

ON QUASI-CONVEX FUNCTIONS AND RELATED TOPICS

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ABSTRACT. Let S be the class of functions f which are analytic and univalent in the unit disc E with $f(0) = 0$, $f'(0) = 1$. Let C , S^* and K be the classes of convex, starlike and close-to-convex functions respectively. The class C^* of quasi-convex functions is defined as follows:

Let f be analytic in E and $f(0) = 0$, $f'(0) = 1$. Then $f \in C^*$ if and only if there exists a $g \in C$ such that, for $z \in E$

$$\operatorname{Re} \frac{(zf'(z))'}{g'(z)} > 0.$$

In this paper, an up-to-date complete study of the class C^* is given. Its basic properties, its relationship with other subclasses of S , coefficient problems, arc length problem and many other results are included in this study. Some related classes are also defined and studied in some detail.

KEY WORDS AND PHRASES. Univalent convex, alpha-convex, quasi-convex, alpha quasi-convex, close-to-convex, arclength, coefficient, radius of convexity, order β type γ , Livingston's and Libera's operators.

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1. INTRODUCTION.

Denote by S the class of functions f which are analytic and univalent in the unit disc E and satisfy $f(0) = 0$, $f'(0) = 1$. The subclasses S^* and C of starlike and convex functions respectively are well-known and have been extensively studied, see [1], [2] and [3]. A function f is said to be in S^* if and only if for $z \in E$

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0. \quad (1.1)$$

S^* and C are related by the Alexander relation [4], that is

$$f \in C \text{ if, and only if } zf' \in S^* \quad (1.2)$$

Hence a function f is said to be in C , if and only if for $z \in E$

$$\operatorname{Re} \frac{(zf'(z))'}{f'(z)} > 0 \quad (1.3)$$

The subclass K of S consisting of close-to-convex functions is also well known [5] and many properties of S^* can be extended to the wider class K . A function f is said to be in K if and only if there exists a convex function g such that, for $z \in E$

$$\operatorname{Re} \frac{f'(z)}{g'(z)} > 0 \quad (1.4)$$

Since $G=zf'$ is starlike for g convex, (1.4) can be written as

$$\operatorname{Re} \frac{zf'(z)}{G(z)} > 0 \quad (1.5)$$

for $z \in E$ and $G \in S^*$. Taking $G(z)=f(z)$ in (1.5) one sees that $S^* \subset K$, which shows that

$$C \subset S^* \subset K$$

2. QUASI-CONVEX FUNCTIONS.

We proceed now to define and discuss a subclass of S which is related to K by an Alexander type relation [6].

DEFINITION 2.1. Let f be analytic in E with $f(0) = 0$, $f'(0) = 1$. Then f is said to be quasi-convex in E if and only if there exists a convex function g with $g(0) = 0$, $g'(0) = 1$ such that, for $z \in E$

$$\operatorname{Re} \frac{(zf'(z))'}{g'(z)} > 0 \quad (2.1)$$

Denote the class of quasi-convex functions by C^* .

It is clear that, when $f(z)=g(z)$ and $g \in C$, then (2.1) holds. Hence $C \subset C^*$. We show now that $C^* \subset K$, so that every quasi-convex function is univalent.

THEOREM 2.1. Let $f \in C^*$. Then, for $z \in E$,

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \quad g \in C$$

and so $C^* \subset K \subset S$, thus, every quasi-convex function is close-to-convex and hence univalent in E .

PROOF. A result of Libera [7] shows that, if s and t are functions analytic in E with $s(0)=t(0)=0$ and teS^* , then for $z \in E$,

$$\operatorname{Re} \frac{s'(z)}{t'(z)} > 0 \quad \rightarrow \quad \operatorname{Re} \frac{s(z)}{t(z)} > 0$$

An immediate application of this with $s(z)=zf'(z)$ and $t(z)=g(z)$ proves the theorem.

It follows at once from the definition (2.1) that

$$f \in C^* \quad \text{if and only if} \quad zf' \in K \tag{2.2}$$

We now extend some results to the class C^* which are known to be true for C , see [1], [2] and [3].

THEOREM 2.2. Let $f \in C^*$ with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then, for $|z| = r < 1$,

$$(i) \quad |a_n| < 1, \quad n=2,3,\dots$$

$$(ii) \quad \frac{1}{(1+r)^2} < |f'(z)| < \frac{1}{(1-r)^2}$$

$$(iii) \quad \frac{r}{1+r} < |f(z)| < \frac{r}{1-r}$$

$$(iv) \quad |\omega| > \frac{1}{2}, \quad \text{where } f(z) \neq \omega \quad \text{in } E.$$

