CHARACTERIZATIONS OF PROJECTIVE
AND $k$-PROJECTIVE SEMIMODULES

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To my late father

This paper deals with projective and $k$-projective semimodules. The results for projective semimodules are generalizations of corresponding results for projective modules.

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1. Introduction. Throughout this paper, $R$ denotes a semiring with identity 1, all semimodules $M$ are left $R$-semimodules and in all cases are unitary semimodules, that is, $1 \cdot m = m$ for all $m \in M$ all left $R$-semimodule $RM$.

We recall here (cf. [1, 2, 3, 4, 5]) the following facts:

(a) let $\alpha : M \to N$ be a homomorphism of semimodules. The subsemimodule Im $\alpha$ of $N$ is defined as follows: Im $\alpha = \{ n \in N : n + \alpha(m') = \alpha(m) \text{ for some } m, m' \in M \}$. The homomorphism $\alpha$ is said to be an isomorphism if $\alpha$ is injective and surjective; to be $i$-regular if $\alpha(M) = \text{Im} \ \alpha$; to be $k$-regular if for $m, m' \in M$, $\alpha(m) = \alpha(m')$ implying that $m + k = m' + k'$ for some $k, k' \in \text{Ker} \ \alpha$; and to be regular if it is both $i$-regular and $k$-regular;

(b) the sequence $K \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is called an exact sequence if $\text{Ker} \ \beta = \text{Im} \ \alpha$, and proper exact if $\text{Ker} \ \beta = \alpha(K)$;

(c) for any two $R$-semimodules $N, M$, Hom$_R(N,M) := \{ \alpha : N \to M \mid \alpha \text{ is an } R\text{-homomorphism of semimodules} \}$ is a semigroup under addition. If $M, N, U$, are $R$-semimodules and $\alpha : M \to N$ is a homomorphism, then Hom$(I_U, \alpha) : \text{Hom}_R(U,M) \to \text{Hom}_R(U,N)$ is given by Hom$(I_U, \alpha)y = \alpha y$ where $I_U$ is the identity;

(d) $P$ is a projective semimodule if and only if for each surjective $R$-homomorphism $\alpha : M \to N$, the induced homomorphism

$$\bar{\alpha} : \text{Hom}(P,M) \to \text{Hom}(P,N)$$

is surjective;

(e) a left $R$-semimodule $P$ is $Mk$-projective if and only if it is projective with respect to every surjective $k$-regular homomorphism $\varphi : M \to N$.

In Section 2, we study the structure of $k$-projective semimodules. Proposition 2.2 shows that for a semimodule $P$, the class of all semimodules $M$ such that $P$ is $Mk$-projective is closed under subtractive subsemimodules, factor semimodules, and gives a sufficient condition for the class to be closed undertaking homomorphic images. Example 2.3 sheds light upon one difference between the structure of projectivity in
module theory and semimodule theory. In Section 3, we characterize projective and $k$-projective semimodules via the Hom functor. Theorems 3.5 and 3.7 assert that $P$ is $M$-projective ($Mk$-projective) if and only if $\text{Hom}_R(P, -)$ preserves the exactness of all proper exact sequences $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$, with $\beta$ $k$-regular (both $\alpha$ and $\beta$ $k$-regular).

2. $k$-projective semimodules. We study the structure of $k$-projective semimodules via the Hom function. We show that the class of all semimodules $M$, such that $P$ is $Mk$-projective, is closed under subtractive subsemimodules, factor semimodule and undertaking homomorphic image for a $k$-regular homomorphism.

For proving Proposition 2.2 we need the following proposition, which is modified from [5, Theorem 2.6].

