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ON M-IDEALS IN $B(\sum_{i=1}^{\infty} \oplus_p \ell_r^{n_i})$

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ABSTRACT. For $1 < p, r < \infty$, $X = (\sum_{i=1}^{\infty} \oplus_p \ell_r^{n_i})$, $\{n_i\}$ bounded, the space K(X) of all compact operators on X is the only nontrivial M-ideal in the space B(X) of all bounded linear operators on X.

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1. INTRODUCTION.

Since Alfsen and Effros [1] introduced the notion of an M-ideal, many authors have studied M-ideals in operator algebras. It is known that K(X), the space of all compact operators on X, is an M-ideal in B(X), the space of all bounded linear operators on X, if X is a Hilbert space or $\ell_p (1 < p < \infty)$. Smith and Ward [2] proved that M-ideals in a C^*-algebra are exactly the closed two sided ideals. Smith and Ward [3], and Flinn [4] proved that, for $1 < p < \infty$, $K(\ell_p)$ is the only nontrivial M-ideal in $B(\ell_p)$. The purpose of this paper is to generalize this result to B(X), where $X = (\sum_{i=1}^{\infty} \oplus_p \ell_r^{n_i})$, for $1 < p, r < \infty$ and $\{n_i\}$ a bounded sequence of positive integers. In this proof, the ideas and results of [4], [2], [5] and [3] are heavily used.

2. NOTATIONS AND PRELIMINARIES.

If X is a Banach space, B(X) (resp. K(X)) will denote the space of all bounded linear operators (resp. compact linear operators) on X.

A closed subspace J of a Banach space X is an L-summand (resp. M-summand) if there is a closed subspace $\tilde{J}$ of X such that X is the algebraic direct sum of J and $\tilde{J}$, and $\|x + y\| = \|x\| + \|y\|$ (resp. $\|x\| = \max\{\|x\|, \|y\|\}$) for $x \in J$, $y \in \tilde{J}$. A projection $P: X \to X$ is an L-projection (resp. M-projection) if $\|x\| = \|Px\| + \|(I - P)x\|$ (resp. $\|x\| = \{\|Px\|, \|(I - P)x\|\}$ for every $x \in X$.

A closed subspace J of a Banach space X is an M-ideal in X if $J^{\perp} = \{x^* \in X^*: x^*|_J = 0\}$ is an L-summand in X^*.

If $(X_i)_{i=1}^{\infty}$ is a sequence of Banach spaces for $1 \leq P \leq \infty$, $\sum_{i=1}^{\infty} \oplus_p X_i$ is the space of all sequences $x = (x_i)_{i=1}^{\infty}$, $x_i \in X_i$, with the norm $\|x\| = (\sum_{i=1}^{\infty} \|x_i\|^p)^{1/p} < \infty$ if $1 \leq p < \infty$ and $\|x\| = \sup_i\{\|x_i\|\} < \infty$ if $p = \infty$.

An element h in a complex Banach algebra A with the identity e is hermitian if $\|e^{i\lambda h}\| = 1$ for all real λ [6].

If J_1 and J_2 are complementary nontrivial M-summands in A (i.e. $A = J_1 \oplus_{\infty} J_2$), P is the M-projection of A onto J_1 and $z = P(e) \in J_1$, then z is hermitian with $z = z^2$ [2, 3.1], $zJ_i \subseteq J_i$ $(i = 1,2)$ and $zJ_2z = 0$ [2, 3.2 and 3.4]. since I - P is the M-projection of A onto J_2, $e - z = (e - z)^2$ is hermitian, $(e - z)J_i \subseteq J_i$ $(i = 1,2)$ and $(e - z)J_1(e - z) = 0$.

If M is an M-ideal in a Banach algebra A, then M is a subalgebra of A [2, 3.6]. If $h \in A$ is hermitian and $h^2 = e$, then $hM \subseteq M$ and $Mh \subseteq M$ [4, Lemma 1].

If A is a Banach algebra with the identify e, then A^{**} endowed with Arens multiplication is a Banach algebra and the natural embedding of A into A^{**} is an algebra isomorphism into [6]. If J is an M-ideal in A, then $A^{**} = J^{\perp\perp} \oplus_{\infty} (J^{\perp\perp})^{\sim}$ and the associated hermitian element $z \in J^{\perp\perp}$ commutes with every other hermitian element of A^{**} [5 .22].