All inequalities are sharp, equality being attained for

$$f_0(z) = \frac{z}{1-z} \tag{2.3}$$

PROOF. (i) Since $f \in C^*$, there exists a convex function g with $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, such that, for $z \in E$

$$\frac{(zf'(z))'}{g'(z)} = h(z), \quad \text{where } \operatorname{Re} h(z) > 0,$$

and $h(z) = \sum_{n=0}^{\infty} c_n z^n, \quad c_0 = 1.$

So

$$(zf'(z))' = g'(z)h(z)$$

Equating the coefficient of z^{n-1} on both sides, we have

$$n^2 a_n = n b_n + (n-1) c_1 b_{n-1} + (n-2) c_2 b_{n-2} + \dots + 2c_{n-2} b_2 + c_{n-1}$$

i.e.

$$n^2 |a_n| < n |b_n| + (n-1) |c_1| |b_{n-1}| + \dots + 2 |c_{n-2}| |b_2| + |c_{n-1}|. \tag{2.4}$$

Now, from the known results [7], $|b_n| < 1, n=2,3,\dots$, and $|c_n| < 2, n=1,2,3,\dots$, we have

$$n^2 |a_n| < n + \frac{2\{n(n-1)\}}{2} = n^2, \quad n=2,3,\dots \tag{2.4}$$

and this implies $|a_n| < 1, n=2,3,\dots$

(ii) $| (zf'(z))' | = |g'(z)h(z)|$

Using known [1,2 and 3] distortion theorems for the functions g and h , we have

$$\frac{1-r}{(1+r)^3} < |(zf'(z))'| < \frac{1+r}{(1-r)^3} \tag{2.5}$$

Integrating the right hand side of (2.5) from 0 to z , we obtain

$$\begin{aligned} |zf'(z)| &< \int_0^z |(zf'(z))'| dz \\ &< \int_0^r \frac{1+r}{(1-r)^3} dr = \frac{r}{(1-r)^2} \end{aligned}$$

In order to obtain a lower bound for $|f'(z)|$, we proceed as follows. Let d_1 be the radius of the open disc contained in the map of E by zf' . Let z_0 be the point of $|z|=r$ for which $|zf'(z)|$ assumes its minimum value. This minimum increases with $\{r$ the image of $|z| < r$ by $w=f'(z)$ expands} and is less than d_1 . Hence the linear-segment connecting the origin with the point $z_0 f'(z_0)$ will be covered entirely by the values of $zf'(z)$ in E . Let λ be the arc in E which is mapped by $w=zf'(z)$ onto this linear-segment. Then

$$\begin{aligned} |zf'(z)| &= \int_{\lambda} |(zf'(z))'| d|z| \\ &> \int \frac{1-r}{(1+r)^3} dr \\ &= \frac{r}{(1+r)^2} \end{aligned}$$

Integrating (ii), we obtain (iii) and by letting $r \rightarrow 1$ in the lefthand side of (iii), we have (iv).

Waadeland [8] proved that every starlike m -fold symmetric function g , with $g(z) = z + \sum_{k=1}^{\infty} b_{mk+1} z^{mk+1}$ satisfies

$$\begin{aligned} |b_{mk+1}| &< \binom{\frac{2}{m} + k-1}{k} \\ &\approx \frac{1}{m^k} k^{\frac{2}{m} - 2} \end{aligned} \tag{2.6}$$

In order to extend this result to C^* , we need only to extend Waadeland's result to K and then use the relationship between C^* and K . However this extension to K was done by Pommerenke [9] and so (2.6) is true for $f \in C^*$.

The following result for the class C^* follows exactly in the same way as for the class C in [10].

THEOREM 2.3. Let $f \in C^*$ with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$. Let $g(z) \prec f(z)$. Then, for all n ,

$$S_n\left(\frac{1}{2}\right) z \prec f(z),$$

where

$$S_n(z) = z + \sum_{k=2}^n b_k z^k$$

(\prec means "subordinate to")

Clunie and Keogh [10] showed that if $f \in C$ with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $f(E)$ has definite area then $n a_n = o(1)$ as $n \rightarrow \infty$. This result has been extended to C^* in [6] as follows.

THEOREM 2.4. Let $f \in C^*$ with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. If $f(E)$ has finite area, then $n a_n = o(1)$ as $n \rightarrow \infty$, the index of n being best possible.

Denote by $C(r)$ the closed curve which is the image of $f(E_r)$ and by $L(r)$ the length of $C(r)$. We prove:

THEOREM 2.5 [6]. Let $f \in C^*$. Then, for $0 < r < 1$,

$$2\sqrt{(\pi A(r))} \leq L(r) \leq \sqrt{2(\pi A(\sqrt{r}))} \left(\log \frac{1}{1-r}\right)^{1/2} \tag{2.7}$$

Further, if $A(r) < \infty$ for $0 < r < 1$, then

$$L(r) = o(1) \left(\log \frac{1}{1-r}\right) \text{ as } r \rightarrow 1 \tag{2.8}$$

The convex function $f(z) = \log \frac{1}{1-r}$ shows that the factor $\left(\log \frac{1}{1-r}\right)^{1/2}$ in (2.7) is the best possible.