**Proposition 2.1.** Let $R$ be a semiring,

(i) if $0 \to M \xrightarrow{\alpha} M' \xrightarrow{\beta} M''$ is a proper exact sequence of $R$-semimodules and $\alpha$ is $k$-regular, then for every $R$-semimodule $E$,

$$0 \to \text{Hom}_R(E,M) \xrightarrow{\bar{\alpha}} \text{Hom}_R(E,M') \xrightarrow{\bar{\beta}} \text{Hom}_R(E,M'')$$

is a proper exact sequence of Abelian semigroups and $\bar{\alpha}$ is regular, where $\bar{\alpha}(\xi) = \alpha\xi$ for $\xi \in \text{Hom}(E,M)$ and $\bar{\beta}(\gamma) = \beta\gamma$ for $\gamma \in \text{Hom}(E,M')$;

(ii) if $M \xrightarrow{\alpha} M' \xrightarrow{\beta} M'' \to 0$ is a proper exact sequence of $R$-semimodules and $\beta$ is $k$-regular, then for every $R$-semimodule $E$,

$$0 \to \text{Hom}(M'',E) \xrightarrow{\bar{\beta}} \text{Hom}(M',E) \xrightarrow{\bar{\alpha}} \text{Hom}(M,E)$$

is a proper exact sequence of Abelian semigroups and $\bar{\beta}$ is regular, where $\bar{\beta}(\xi) = \beta\xi$.

**Proof.** (i) Since the sequence $0 \to M \xrightarrow{\alpha} M' \xrightarrow{\beta} M''$ is proper exact, then the sequence is exact with $\alpha$ being $i$-regular.

Using [5, Theorem 2.6], the sequence

$$0 \to \text{Hom}(E,M) \xrightarrow{\bar{\alpha}} \text{Hom}(E,M') \xrightarrow{\bar{\beta}} \text{Hom}(E,M'')$$

is exact with $\bar{\alpha}$ being regular. This means that the sequence is proper exact. (ii) can be proved by the same argument. \qed

**Proposition 2.2.** Let $P$ be a left $R$-semimodule. If $0 \to M' \xrightarrow{\theta} M \xrightarrow{\eta} M'' \to 0$ is a proper exact sequence with $\theta$ being regular, $\eta$ being $k$-regular, and $P$ is $Mk$-projective, then $P$ is $k$-projective relative to both $M'$ and $M''$.

**Proof.** Let $\Psi : M'' \to N$ be surjective $k$-regular homomorphism and $\alpha : P \to N$ be homomorphism. Since $\eta$ is surjective $k$-regular, then $\Psi \eta$ is $k$-regular. Since $P$ is $Mk$-projective, then there exists a homomorphism $\varphi : P \to M$ such that the following diagram commutative:

$$\begin{array}{ccc}
M & \xrightarrow{\eta} & M'' \\
\downarrow{\eta} & & \downarrow{\Psi} \\
N & \xrightarrow{\Psi} & 0
\end{array}$$

Therefore $P$ is $M''k$-projective.
To prove that $P$ is $M$-projective. Let $\Psi : M' \to N$ be a surjective $k$-regular homomorphism and set $K = \text{Ker} \Psi$. Since $\Psi$ is surjective $k$-regular homomorphism, then $M'/K \cong N$. Define $\bar{\theta} : M'/K \to M/\theta(K)$ by the rule $\bar{\theta}(m'/K) = \theta(m'/\theta(K))$, and $\bar{\eta} : M/\theta(K) \to M''$ by the rule $\bar{\eta}(m/\theta(K)) = \eta(m)$. Clearly, both $\bar{\theta}$ and $\bar{\eta}$ are well defined homomorphisms. Consider the sequence $0 \to M'/K \xrightarrow{\bar{\theta}} M/\theta(K) \xrightarrow{\bar{\eta}} M'' \to 0$. Let $m/\theta(K) \in \text{Ker} \bar{\eta}$, then $\eta(m) = 0$, hence $m \in \text{Ker} \eta = \theta(M')$. Hence $\text{Ker} \bar{\eta} = \bar{\theta}(M'/K)$. Clearly $\bar{\eta}$ is surjective, and $\bar{\theta}$ is injective. Since $\theta$ is $i$-regular, then $\bar{\theta}$ is $i$-regular. Now consider the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \to & M' & \xrightarrow{\bar{\theta}} & M & \xrightarrow{\eta} & M'' & \to & 0 \\
\downarrow{\pi_k} & & \downarrow{\pi_{h(k)}} & & \downarrow{\pi} & & \downarrow{\pi} & & \downarrow{\pi} \\
0 & \to & M'/K & \xrightarrow{\bar{\theta}} & M/\theta(K) & \xrightarrow{\bar{\eta}} & M'' & \to & 0 \\
0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\] (2.5)