From now X, will always denote $\sum_{i=1}^{\infty} \oplus_p \ell_r^{n_i}$, where $1 < p, r < \infty$ and $\{n_i\}_{i=1}^{\infty}$ a bounded sequence of posititve integers. An operator $T \in B(X)$ has a matrix representation with respect to the natural basis of X. From the definition, it is obvious that any diagonal matrix $T \in B(X)$ with real entries is hermitian.

Flinn [4] proved that if M is an M-ideal in $B(\ell_p)$ and $h \in B(\ell_p)$ is a diagonal matrix, then $hM \subseteq M$ and $Mh \subseteq M$. His proof is valid for X. He also proved that if M is a nontrivial M-ideal in $B(\ell_p)$, then $M \supseteq K(\ell_p)$. Again his proof with a small modification is valid for X.

Thus we have observed that if M is a nontrivial M-ideal in B(X), then $M \supseteq K(X)$.

If M is an M-ideal in a Banach algebra A and $h \in M$ is hermitian, then $hAh \subseteq M$. Indeed, $(e - z)h = (e - z)^2h = (e - z)h(e - z) = 0 = h(e - z)$ and so $zh = hz = h$. Since $zA^{**}z \subseteq M^{\perp\perp}$ [2: 3.4], $zAz \subseteq M^{\perp\perp}$ and hence $hAh = hzAzh \subseteq M^{\perp\perp}$. Since $h \in M$, $hAh \subseteq A \cap M^{\perp\perp} = M$. Thus if $e \in M$, then $A = M$.

3. MAIN THEOREM.

We may assume that $X = (\ell_r^{m_1} \oplus_p \dots \oplus_p \ell_r^{m_s}) \oplus_p (\ell_r^{n_1} \oplus_p \dots \oplus_p \ell_r^{n_k}) \oplus_p (\ell_r^{n_1} \oplus_p \dots \oplus_p \ell_r^{n_k}) \oplus_p \dots$

Set $\alpha = m_1 + \dots + m_s$ and $\beta = n_1 + \dots + n_k$. Let N be the set of all natural numbers, $S_o = \{1, 2, \dots \alpha\}$ and, for $1 \le j \le k$, $S_j = \bigcup_n (n + \beta N)$, , where n runs over $\alpha + n_o + n_{j-1} < n \le \alpha + n_o + \dots + n_j$, $n_o = 0$. Let P_j be the projection on X defined by $P_j x = 1_{S_j} x$ for every $x \in X$, where i_{S_j} is the indicator function of the set S_j. Let $(e_i)_{i=1}^{\infty}$ be the unit vector basis for X. $A = \sum_{ij} a_{ij} e_j \otimes e_i \in B(X)$ is the operator with matrix (a_{ij}) with respect to $(e_i)_{i=1}^{\infty}$.

LEMMA 1. If M is an M-ideal in B(X) and contains $A = \sum a_{ij} e_j \otimes e_i$ such that $(a_{ii})_{i \geq 1} \in \ell_\infty \setminus c_o$, then $M = B(X)$.

PROOF. By multiplying by diagonal matrices from both sides, and as in Lemma 2 [4], we may assume that $A = \sum_{i=1}^{\infty} e_{f(i)} \otimes e_{f(i)}$, where $f(i+1) - f(i) \geq \beta$, $f(i) \in S_j$ for all i and a fixed $j (1 \le j \le k)$. Fix $\ell (\ell \neq j, 1 \le \ell \le k)$ and s

$(\alpha + n_o + \dots + n_{\ell-1} < s \le \alpha + n_o + \dots + n_\ell)$, and let $g(i) = s + (i-1)\beta$ $(i = 1,2,3\dots)$.

CLAIM: $B = \sum_{i=1}^{\infty} e_{g(i)} \otimes e_{f(i)} \in M$. Suppose $B \notin M$. Choose $\Phi \in M^\perp$ so that $\|\Phi\| = 1 = \Phi(B)$. Since $\|B\| = 1$ and $AB = B$, $\Psi \in B(X)^*$ defined by $\Psi(G) = \Phi(GB)$ has norm one and attains its norm at $A \in M$. Hence $\Psi \in \tilde{M}$ and $\|\Phi + \Psi\| = 2$, where $B(X)^* = M^\perp \oplus_1 \tilde{M}$. Since $|(\Phi + \Psi)(G)| = |\Phi(G + GB)| \le \|\Phi\| \|G\| \|I + B\|$, $\|\Phi + \Psi\| \le \|I + B\|$. To draw a contradiction, we will show that $\|I + B\| < 2$. Let j and ℓ be as above. For $x \in X$ with $\|x\| = 1$, $\|x\|^p = \|P_j x\|^p + \|(I - P_j)x\|^p$. Let $t = \|P_j x\|^p$, then $1 - t = \|(I - P_j)x\|^p$. Since Bx has support in S_j and $\|Bx\| \le \|(I - P_j)x\|$, we have