PROOF. The left hand inequality follows at once from the isoperimetric inequality. Since $f \in C^*$, $F(z) = z f'(z)$ is close-to-convex. Thus

$$\begin{aligned} L(r) &= \int_0^{2\pi} |z f'(z)| d\theta = \int_0^{2\pi} |F(z)| d\theta \\ &< 2\pi \int_0^r M(\rho, z f') \frac{d\rho}{\rho}, \text{ see [1, p. 45]} \end{aligned}$$

$$\begin{aligned}
 &< 2\pi \sum_{n=1}^{\infty} |a_n| r^n && (2.9) \\
 &< 2\pi \left(\sum_{n=1}^{\infty} n |a_n|^2 r^n \right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{r^n}{n} \right)^{1/2} \\
 &= \overline{2\sqrt{(\pi A(\sqrt{r}))}} \quad \left(\log \frac{1}{1-r} \right)^{1/2}
 \end{aligned}$$

We can show (2.8) easily by taking $A(r) < \infty$.

REMARK 2.1. For $f \in C$, it is well-known that $L(r) < 2\pi M(r)$. It follows from (2.7) that for $f \in C^*$, $L(r) = O(1) M(r) \left(\log \frac{1}{1-r} \right)^{1/2}$ as $r \rightarrow 1$. The question of whether the factor $\left(\log \frac{1}{1-r} \right)^{1/2}$ can be removed is still open.

It is well-known [11] that

$$\operatorname{Re} \frac{(zf'(z))'}{f'(z)} > 0 \quad \rightarrow \quad \operatorname{Re} \frac{zf'(z)}{f(z)} > \frac{1}{2}, \quad z \in E,$$

that is, every convex function is starlike of order $\frac{1}{2}$. It is natural to ask if such a relationship exists between C^* and K . The following example shows that this is not in fact the case.

EXAMPLE 2.1. Take $f(z) = z$, $g(z) = \frac{z}{1-\alpha z}$, $\frac{1}{2} < \alpha < \frac{1}{\sqrt{2}}$.

Then

$$\operatorname{Re} \frac{(zf'(z))'}{g'(z)} = \operatorname{Re}(1-\alpha z)^2 > 0, \quad z \in E,$$

but

$$\operatorname{Re} \frac{zf'(z)}{g(z)} = \operatorname{Re}(1-\alpha z),$$

and

$$\operatorname{Inf} \operatorname{Re} \frac{zf'(z)}{g(z)} < \frac{1}{2} \quad \text{for} \quad \frac{1}{2} < \alpha < \frac{1}{\sqrt{2}}, \quad z \in E.$$

Now, following the same method as in [12], we have

THEOREM 2.6. Let $f \in C^*$ and $g(z) = \frac{f'(z)}{f'(-z)}$. Then $\operatorname{Re}(g(z))^{1/3} > 0$, for $z \in E$.

This result is sharp as can be seen from the function

$$zf'_1(z) = [z(1-\mu z)] / [(1+z)^2],$$

where $\mu = (\cos \gamma)e^{i\gamma}$, $0 < \gamma < \pi$, and $f_1 \in C^*$ with respect to the convex function $\phi(z) = \frac{[-ie^{i\gamma}z]}{(1+z)}$.

3. RELATIONSHIP OF C^* WITH OTHER SUBCLASSES OF S .

(1) The class C .

The class C of convex functions is a proper subclass of C^* . In fact if we write

$$F(z) = \frac{f[(x+z)/(1+\bar{x}z)] - f(x)}{f'(x) (1-|x|^2)}, \quad x \in E, z \in E,$$

where $f(z) = \frac{z}{(1-z)^2}$,

then the function $f_*(z)$ defined in E by

$$f_*(z) = \int_0^z \frac{F(\xi)}{\xi} d\xi$$

belongs to C^* but not to C , see [6] for more details.

(ii) The Class S^* .

The class C^* , while a proper subclass of the class K of close-to-convex functions, is not contained in S^* . For example, the function

$$f(z) = \frac{1-i}{2} \frac{z}{1-z} - \frac{1+i}{2} \log(1-z)$$

belong to C^* but for sufficiently small ϵ , $\text{Re} \frac{zf'(z)}{f(z)} < 0$, when $z = e^{i\theta}$, $-\epsilon < \theta < 0$.

This means f is not starlike. Also the Koebe function $k(z) = \frac{z}{(1-z)^2}$ is starlike but does not belong to C^* . It is clear from the coefficient result and the distortion theorems for the class C^* .

(iii) The Class R_α .

The class R_α of univalent functions was introduced by Reade [13] and studied by Pommerenke [9]. We define R_α as follows:

An analytic function f with $f'(z) \neq 0$ belongs to R_α , $0 < \alpha < 1$, if and only if

$$\int_{\theta_1}^{\theta_2} \text{Re} \left\{ \frac{(zf'(z))'}{f'(z)} \right\} d\theta > -\alpha\pi,$$

for all θ_1, θ_2 such that $0 < \theta_1 < \theta_2 < 2\pi$, $0 < r < 1$.

Before establishing a relationship between C^* and R_α , we need the following necessary condition for C^* .