Applying $\text{Hom}_R(P, -)$ to this diagram we have the commutative diagram

\[
\begin{array}{cccccc}
0 & \to & \text{Hom}(P, M') & \xrightarrow{\bar{\theta}^*} & \text{Hom}(P, M) & \xrightarrow{\eta^*} & \text{Hom}(P, M'') & \to & 0 \\
\downarrow{\text{(\pi_k)^*}} & & \downarrow{\text{(\pi_{h(k)})^*}} & & \downarrow{I_s} & & \downarrow{\text{(\pi)^*}} & & \downarrow{\text{(\pi)^*}} \\
0 & \to & \text{Hom}(P, M'/K) & \xrightarrow{\bar{\theta}^*} & \text{Hom}(P, M/\theta(K)) & \xrightarrow{\eta^*} & \text{Hom}(P, M'') & \to & 0 \\
0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\] (2.6)

Using Proposition 2.1, and since $P$ is $M$-projective, then all rows and columns are proper exact sequence. We should show that $(\pi_k)_*$ is surjective. Let $\alpha \in \text{Hom}(P, M'/K)$. Since $(\pi_{h(K)})_*$ is surjective, then there exists $\beta \in \text{Hom}(P, M)$ such that $(\pi_{h(K)})_*(\beta) = \bar{\theta}_*(\alpha)$. Now $\bar{\eta}_* \bar{\theta}_* (\alpha) = \bar{\eta}_* ((\pi_{h(K)})_*(\beta)) = I_s \eta_* (\beta) = 0$. Hence $\eta_* (\beta) \in \text{Ker} I_s = 0$. Hence $\beta = \theta_*(y)$ where $y \in \text{Hom}(P, M')$. Thus $\bar{\theta}_* (\alpha) = (\pi_{h(K)})_* (\beta) = (\pi_{h(K)})_* \theta_*(y) = \bar{\theta}_*(\pi_k)_*(y)$. Again by Proposition 2.1, $\bar{\theta}_*$ is injective, hence $\alpha = (\pi_k)_*(y)$. Thus $(\pi_k)_*$ is surjective. Therefore $P$ is $M$-projective.

Let $\Omega(P)$ be the collection of all semimodules $M$ such that $P$ is $M$-projective. The above results show that this class is closed under subtrusive subsemimodules and
give us a sufficient condition to be closed undertaking a homomorphic image. Since for every subsemimodule \( K \) of \( M \), the canonical surjection \( \pi_K : M \to M/K \) is \( k \)-regular surjective, then the class \( \Omega(P) \) is closed under factor semimodules.

We know that in module theory any projective module is a direct summand of a free module. However, for arbitrary semirings this is not true.

**Example 2.3.** Let \( R \) be the field \( \mathbb{Z}/\langle p \rangle \) for any prime integer. Let \( S = \{\{0\}, \mathbb{Z}/\langle p \rangle \} \) set \( R' = \{(\hat{a}, I) : \hat{a} \in I \in S\} \), that is

\[
R' = \{(\hat{0}, \{0\}), (\hat{a}, \mathbb{Z}/\langle p \rangle), \hat{a} \in \mathbb{Z}/\langle p \rangle\}.
\]  

Define operations \( \oplus \) and \( \otimes \) on \( R' \) by setting

\[
(\hat{a}, I) \oplus (\hat{b}, H) = (\hat{a} + \hat{b}, I + H),
\]

\[
(\hat{a}, I) \otimes (\hat{b}, H) = (\hat{a}\hat{b}, IH).
\]  

