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ON THE q-KONHAUSER BIORTHOGONAL POLYNOMIALS

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ABSTRACT. Recently, Al-Salam and Verma discussed two polynomial sets $\{Z_n^{(\alpha)}(x,k|q)\}$ and $\{Y_n^{(\alpha)}(x,k|q)\}$, which are biorthogonal on $(0,\infty)$ with respect to a continuous or discrete distribution function. For the polynomials $Y_n^{(\alpha)}(x,k|q)$ the operational formula is derived.

KEY WORDS AND PHRASES. q-Konhauser polynomials, Biorthogonality, q-derivative, q-binomial theorem, q-Laguerre polynomials, Operational formula.
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1. INTRODUCTION.

For $|q| < 1$, let

$$(a;q)_\infty = \prod_{j=o}^{\infty} (1-aq^j)$$

and for arbitrary complex n,

$$(a;q)_n = (a;q)_\infty/(aq^n;q)_\infty \quad ,$$

so that, we have

$$(a;q)_n = \begin{cases} 1, \text{ if } n=0, \\ (1-a)(1-aq)\dots(1-aq^{n-1}), \text{ if } n = 1,2,\dots \end{cases} .$$

For convenience, we shall write $[a]_n$ to mean $(a;q)_n$. If the base is not q but, say p, then we shall mention it explicitly as $(a;p)_n$.

Let δ be the q-derivative defined by means of the following

$$\delta f(x) = \{f(x) - f(qx)\}/x$$

By induction it is fairly easy to verify the relation

$$(x^{k+1}\delta)^n x^\alpha = (q^\alpha;q^k)_n x^{\alpha+nk}. \tag{1.1}$$

Using the q-binomial theorem (Slater [1]),

$$\sum_{n=0}^{\infty} \frac{[a]_n}{[q]_n} x^n = \frac{[ax]_\infty}{[x]_\infty} \quad ,$$

One can easily show that

$$\sum_{n=0}^{\infty} \frac{x^n}{[q]_n} q^{n(n-1)/2} = [-x]_\infty \text{ ; (see Askey [2].)} \tag{1.2}$$

Al-Salam and Verma [3] introduced the following pair of biorthogonal polynomials.

$$Z_n^{(\alpha)}(x,k|q)$$

$$= \frac{[q^{1+\alpha}]_{nk}}{(q^k;q^k)_n} \sum_{j=o}^{n} \frac{(q^{-nk};q^k)_j}{(q^k;q^k)_j \; [q^{1+\alpha}]_{kj}} q^{(1/2)kj(kj-1)+kj(n+\alpha+1)} , \tag{1.3}$$

$$Y_n^{(\alpha)}(x,k|q)$$

$$= \frac{1}{[q]_n} \sum_{r=o}^{n} \frac{x^r}{[q]_r} q^{r(r-1)/2} \sum_{j=o}^{r} \frac{[q^{-r}]_j}{[q]_j} q^j (q^{1+\alpha+j};q^k)_n . \tag{1.4}$$

For $k=1$, both $Z_n^{(\alpha)}(x,k|q)$ and $Y_n^{(\alpha)}(x,k|q)$ get reduced to the q-Laguerre polynomials $L_n^{(\alpha)}(x|q)$ discovered by Hahn [4].

2. OPERATIONAL FORMULA.

In order to obtain operational representation for the polynomials $Y_n^{(\alpha)}(x,k|q)$, we can write from (1.4)

$$Y_n^{(\alpha)}(x,k|q)$$

$$= \frac{1}{[q]_n} \sum_{r=o}^{\infty} \sum_{s=o}^{r} \frac{[q^{-r}]_s}{[q]_r [q]_s} x^r (q^{1+\alpha+s};q^k)_n \, q^{(1/2)r(r-1)+s}$$

$$= \frac{1}{[q]_n} \sum_{r=o}^{\infty} \frac{x^r}{[q]_r} q^{r(r-1)/2} \sum_{s=o}^{\infty} \frac{(-x)^s}{[q]_s} (q^{1+\alpha+s};q^k)_n$$

$$= \frac{[-x]_\infty}{[q]_n} \sum_{s=o}^{\infty} \frac{(-x)^s}{[q]_s} (q^{1+\alpha+s};q^k)_n .$$

This may be put in the form

$$Y_n^{(\alpha)}(x,k|q) = \frac{[-x]_\infty}{[q]_n} \sum_{s=o}^{\infty} \frac{(-x)^s}{[q]_s} x^{-1-\alpha-s-nk} (x^{k+1}\delta)^n \, x^{1+\alpha+s}$$

where property (1.1) of the operator δ is used. Finally, we shall have

$$Y_n^{(\alpha)}(x,k|q) = \frac{1}{[q]_n} x^{-1-\alpha-nk} [-x]_\infty (x^{k+1}\delta)^n \left\{ \frac{x^{1+\alpha}}{[-x]_\infty} \right\} . \tag{2.1}$$

More generally, one can obtain

$$(x^{k+1}\delta)^m \left\{ \frac{x^{1+\alpha+nk}}{[-x]_\infty} Y_n^{(\alpha)}(x,k|q) \right\}$$

$$= [q^{n+1}]_m \frac{x^{1+\alpha+nk+mk}}{[-x]_\infty} Y_{m+n}^{(\alpha)}(x,k|q) . \tag{2.2}$$

For $m=1$, this reduces to a recurrence relation

$$(1-q^{n+1})\, Y_{n+1}^{(\alpha)}(x,k|q) = Y_n^{(\alpha)}(x,k|q) - q^{1+\alpha+nk}(1+x)Y_n^{(\alpha)}(xq,k|q) . \tag{2.3}$$

One notes that (2.1), (2.2) and (2.3) reduce, when $k=1$, to corresponding properties for the q-Laguerre polynomials $L_n^{(\alpha)}(x|q)$.

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