THEOREM 3.1. Let $f \in C^*$ and $z = re^{i\theta}$, $0 < r < 1$. Then, for $0 < \theta_1 < \theta_2 < 2\pi$,

$$\int_{\theta_1}^{\theta_2} \text{Re} \left\{ \frac{(zf'(z))'}{f'(z)} \right\} d\theta > -\pi + \frac{\theta_2 - \theta_1}{2} \tag{3.1}$$

PROOF. It has been proved in [14] that for $zf' = F \in K$, and for all θ_1, θ_2 such that $0 < \theta_1 < \theta_2 < 2\pi$

$$-\pi + \frac{\theta_2 - \theta_1}{2} < \int_{\theta_1}^{\theta_2} \text{Re} \left\{ \frac{re^{i\theta} F'(re^{i\theta})}{F(re^{i\theta})} \right\} d\theta < 2\pi + \frac{\theta_2 - \theta_1}{2}$$

using this and the fact that $f \in C^*$ if and only if $zf' \in K$, we obtain the required result.

We note that in (3.1), $\frac{\theta - \theta_1}{2}$ can be very small and we can take $\frac{\theta - \theta_1}{2} = \epsilon\pi$, where $0 < \epsilon < 1$. Thus we conclude that

$$C^* \subset R_\alpha,$$

for some α , ($0 < \alpha < 1$).

REMARK 3.1. It is an open problem to find the exact value of $\alpha \in (0,1)$ that goes with C^* . It should be some fixed number determined by C^* .

(iv) The class of functions convex in one direction.

Robertson [15] introduced the class C_1 of convex functions in one direction. These are the functions for which the intersection of the image region with each line of certain fixed direction is either empty or one interval. He has also shown that if f has real coefficients, then $f \in C_1$ if and only if $zf' \in T$, where T is the class of typically real functions, that is, the functions with real coefficients.

We prove the following:

THEOREM 3.2. If $f \in C^*$ in E and has real coefficients, then it is convex in one direction.

PROOF. Let $C^*(R)$, $K(R)$ and $C_1(R)$ be the classes of functions which are in C^* , K and C_1 respectively and have real coefficients. Let $f \in C^*(R)$. This implies $zf' \in K(R)$. But $K(R) \subset T$. Hence $zf' \in T$ and so $f \in C_1(R)$. Hence $C^*(R) \subset C_1(R)$ and this proves the theorem.

From Theorem 3.2 and the results for the class $C_1(R)$ in [15], we have:

THEOREM 3.3. Let $f \in C^*(R)$. Then

$$(1) \quad \operatorname{Re} \frac{f(z)}{z} > \frac{1}{2} \quad \text{and} \quad \frac{f(z)}{z} \text{ is subordinate to } (1+z)^{-1}.$$

$$(ii) \quad \frac{1 + |a_2|r}{1 + 2|a_2|r + r^2} < \operatorname{Re} \left\{ \frac{f(re^{i\theta})}{re^{i\theta}} \right\} < \frac{1 + |a_2|r}{1 - r^2},$$

where $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$.

(iii) $L(r) < \frac{2\pi r}{(1-r^2)}$, where $L(r)$ is the length of the closed curve $f(E_r)$. The equality is obtained for $f(z) = \frac{z}{1-z}$.

$$\left. \begin{aligned} (iv) \quad & \left| \arg \frac{f(z)}{z} \right| < \arcsin |z| \\ & \text{and} \\ & \left| \arg f'(z) \right| < 2 \arcsin |z| \end{aligned} \right\} , z \in E$$

(v) $f \in C^*(R)$, where

$$F(z) = \int_0^1 f(tz)d\phi(t) = z + \sum_{n=2}^{\infty} \mu_n a_n z^n,$$

$\phi(t)$ is any real function monotonic increasing in the interval $(0,1)$ and the moments sequence $\{\mu_n\}$ is given by

$$\mu_n = \int_0^1 t^n d\phi(t), \mu_1 = 1$$

Thus we have seen that

$$(1) \quad C \subset C^* \subset R_\alpha \subset K \subset S$$

$$(2) \quad C^*(R) \subset C(R) \subset T$$

We now discuss the relationship of C^* with other subclasses in a different way. We have the following:

THEOREM 3.4. Let $f \in C^*$ in E . Then f maps $|z| < r = 4\sqrt{2} - 5 \approx 0.6568$ onto a convex domain, and this result is sharp.

This follows at once from the result of Lewandowski [16] where he proved that the exact radius r_1 such that the image of $|z| < r_1$ by $f \in K$ is a starshaped domain (with respect to the origin) is

$$r_1 = 4\sqrt{2} - 5 \approx 0.6568$$

We see that, from this result $f \in C^* \rightarrow zf' \in K \rightarrow zf' \in S^*$ for $|z| < 4\sqrt{2} - 5 \rightarrow f \in C$ for $|z| < 4\sqrt{2} - 5$.

Lewandowski's method yields the existence of an extremal function which maps E onto the w -plane cut along a half-line not passing through the origin consequently we have the extremal function for theorem 3.4.