Clearly \( R' \) is semiring. Let \( I^+(R') \) be the set of all additively idempotent elements of \( R' : I^+(R') = \{(\hat{0}, \{0\}), (\hat{0}, \mathbb{Z}/\langle p \rangle)\} \). We note that the function \( \alpha : R' \to I^+(R') \), defined by \( \alpha(\hat{0}, \{0\}) = (\hat{0}, \{0\}) \) and \( \alpha(\hat{a}, \mathbb{Z}/\langle p \rangle) = (\hat{0}, \mathbb{Z}/\langle p \rangle) \), is a surjective \( R' \)-homomorphism of left \( R' \)-semimodules. Furthermore, the restriction of \( \alpha \) to \( I^+(R') \) is the identity map. Therefore \( I^+(R') \) is a retract of \( R' \). Since \( R' \) is projective, as a left semimodule over itself, by [5, Corollary 15.13] we see that \( I^+(R') \) is also projective. If \( I^+(R') \) is a free semimodule, then \( (\hat{0}, \mathbb{Z}/\langle p \rangle) \) is a basis, but \( (\hat{a}, \mathbb{Z}/\langle p \rangle) = (\hat{0}, \mathbb{Z}/\langle p \rangle) \) for every \( \hat{a} \in \mathbb{Z}/\langle p \rangle \). Hence \( I^+(R') \) is not free. Now, suppose that \( I^+(R') \) is a direct summand of a free \( R' \)-semimodule, say \( F \). Then \( F \cong K \oplus I^+(R') \) for an \( R' \)-semimodule \( K \). Let \( \varphi : K \oplus I^+(R') \to F = \oplus_{\alpha} R'_{\alpha} \) be an isomorphism, where \( R'_{\alpha} = R' \) for all \( \alpha \). Since \( (0, (\hat{0}, \mathbb{Z}/\langle p \rangle) ) \) is an idempotent element where \( 0 \in K \), then \( \varphi(0, (\hat{0}, \mathbb{Z}/\langle p \rangle)) \) is an idempotent element in \( F \). Since the only idempotent elements of \( R' \) are \( (\hat{0}, \{0\}) \) and \( (\hat{0}, \mathbb{Z}/\langle p \rangle) \), then \( \varphi(0, (\hat{0}, \mathbb{Z}/\langle p \rangle)) = (x_{\alpha}) \), where only finite numbers of \( x_{\alpha} \) are nonzero and \( x_{\alpha} = (\hat{0}, \mathbb{Z}/\langle p \rangle) \). Let \( y_{\alpha} \) have components \( y_{\alpha} = (\hat{1}, \mathbb{Z}/\langle p \rangle) \) for \( \alpha \), with \( x_{\alpha} = (\hat{0}, \mathbb{Z}/\langle p \rangle) \), and otherwise \( y_{\alpha} = (\hat{0}, \{0\}) \). Suppose \( \varphi(k, (\hat{0}, \mathbb{Z}/\langle p \rangle)) = (y_{\alpha}) \), where \( 0 \neq k \in K \). Clearly, \( p(y_{\alpha}) = (x_{\alpha}) \), \( p\varphi(k, (\hat{0}, \mathbb{Z}/\langle p \rangle)) = \varphi(0, (\hat{0}, \mathbb{Z}/\langle p \rangle)) \). Hence \( p(k, (\hat{0}, \mathbb{Z}/\langle p \rangle)) = (0, (\hat{0}, \mathbb{Z}/\langle p \rangle)) \). Hence \( pk = 0 \). Therefore \( p(k, (\hat{0}, \{0\})) = 0 \). Hence \( (k, (\hat{0}, \{0\})) \) has an additive inverse. Thus, \( \varphi(k, (\hat{0}, \{0\})) \) also has an additive inverse.

Now, every element of \( F \) is of the form \( (u_{\alpha}) \), \( u_{\alpha} \in R' \). Since every nonzero element of \( R' \) has no additive inverse, then every nonzero element of \( F \) has no additive inverse. Thus we have a contradiction. Therefore, \( I^+(R') \) is not a direct summand of a free \( R' \)-semimodule.

### 3. Characterizations of projective and \( k \)-projective semimodules

We characterize projective and \( k \)-projective semimodules via the Hom functor.

We state and prove the following lemma and corollaries which are needed in the proof of **Theorem 3.5.**
**Lemma 3.1.** Let $R$ be a semiring and let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a sequence of $R$-semimodules. Then, the sequence is exact if there exists a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
N & \rightarrow & 0 \\
\downarrow \phi & & \downarrow \\
M' & \rightarrow & M \\
\downarrow \Psi & & \downarrow \theta \\
K & \rightarrow & 0 \\
\downarrow \eta & & \downarrow \\
0 & \rightarrow & 0 \\
\end{array}
\]

(3.1)

of $R$-semimodule in which the nonhorizontal sequences are all exact.

