dunham [4] characterized the minimal $T_{\frac{1}{2}}$ topologies on a set X as those of the form $\{0 \subset X \mid 0 \subset A \text{ or } A \subset 0 \text{ and } 0' \text{finite }\}$, for some proper subset A of X. (When X is finite with more than one element, A must also be nonempty.) Obviously, any maximal ultraconnected space is minimal $T_{\frac{1}{2}}$.

THEOREM 2.1. Any ultraconnected $T_{\frac{1}{2}}$ space is maximal ultraconnected and minimal $T_{\frac{1}{2}}$.

PROOF. Let (X,τ) be an ultraconnected $T_{\frac{1}{2}}$ space. Since (X,τ) is $T_{\frac{1}{2}}$ the induced order $\rho(\tau)$ is a partial order. Suppose there exist $x,y,z \in X$ such that $x \rho(\tau)y$ and $y \rho(\tau)z$. If $\{y\}$ is open, then $x \rho(\tau)y \implies x \in \{y\}$; i.e., $x = y$. On the other hand, if $\{y\}$ is closed, then $y \rho(\tau)z \implies z \in \overline{\{y\}} = \{y\}$; i.e., $z = y$. Since the singletons are either open or closed, it is evident that the length of $(X, \rho(\tau))$ is at most 1.

If $\{x\}$ is open and $y \rho(\tau)x$, then $y = x$ and hence x is minimal in $(X, \rho(\tau))$. Similarly if $\{x\}$ is closed, then x is maximal in $(X, \rho(\tau))$. Since (X,τ) is ultraconnected any two minimal elements have an upper bound and there exists only one maximal element which will be the greatest element in $(X, \rho(\tau))$. Moreover, if x is minimal in $(X, \rho(\tau))$, then $\{x\}$ is open and not closed. Hence $\tau = V(\rho(\tau))$. Thus (X,τ) is maximal ultraconnected, and by the above remark it is minimal $T_{\frac{1}{2}}$ too.

NOTE 2.1. Though every maximal ultraconnected space is minimal $T_{\frac{1}{2}}$, there are minmal $T_{\frac{1}{2}}$ spaces which are not even ultraconnected. However, every minimal $T_{\frac{1}{2}}$ space is connected [4].

Let X be a set with 3 or more elements and $\phi \neq A \subset X$ such that $|X\setminus A| > 2$. Then $\tau = \{0 \subset X \mid 0 \subset A \text{ or } A \subset 0 \text{ and } 0' \text{ finite }\}$ is a minimal $T_{\frac{1}{2}}$ topology, which is not ultraconnected. For if $x,y \in X\setminus A$, then $\{x\}$ and $\{y\}$ are closed subsets of (X,τ) with empty intersection.

DEFINITION 2.2. A subset of a topological space is called ultraconnected if it is ultraconnected as a subspace.

REMARK 2.2. We will call two subsets A and B of a topological space (X, τ) equivalent ($A \equiv B$) if every open set containing A contains B and conversely. $A^* = \bigcap\{0 \in \tau \mid 0 \supset A\}$ is the largest subset of X equivalent to A. Note that, if $A \subset B \subset C$ and $A \equiv C$, then $A \equiv B$ and $B \equiv C$.

THEOREM 2.2. Let A and B be subsets of a topological space (X,τ) and $A \equiv B$. Then A is ultraconnected iff B is ultraconnected.

PROOF. Suppose A is ultraconnected, but B is not. Then there exist two nonempty disjoint closed sets C_1, C_2 in B. Let $C_i = D_i$ B; $i = 1,2$; D_i closed in (X, τ).

$$C_1 \cap C_2 \neq \phi \Longrightarrow D_1 \cap D_2 \cap B = \phi \Longrightarrow B \subset D_1' \cup D_2'$$

Since $A \equiv B$, $A \subset D_1' \cup D_2'$ and hence $D_1 \cap D_2 \cap A = \phi$. But $D_1 \cap A \neq \phi$, for otherwise $A \subset D_1' \Longrightarrow B \subset D_1' \Longrightarrow C_1 = D_1 \cap B = \phi$. Similarly $D_2 \cap A \neq \phi$. Since $D_1 \cap A$, $D_2 \cap A$ are nonempty disjoining closed sets in A, we get a contradiction. Hence the result.

DEFINITION 2.3. A subset A of a topological space is generalized closed if $\overline{A} \subset 0$ and 0 whenever $A \subset 0$ and 0 is closed [3].

COROLLARY 2.1. If A is a generalized closed subset of (X, τ), then A is ultraconnected iff $\overline{A}$ is ultraconnected.

PROOF. In view of Theorem 2.2, it is sufficient to show that $A \equiv \overline{A}$. Since A is generalized closed, if $A \subset 0 \in \tau$, then $\overline{A} \subset 0$. The other implication is trivial.

COROLLARY 2.2. If A and B are subsets of a space (X, τ) such that $A \subset B \subset A^*$, then A is ultraconnected iff B is ultraconnected.

PROOF. Since $A \subset B \subset A^*$ and $A \equiv A^*$, it follows that $A \equiv B$ (see the previous remark). Thus the conclusion is an immediate consequence of Theorem 2.2.

DEFINITION 2.4. A subset A of a space X is called semi-open if there exists an open set 0 such that $0 \subset A \subset \overline{0}$ (Levine [5]). A semi-homeomorphism is a bijection under which both images and inverse images of sem-open sets are semi-open. A topological property invariant under semi-homeomorphisms is called a semi-topological property by Crossley and Hildebrahd [6].