THEOREM 3.5. Let $f \in C^*$ and $g \in C$ in E . If $\text{Re} \frac{zf'(z)}{g(z)} > 0, z \in E$, then $\text{Re} \frac{(zf'(z))'}{g'(z)} > 0$, for $|z| < \frac{1}{3}$ For the proof, see [17].

4. APPLICATIONS OF THE CLASS C^* .

(a) The Class K_1 .

We now introduce a new class K_1 by replacing convex function g in (1.4) with quasi-convex function. This generalizes the concept of quasi-convexity and close-to-convexity both.

DEFINITION 4.1. Let f be analytic in E and $f(0) = 0, f'(0) = 1$. Then $f \in K_1$, if and only if, there exists a $g \in C^*$ such that for $z \in E$,

$$\text{Re} \frac{f'(z)}{g'(z)} > 0.$$

Clearly $C^* \subset K \subset K_1$.

We state some basic properties of the class K_1 . We refer to [18] for the proofs.

THEOREM 4.1. Let $f \in K_1$ and be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then

- (i) $|a_n| < n$, for all n .
 - (ii) $\frac{(1-r)}{(1+r)^3} < |f'(z)| < \frac{(1+r)}{(1-r)^3}$
 - (iii) $\frac{r}{(1+r)^2} < |f(z)| < \frac{r}{(1-r)^2}$
 - (iv) $|w| > \frac{1}{4}$, where $f(z) \neq w$ in E .
- } for $|z| = r < 1$

All inequalities are sharp, equality being attained for

$$f_0(z) = \frac{z}{(1-z)^2} \in K_1$$

(v) $||a_{n+1}| - |a_n|| = o(1)$, for all n ,

where $O(1)$ denotes a constant.

(vi) $L(r) = o(1) [A(\sqrt{r}) \frac{1}{1-r}]^{1/2}$, $0 < r < 1$.

The question whether the factor $(\frac{1}{1-r})^{1/2}$ can be improved is unsettled and remains open.

(vii) For $f \in K_1$ implies that zf' is univalent in $|z| < \frac{1}{5}$.

(b) Alpha-quasi-convex functions.

Mocanu [19] introduced the class M_α of alpha-convex functions as follows:

Let α be real and suppose that $f: f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in E with

$f(z).f'(z) \neq 0$. Then $f \in M_\alpha$ if, for $z \in E$,

$$\text{Re} \left\{ (1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \frac{(zf'(z))'}{f'(z)} \right\} > 0.$$

It has been shown [20] that all α -convex functions are univalent and starlike and they unify the classes of starlike ($\alpha=0$) and convex ($\alpha=1$) functions.

Using the concept of quasi-convexity, we now define the following:

DEFINITION 4.2. Let α be real and $f: f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in E .

Then f is said to be alpha-quasi-convex, if and only if there exists a convex function g such that, for $z \in E$

$$\operatorname{Re} \left\{ (1-\alpha) \frac{f'(z)}{g'(z)} + \alpha \frac{(zf'(z))'}{g'(z)} \right\} > 0.$$

We denote the class of α -quasi-convex functions as Q_α . We note that $Q_0=K$ and $Q_1=C^*$. Thus alpha-quasi-convex functions connect the classes K and C^* in the same way as alpha-convex functions do S^* and C .

In [21], we proved:

THEOREM 4.2.

(i) Let $F(z) = (1-\alpha) f(z) + \alpha zf'(z)$, and α be real, $\alpha > 0$, $z \in E$. Then $f \in Q_\alpha$, if and only if, $F \in K$.

(ii) $f \in Q_\alpha$, if and only if, for $\alpha > 0$, there exists a close-to-convex function F such that, for $z \in E$

$$f(z) = \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \int_0^z t^{-2+\frac{1}{\alpha}} F(t) dt$$

(iii) Every α -quasi-convex function, for $0 < \alpha < 1$ is close-to-convex.

(iv) Let $F \in K$ in E . Then F will be α -quasi-convex in $|z| < r_0 = 1/(2\alpha + \sqrt{4\alpha^2 - 2\alpha + 1})$. This result is sharp.

(v) Let $f \in Q_\alpha$ and be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then, for $n > 2$,

$$|a_n| < \frac{n}{1+\alpha(n-1)}$$

This result is sharp as can be seen from the function

$$f_0(z) = \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \int_0^z t^{\frac{1}{\alpha}-1} (1-t)^{-2} dt$$

(c) The class $C^*(\beta, \gamma)$ of quasi-convex functions of order β type γ .

A function $f \in S$ is called a convex function of order β , $0 < \beta < 1$ if, for $z \in E$,

$$\operatorname{Re} \frac{(zf'(z))'}{f'(z)} > \beta,$$

We note this class by $C(\beta)$.

Also $f \in S$ is a starlike function of order β , $0 < \beta < 1$, for $z \in E$

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \beta,$$

and we call this class as $S^*(\beta)$. These two classes were introduced by Robertson [22].

In [23], Libera introduced the close-to-convex functions of order β type γ .