**Proof.** Let $x \in \text{Ker} \beta$, then $\eta \phi(x) = \beta(x) = 0$, hence $\phi(x) \in \text{Ker} \eta$. Since $\text{Ker} \eta = 0$, then $x \in \text{Ker} \phi = \text{Im} \theta$. Hence $x + \theta(k_1) = \theta(k_2)$, therefore $x + \theta(\Psi(m_1)) = \theta(\Psi(m_2))$. Since $\theta \Psi = \alpha$, then $x + \alpha(m_1) = \alpha(m_2)$. Thus $x \in \text{Im} \alpha$. Conversely, let $x \in \text{Im} \alpha$, then $x + \alpha(m_1) = \alpha(m_2)$ for some $m_1, m_2 \in M$. Again since $\theta \Psi = \alpha$, then $x + \theta(\Psi(m_1)) = \theta(\Psi(m_2))$, hence $x \in \text{Im} \theta$. But $\text{Im} \theta = \text{Ker} \phi$, hence $\beta(x) = \eta \phi(x) = 0$. Thus $\text{Im} \alpha = \text{Ker} \beta$. \qed

Since every proper exact sequence is exact sequence. Then we have the following corollary.

**Corollary 3.2.** Let $R$ be a semiring and let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a sequence of $R$-semimodules. Then the sequence is proper exact if there exists a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
N & \rightarrow & 0 \\
\downarrow \phi & & \downarrow \\
M' & \rightarrow & M \\
\downarrow \Psi & & \downarrow \theta \\
K & \rightarrow & 0 \\
\downarrow \eta & & \downarrow \\
0 & \rightarrow & 0 \\
\end{array}
\]

(3.2)

of $R$-semimodules in which the nonhorizontal sequences are all proper exact.
**Proof.** Since every proper exact sequence is exact sequence, then using Lemma 3.1, we have $\text{Ker} \beta = \text{Im} \alpha$, hence $\alpha(M') \subset \text{Ker} \beta$. Now let $x \in \text{Ker} \beta$, then $\beta(x) = \eta \varphi(x) = 0$, hence $\varphi(x) \in \text{Ker} \eta$. But $\text{Ker} \eta = 0$, therefore $x \in \text{Ker} \varphi = \theta(K)$. Hence $x = \theta(k) = \theta(\Psi(m')) = \alpha(m')$. Thus $\alpha(M') = \text{Ker} \beta$.

**Corollary 3.3.** Let $R$ be a semiring and let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a sequence of $R$-semimodules with $\beta$ being $k$-regular. Then the sequence is proper exact if and only if there exists a commutative diagram

![Diagram](image)

of $R$-semimodules in which the nonhorizontal sequences are all proper exact.

**Proof.** Let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a proper exact sequence with $\beta$ being $k$-regular. Consider the following diagram:

![Diagram](image)

where $\Psi(m') = \alpha(m')$ for all $m' \in M'$, $i(x) = x$ for all $x \in \text{Ker} \beta$, $\varphi(m) = m/\text{Ker} \beta$ for all $m \in M$, and $\eta(m/\text{Ker} \beta) = \beta(m)$ for all $m/\text{Ker} \beta \in M/\text{Ker} \beta$. Let $m_1/\text{Ker} \beta = m_2/\text{Ker} \beta$, then $m_1 + k_1 = m_2 + k_2$, $k_1, k_2 \in \text{Ker} \beta$. Hence $\beta(m_1) = \beta(m_2)$, therefore $\eta$ is well defined. Now if $\beta(m_1) = \beta(m_2)$, and since $\beta$ is $k$-regular, we have $m_1 + k_1 = m_2 + k_2$, $k_1, k_2 \in \text{Ker} \beta$. 


Hence $m_1 / \ker \beta = m_2 / \ker \beta$, therefore $\eta$ is injective. Clearly the sequence $0 \to \ker \beta \xrightarrow{i} M \xrightarrow{\varphi} M / \ker \beta \to 0$ is proper exact sequence. Thus diagram (3.4) is a commutative diagram in which the nonhorizontal sequences are all proper exact.