REMARK 2.3. Ultraconnectedness is not semitopological. Let $X = \{a,b,c\}$. $\tau_1 = \{\phi, \{a\}, \{a,b\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{a,b\}, \{a,c\}, X\}$. Now (X, τ_1) is ultraconnected, but (X, τ_2) is not, while τ_1 and τ_2 yield the same collection of semi-open sets and hence are semi-homeomorphic.

3. F-CONNECTED SPACES.

A topological space in which the intersection of any two nonempty open sets is nonempty is called hyperconnected [1]. We define a topological space to be F-connected if it is both hyperconnected and ultraconnected.

REMARK 3.1. In the above remark (X, τ_1) is F-connected while (X, τ_2) is not. Hence F-connectedness is not a semi-topological property. Neither the join nor the product of two F-connected topologies on a set are F-connected. Let $\tau_1 = \{\phi, A, X\}$ and $\tau_2 = \{\phi, B, X\}$ where $A \cap B = \phi$. Then $\tau_1 \vee \tau_2$ and $\tau_1 \times \tau_2$ are not F-connected but τ_1 and τ_2 are F-connected.

THEOREM 3.1. Every subspace of a topological space (X, τ) is F-connected iff τ is nested.

PROOF. Necessity: Assume τ is not nested. Then there exist $A, B \subset X$ such that $A \not\subset B$ and $B \not\subset A$. Choose $x \in A \setminus B$ and $y \in B \setminus A$. Then the subspace $\{x,y\}$ has the discrete topology which is obviously not F-connected.

Sufficiency: Let τ be nested and $A \subset X$. Let 0_1, 0_2 be nonempty open sets in A. Then there exist B_1, B_2 ε τ such that $0_1 = A \cap B_1$ and $0_2 = A \cap B_2$. Since τ is nested, $B_1 \subset B_2$ or $B_2 \subset B_1$. Assume $B_1 \subset B_2$. Then $0_1 \subset 0_2$ and hence $0_1 \cap 0_2 \neq \phi$. Similarly, the intersection of any two nonempty closed sets in A is also nonempty. Thus A is F-connected.

THEOREM 3.2. If $\mathbf{U}$ is an ultrafilter on $X \setminus \{a\}$, for some $a \varepsilon X$, then $\tau = \{\phi, X\} \cup \mathbf{U}$ is a maximal F-connected topology on X.

PROOF. Obviously, (X, τ) is F-connected. Suppose (X, τ_1) is F-connected and $\tau_1 > \tau$. Let $A \varepsilon \tau_1 \setminus \tau$. Since $A \varepsilon \tau_1 \setminus \tau$, $a \notin A$. For if $a \varepsilon A$, then $\{a\} \cap (X \setminus A) = \phi$, a contradiction since (X, τ_1) is ultraconnected. Now $a \notin A$ and $A \notin \mathbf{U}$ implies $(X \setminus \{a\}) \setminus A \;\varepsilon$. Thus A and $(X \setminus \{a\}) \setminus A$ are two nonempty disjoint open sets in (X, τ_1), a contradiction. Hence the result.

THEOREM 3.3. Any compact, maximal F-connected topology on a set X is of the form $\tau_a = \{\phi, X\} \cup \mathbf{U}_a$, where $\mathbf{U}_a$ is an ultrafilter on $X \setminus \{a\}$, for some $a \varepsilon X$.

PROOF. Let (X, τ) be compact and maximal F-connected. Since the family of all the nonempty closed sets has finite intersection property and (x, τ) is compact, it has nonempty intersection. Choose $a \;\varepsilon \bigcap \{C \subset X \mid C$ is closed and nonempty $\}$. Thus the proper open sets are subsets of $X \setminus \{a\}$ and they form a filter base F. Let $\mathbf{U}_a$ be an ultrafilter on $X \setminus \{a\}$ containing F. Then $\tau \subset \{\phi, X\} \cup \mathbf{U}_a = \tau_a$. Since (X, τ) is maximal F-connected in view of Thoerem 3.2, $\tau = \tau_a$.

ACKNOWLEDGMENT. The author wishes to thank Professor T. Thrivikraman for his guidance during the preparation of this paper. He also wishes to thank the referee for the valuable comments which improved the presentation considerably.

REFERENCES

1. STEEN, L.A. and SEEBACH, J.A., Jr., Counter Examples in Topology, Springer Verlag New York, 1978.
2. ANDIMA, S. J. and THRON, W.J., Order-induced Topological Properties, Pacific J. Math. 75 (1978), 297-318.
3. LEVINE, N., Generalized Closed Sets in Topology, Rend. del. Circ. Mat. Di. Palermo 19 (1970), 89-96.
4. DUNHAM, W., $T_{\frac{1}{2}}$ spaces, Kyumpook Math. J. 17(2) (1977), 161-169.
5. LEVINE, N., Semi-open Sets and Semi-continuity in Topological Spaces, Amer. Math. Monthly 70 (1963), 36-41.
6. CROSSLEY, S.G. and HILDEBRAND, S.K., Semitopologial Properties, Fund. Math. LXXIV (1972), 232-254.
7. NOIRI, T., Functions which Preserve Hyperconnected Spaces, Rev. Roumaine Math. Pures Appl. 25 (1980), 1091-1094.

Special Issue on Intelligent Computational Methods for Financial Engineering

Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems)

This special issue will include (but not be limited to) the following topics:

- **Computational methods**: artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning
- **Application fields**: asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects**: decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site http://www.hindawi.com/journals/jamds/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Lean Yu, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; yulean@amss.ac.cn

Shouyang Wang, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; sywang@amss.ac.cn

K. K. Lai, Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; mskklai@cityu.edu.hk