DEFINITION 4.3. A function f analytic in E , normalized by the conditions $f(0)=0, f'(0)=1$, is said to be close-to-convex of order β type γ where $0 < \beta < 1$ and $0 < \gamma < 1$, if and only if there exists a function $g \in S^*(\gamma)$ such that, for $z \in E$

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > \beta.$$

We denote such a class of functions as $K(\beta, \gamma)$. It is clear that $K(0,0)=K$.

We now introduce terminology of order and type together in the class C^* as:

DEFINITION 4.4. A function f , analytic in E , normalized by the conditions $f(0)=0, f'(0)=1$, is said to be quasi-convex of order β type γ , if and only if there exists a function $g \in C(\gamma)$ such that for $z \in E$

$$\operatorname{Re} \frac{(zf'(z))'}{g'(z)} > \beta,$$

where $0 < \beta < 1$ and $0 < \gamma < 1$. We call such a class as $C^*(\beta, \gamma)$. Clearly $C^*(0,0)=C^*$.

We shall now state some results on the class $C^*(\beta, \gamma)$. For the proofs, we refer to [24].

THEOREM 4.3. Every quasi-convex function of order β type γ is close-to-convex of the same order and hence univalent.

REMARK 4.1. From the definition of $C^*(\beta, \gamma)$, we can see that an Alexander-type relation holds between the classes $C^*(\beta, \gamma)$ and $K(\beta, \gamma)$, that is

$$f \in C^*(\beta, \gamma) \text{ if and only if } zf' \in K(\beta, \gamma). \tag{4.1}$$

THEOREM 4.4. Let $f \in C^*(\beta, \gamma)$ and be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then we have

$$(i) \quad |a_n| < \frac{2(3-\gamma) \dots (n-2\gamma)[n(1-\beta) + (\beta-\gamma)]}{n \cdot n!}$$

$$(ii) \quad \frac{1}{r} \int_0^r \frac{(1-r)dr}{(1+r)^{2-2\gamma} [1+(1-2\beta)r]} < |f'(z)|$$

$$< \begin{cases} r(1-\gamma)(1-2\beta) + (\beta-\gamma)[1-(1-r)^{2-2\gamma}] \\ \quad \quad \quad \gamma \neq 2, \gamma \neq 1 \\ (1-2\beta) \frac{1}{r} \log(1-r) + \frac{2(1-\beta)}{r}, \gamma = \frac{1}{2} \\ 2(\beta-1) \frac{1}{r} \log(1-r) + (2\beta-1), \gamma = 1, \end{cases}$$

where $|z|=r, 0 < r < 1$. The first result and right-hand side of the second are sharp.

THEOREM 4.5. Let $f \in S$ and $g \in C$. Let $\operatorname{Re} \frac{zf'(z)}{g(z)} > \beta$, for $z \in E$. Then $f \in C^*(\beta, 0)$ for $|z| < r_0 = \frac{1}{3}$, and $f \in C^*$ for $|z| < r_1 = \frac{1}{3-4\beta}$.

We can also define the class $Q(\alpha, \beta, \gamma)$ of alpha-quasi-convex functions of order β type γ as:

DEFINITION 4.6. Let $\alpha > 0$. A function f , analytic in E , is said to be alpha-quasi-convex of order β type γ , if and only if there exists a function $g \in C(\gamma)$ such that

$$\operatorname{Re} \left[(1-\alpha) \frac{f'(z)}{g'(z)} + \alpha \frac{(zf'(z))'}{g'(z)} \right] > \beta,$$

for $z \in E$, and $\beta, \gamma \in [0, 1]$. We denote this class of functions as $Q(\alpha, \beta, \gamma)$. It is clear that $Q(\alpha, 0, 0) = Q_\alpha$. For different values of α, β , and γ , we have

$$\begin{aligned} Q(0, \beta, \gamma) &= K(\beta, \gamma) & Q(0, 0, 0) &= K \\ Q(1, \beta, \gamma) &= C^*(\beta, \gamma), & Q(1, 0, 0) &= C^* \end{aligned}$$

We notice that this class unifies the two classes $K(\beta, \gamma)$ and $C^*(\beta, \gamma)$, and it follows from the definition that

$$f \in Q(\alpha, \beta, \gamma) \text{ if and only if } \{(1-\alpha)f + \alpha zf'\} \in K(\beta, \gamma)$$

Integral representation and coefficient problem can be solved in the same way as we did for the class Q_α .

(d) OPERATORS ON THE CLASSES $C^*(\beta, \gamma)$.

Let $f = T(F)$, where T is an operator. Now we shall be dealing with the mapping properties of f when $F \in C^*(\beta, \gamma)$, $\beta, \gamma \in [0, 1]$ and T is a differential or integral operator.