Conversely, see Corollary 3.2.

**Corollary 3.4.** Let $R$ be a semiring and let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a sequence of $R$-semimodules with $\beta$ being $k$-regular. Then the sequence is exact if and only if there exists a commutative diagram

$$
\begin{array}{cccccc}
0 & 0 \\
& N \\
M' & \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \\
& K \\
& 0 & 0
\end{array}
$$

(3.5)

of $R$-semimodules in which the nonhorizontal sequences are all exact.

**Proof.** Let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be an exact sequence with $\beta$ being $k$-regular. Consider the following diagram:

$$
\begin{array}{cccccc}
0 & 0 \\
& M / \im \alpha \\
M' & \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \\
& \im \alpha \\
& 0 & 0
\end{array}
$$

(3.6)

where $\Psi(m') = \alpha(m') \in \im \alpha$ for all $m' \in M'$, $i(x) = x$ for all $x \in \im \alpha$, $\varphi(m) = m / \im \alpha$ for all $m \in M$, and $\eta(m / \im \alpha) = \beta(m)$ for all $m / \im \alpha \in M / \im \alpha$. Let $m_1 / \im \alpha = m_2 / \im \alpha$, then $m_1 + t_1 = m_2 + t_2$, $t_1, t_2 \in \im \alpha = \ker \beta$. Hence $\beta(m_1) = \beta(m_2)$, therefore $\eta$ is well defined. Now if $\beta(m_1) = \beta(m_2)$, then since $\beta$ is $k$-regular we have $m_1 + k_1 = m_2 + k_2$, $k_1, k_2 \in \ker \beta = \im \alpha$. Hence $m_1 / \im \alpha = m_2 / \im \alpha$, therefore $\eta$ is
injective. Clearly, the sequence $0 \to \text{Im } \alpha \overset{i}{\to} M \overset{\varphi}{\to} M/\text{Im } \alpha \to 0$ is exact. Thus diagram (3.6) is a commutative diagram in which the nonhorizontal sequences are all exact. Conversely, see Lemma 3.1.

**Theorem 3.5.** The following statements about left $R$-semimodule $P$ are equivalent:

(i) $P$ is projective;

(ii) for every proper exact sequence of left $R$-semimodules $M' \overset{\alpha}{\to} M \overset{\beta}{\to} M''$ with $\beta$ being $k$-regular the sequence

$$\text{Hom}(P,M') \overset{\bar{\alpha}}{\to} \text{Hom}(P,M) \overset{\bar{\beta}}{\to} \text{Hom}(P,M'')$$

is proper exact.

**Proof.** (ii)$\Rightarrow$(i). Let $\alpha : N \to M$ be surjective homomorphism. Since $N \overset{\alpha}{\to} M \to 0$ is proper exact sequence with $M \to 0$ being regular, then by (ii) $\text{Hom}(P,N) \to \text{Hom}(P,M) \to 0$ is proper exact. Therefore $P$ is projective.

(i)$\Rightarrow$(ii). Let $P$ be a projective semimodule. Suppose that $M' \overset{\alpha}{\to} M \overset{\beta}{\to} M''$ is proper exact with $\beta$ being $k$-regular. Consider the sequence $0 \to \text{Ker } \beta \overset{i}{\to} M \overset{\pi}{\to} M/\text{Ker } \beta \to 0$, where $\pi(m) = m/\text{Ker } \beta$ is the canonical surjection. Clearly $i$ is injective. Since $P$ is projective, and using [1, Theorem 10], the sequence

$$0 \to \text{Hom}_R(P,\text{Ker } \beta) \overset{i}{\to} \text{Hom}(P,M) \overset{\pi}{\to} \text{Hom}(P,M/\text{Ker } \beta) \to 0$$

is proper exact sequence. Define $\Psi : M' \to \text{Ker } \beta$ and $\eta : M/\text{Ker } \beta \to M''$, where $\Psi(m') = \alpha(m')$ and $\eta(m/\text{Ker } \beta) = \beta(m)$. Let $\eta(m/\text{Ker } \beta) = \eta(m'/\text{Ker } \beta)$, then $\beta(m) = \beta(m')$. Since $\beta$ is $k$-regular, then $m + k = m' + k'$, where $k,k' \in \text{Ker } \beta$. Hence $m/\text{Ker } \beta = m'/\text{Ker } \beta$. Therefore $\eta$ is injective. Now consider the commutative diagram