Here we shall discuss the case when T is an integral operator I . In [7], Libera considered the operator $I: H_0 \rightarrow H_0$,

$$H_0(E) = \{f \mid f(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and analytic in } E\}$$

where for $F \in H_0$, $I(F) = f$ and

$$f(z) = \frac{2}{z} \int_0^z F(t) dt.$$

He proved that

$$\left. \begin{aligned} I(C) &\subset C \\ I(S^*) &\subset S^* \\ I(K) &\subset K \end{aligned} \right\} \tag{4.2}$$

A generalization of (4.2) has been considered in [25] by taking operator I_n defined as $I_n : H_0 \rightarrow H_0$, $I_n(F) = f$ and

$$f(z) = nz^{-n+1} \int_0^z t^{n-2} F(t) dt, \quad n=1,2,3,\dots \tag{4.3}$$

A simple proof of (4.2) is given by Mocanu, see [19], where it is also shown that

$$\left. \begin{aligned} I(S^*) &\subset S^*(\gamma) \\ I(C) &\subset C(\gamma) \end{aligned} \right\} \tag{4.4}$$

where $\gamma = (-3 + \sqrt{17})/4$ and is the same in both expressions.

Pascu [26] considered the operator I_λ , $0 < \lambda < 1$, $I : H_0 \rightarrow H_0$, $I_\lambda(F) = f$,

$$f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \frac{1}{z^{\frac{1}{\lambda}-2}} F(z) dz \tag{4.5}$$

which generalizes the results in (4.2) and (4.4).

In [27], Salagean studied the operators (4.5) for the classes $S^*(\gamma)$ and $C(\gamma)$. By using the same techniques used in [26] we obtain the similar results for the classes $K(\beta, \gamma)$ and thus consequently we have:

THEOREM 4.6. Let $0 < \lambda < 1$. Let f be defined by (4.5) where $F \in C^*(\beta, \gamma)$, $\beta, \gamma \in [0, 1]$. Then $f \in C^*(\mu, \sigma)$ where $\beta < \mu < 1$ and σ is defined as follows:

(i) If $0 < \lambda < \frac{1}{2}$ and $\frac{\lambda}{2(\lambda-1)} < \gamma < 1$, then

$$\sigma = \sigma_1 = [2\lambda\gamma + \lambda - 2 + \sqrt{4\lambda^2\gamma^2 - 12\lambda^2\gamma + 8\lambda\gamma + 9\lambda^2 - 4\lambda + 4}] / 4\lambda > 0$$

and

(ii) If $\frac{1}{2} < \lambda < 1$ and $\frac{\lambda-1}{2\lambda} < \frac{3\lambda-\sqrt{8\lambda}}{2\lambda} < \gamma$, then

$$\sigma = \sigma_2 = [2\lambda\gamma + \lambda - \sqrt{4\lambda^2\gamma^2 - 12\lambda^2\gamma + 9\lambda^2 - 8\lambda}] / 4\lambda > 0$$

and

(iii) If $\frac{1}{2} < \lambda < 1$ and $\frac{\lambda-1}{2\lambda} < \frac{3\lambda-\sqrt{8\lambda}}{2\lambda} < \gamma < 1$, then $\sigma = \sigma_1$

Special Cases:

(i) For $\lambda = \frac{1}{2}$, $\gamma = 0$, $\beta = 0$, we obtain a known result for the class C^* , see [16].

(ii) For $\beta = 0$, $\gamma = 0$, we see that

$$F \in C^* \rightarrow f \in C^*$$

THEOREM 4.7. Let $0 < \lambda < 1$ and $0 < \beta < 1$. Let f be given by (4.5) and $f \in Q(\alpha, \beta, \gamma)$ where $0 < \gamma < 1, \alpha > 0$. Then $f \in Q(\alpha, \beta, \gamma)$.

For the proofs of the above results, we refer to [28].

Special Cases:

- (1) For $\lambda = \frac{1}{2}, \gamma = 0, \beta = 0$, we obtain this result for the class Q_α , see [21].
- (ii) For $\gamma = 0, \beta = 0$, we see that

$$f \in Q_\alpha \rightarrow f \in Q_\alpha$$

REMARK 4.2. Using the integral representation of $f \in Q(\alpha, \beta, \gamma)$ and theorem 4.7, we notice that, for $0 < \alpha < 1$

$$Q(\alpha, \beta, \gamma) \subset K(\beta, \gamma).$$

In [29], Livingston has studied the converse question considered by Libera [7]. In fact he studied the mapping properties of the function f defined by

$$f(z) = D(F(z)) = \frac{1}{2} (zF(z))', \tag{4.6}$$

where D is a differential operator and F is one of the subclasses of S . For example he has proved that if $F \in S^*$, then f , given by (4.6), is starlike for $|z| < \frac{1}{2}$ and, in general, in no larger disc centered at the origin.

Padmanabhan [30] has refined the results of Livingston by imposing further restrictions on the character of F . His main theorem shows that if $F \in S^*(\gamma)$, for $0 < \gamma < \frac{1}{2}$, then f , defined by (4.6), is starlike of the same order for $|z| < \{(\gamma - 2) + (\gamma^2 + 4)^{1/2}\} / 2\gamma$. He obtains analogous result when F is a convex function of order γ . Libera and Livingston [31] extended and generalized the results of Padmanabhan in the following ways. They extended to include the range of γ when $\frac{1}{2} < \gamma < 1$ and generalized by finding, the sharp radius of the disc in which $\{ \operatorname{Re} \frac{zf'(z)}{f(z)} > \sigma \}$ when $F \in S^*(\gamma), 0 < \gamma < 1, 0 < \sigma < 1$ and $\sigma > \gamma$. They were not able to obtain suitable results for the complimentary case when $\sigma < \gamma$, but in [32] Bajpai and Singh gave a method which covers both of the cases and their result is the best possible.

We can generalize the Livingston differential operator D as following:

$$D_\lambda(f) = f(z) = (1 - \lambda) F(z) + \lambda zF'(z), \tag{4.7}$$

where $\lambda > 0$ and $z \in E$. The mapping properties of the function f , when F is in one of the subclasses of S have been studied in [33].

We generalize Libera and Livingston's result by replacing Livingston's operator (4.6) by the operators (4.7) and have the following:

THEOREM 4.8. Let $0 < \beta < 1, 0 < \gamma < 1, \gamma < \sigma < 1$ and $\beta < \mu < 1$. Let $F \in C^*(\beta, \gamma)$ and f be given by (4.7). Then $f \in C^*(\mu, \sigma)$ for $|z| < r_1$ where r_1 is given as

$$r_1 = \min (r_0, r_2),$$

where r_0 is the smallest positive root of

$$(1-\sigma) + 2\{(\gamma-\sigma) + \lambda(1-\gamma) (\sigma-2)\}r + (2\gamma-\sigma-1)(1-2\lambda(1-\gamma))r^2 = 0,$$

and r_2 is the smallest positive root of the equation

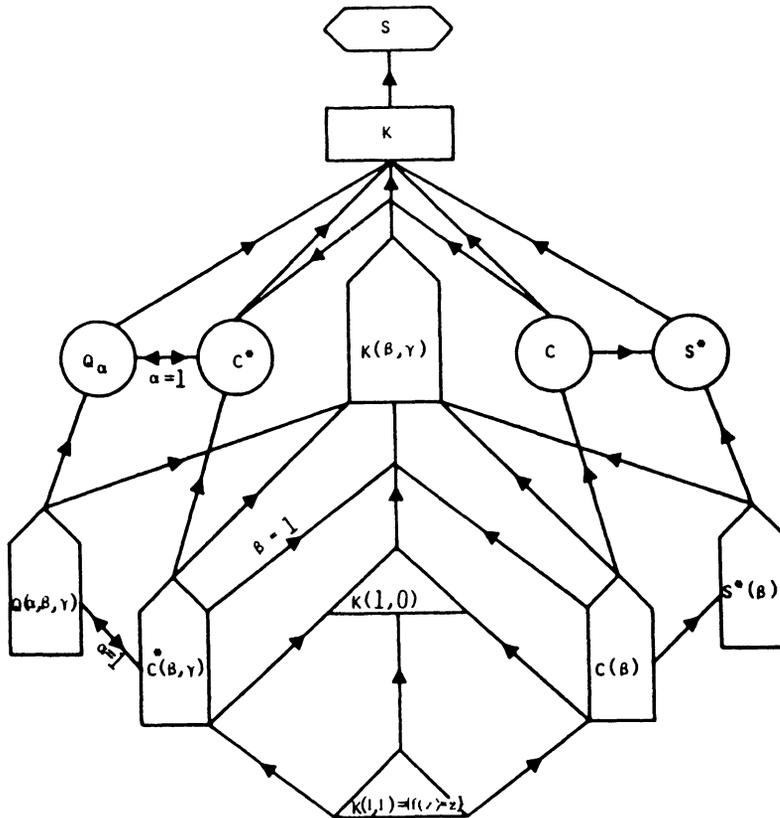
$$[1-(1-2\lambda(1-\gamma))r][\{(1-\mu) + 2\{(\beta-\mu) + \lambda(\beta+\gamma+\mu(1-\gamma) -2)\}r + (2\beta-\mu-1)(1-2\lambda(1-\gamma))r^2\}] = 0$$

For $\beta=\gamma=0$, this result reduces to one for the class C^* , see [34].

THEOREM 4.9. Let $\alpha>0$ and $\lambda>0$. Let $0<\beta, \gamma<1, \beta<\mu<1$ and $\gamma<\sigma<1$. If $f \in Q(\alpha, \beta, \gamma)$ and f is given by (4.7), then $f \in Q(\alpha, \mu, \sigma)$ for $|z|<r_1$ where r_1 is defined as in theorem 4.8.

When $\beta=\gamma=0$, we obtain this result for the class Q_α , see [34]. For the proofs of the above theorems we refer to [35].

We can demonstrate the relationship between all the subclasses of S as follows:



$E: \{z: |z|<1\}, \longrightarrow : \text{Set inclusion, } 0<\alpha<1, \text{ and } \beta, \gamma \in [0, 1].$

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