$$
\begin{array}{ccc}
0 & \overset{0}{\longrightarrow} & 0 \\
\downarrow & & \downarrow \\
\text{Hom}_R(P,M/\text{Ker } \beta) & \overset{\pi}{\longrightarrow} & \text{Hom}_R(P,M/\text{Ker } \beta) \\
\downarrow \Psi & & \downarrow \eta \\
\text{Hom}_R(P,M') & \overset{\alpha}{\longrightarrow} & \text{Hom}_R(P,M) \\
\downarrow i & & \downarrow \beta \\
\text{Hom}_R(P,\text{Ker } \beta) & \overset{0}{\longrightarrow} & 0 \\
\end{array}
$$

where $\bar{\Psi}(\xi) = \Psi \xi$ and $\bar{\eta}(\gamma) = \eta \gamma$, $\xi \in \text{Hom}_R(P,M')$ and $\gamma \in \text{Hom}_R(P,M/\text{Ker } \beta)$. Now let $\xi,\gamma \in \text{Hom}_R(P,M/\text{Ker } \beta)$ such that $\bar{\eta}(\xi) = \bar{\eta}(\gamma)$. Since $\eta$ is injective, then $\xi = \gamma$. Let $\xi \in \text{Hom}(P,\text{Ker } \beta)$. Since $P$ is projective, then there exist $\theta : P \to M'$ such that the
The following diagram is commutative:

\[
\begin{array}{ccc}
\theta & \downarrow \xi \\
M' & \xrightarrow[\Psi]{\alpha} & \text{Ker} \beta & \xrightarrow{} & 0
\end{array}
\]

Thus \( \Psi \) is surjective. Thus the nonhorizontal sequences are all proper exact. Using Corollary 3.2, the sequence

\[
\text{Hom}_R (P, M') \xrightarrow{\alpha} \text{Hom}_R (P, M) \xrightarrow{\beta} \text{Hom}_R (P, M'')
\]

is proper exact. \( \Box \)

**Corollary 3.6.** The following statement about left \( R \)-semimodule \( P \) are equivalent:

(i) \( P \) is projective;

(ii) for every proper exact sequence of left \( R \)-semimodules \( M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \) with \( \beta \) being regular, the sequence

\[
\text{Hom} (P, M') \xrightarrow{\alpha} \text{Hom}(P, M) \xrightarrow{\beta} \text{Hom} (P, M'')
\]

is proper exact.

**Proof.** It is a consequence of Theorem 3.5. \( \Box \)

**Theorem 3.7.** The following statements about left \( R \)-semimodule \( P \) are equivalent:

(i) \( P \) is \( k \)-projective;

(ii) for every proper exact sequence of left \( R \)-semimodules \( M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \) with both \( \alpha \) and \( \beta \) being \( k \)-regular, the sequence

\[
\text{Hom} (P, M') \xrightarrow{\alpha} \text{Hom}(P, M) \xrightarrow{\beta} \text{Hom} (P, M'')
\]

is proper exact.

**Proof.** (ii)\( \Rightarrow \) (i). Let \( \alpha : N \rightarrow M \rightarrow 0 \) be \( k \)-surjective homomorphism. Since \( N \rightarrow M \rightarrow 0 \) is proper exact sequence with \( \alpha \) being \( k \)-regular, then by (ii) \( \text{Hom}(P, N) \rightarrow \text{Hom}(P, M) \rightarrow 0 \) is proper exact. Therefore \( P \) is \( k \)-projective.

(i)\( \Rightarrow \) (ii). It is similar to the proof of Theorem 3.5 and using [2, Theorem 8]. \( \Box \)

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**References**


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As a multidisciplinary field, financial engineering is becoming increasingly important in today’s economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions. However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithms, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based systems, and web-based systems).

This special issue will include (but not be limited to) the following topics:

- **Computational methods**: artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning
- **Application fields**: asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects**: decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

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