$$\|(I + B)x\| \le 1 + \|Bx\| \le 1 + (1 - t)^{1/p} \tag{3.1}$$

$\|(I - P_j)x + Bx\| \le (2\|(I - P_j)x\|^p)^{1/p} = 2^{1/p}(1 - t)^{1/p}$. Hence

$$\|(I + B)x\| = \|x + Bx\| \le \|P_j x\| + \|(I - P_j)x + Bx\| \le t^{1/p} + 2^{1/p}(1-t)^{1/p} \tag{3.2}$$

Obviously, $F(t) = t^{1/p} + 2^{1/p}(1 - t)^{1/p}$ is continuous on [0,1] and $F(0) = 2^{1/p} < 2$ so $F(t) < 2$ for all $0 \le t \le \delta$. For $\delta \le t \le 1$, $1 + (1 - t)^{1/p} < 2$. By (3.1) and (3.2) above, $\|(I + B)\| < 2$. Contradiction! Hence $B \in M$.

Similarly $C = \sum_{i=1}^{\infty} e_{f(i)} \otimes e_{g(i)} \in M$ (use $\|C\| = 1$, $CA = C$, $\Psi(G) = \Phi(CG)$, $I + C$ is the adjoint of $I + B$. Hence $\|I + C\| < 2$).

Since M is an algebra, $1_{s+\beta N} \cdot I = CB \in M$. Thus for all $i = \alpha+1, \alpha+2, \dots, \alpha+\beta$, $1_{i+\beta N} \cdot I \in M$. Since $1_{S_o} \cdot I$ is compact, $1_{S_o} \cdot I \in M$. This proves $M = B(X)$.

COROLLARY 2. If M is an M-ideal in B(X) and there exists an isometry τ: $B(X) \to B(X)$ so that $\tau(M)$ contains an $A = \Sigma a_{ij}e_j \otimes e_i$ with $(a_{ii})_{i>1} \in \ell_\infty \setminus c_o$, then $M = B(X)$.

PROOF. Since $\tau(M)$ is an M-ideal in B(X) and $A \in \tau(M)$, by the lemma $\tau(M) = B(X)$. Hence $M = B(X)$

THEOREM 3. If M is an M-ideal in B(X) and contains a noncompact $T = \Sigma\, t_{ij}e_j \otimes e_i$, then $M = B(X)$.

PROOF. Suppose $T \in M$ and T is not compact. Wlog we may assume

$T = \sum_{k=1}^{\infty} T_k$, $T_k = \sum_{ij=m_k+1}^{m_k+n_k} t_{ij}e_j \otimes e_i$, $||T_k|| = 1$ where $m_k \in \alpha + \beta N$. $n_k \in \beta N$, and $m_k + n_k + \beta < m_{k+1}$.

Since each T_k has norm one, there exists norm one vectors

$x_k = (x_i^k) \in X$, $y_k = (y_i^k) \in X^*$, $z_k = (z_i^k) \in X^*$ all with supports in $\sigma_k = \{i: m_k < i \le m_k + n_k\}$ so that $y_k(T_k x_k) = 1 = z_k(x_k)$.

Let $B_k = \sum_{j\geq 1} x_j^k e_{m_k+1} \otimes e_j$, $C_k = \sum_{j\geq 1} y_j^k e_j \otimes e_{m_k+1}$, $D_k = \sum_{j\geq 1} z_j^k e_j \otimes e_{m_k+1}$,

$A = \sum_{k\geq 1} e_{m_k+1} \otimes e_{m_k+1}$, $B = \sum_{k\geq 1} B_k$, $C = \sum_{k\geq 1} C_k$ and $D = \sum_{k\geq 1} D_k$. Then all of these operators have norm one and $DB = CTB = A$

Let P be the matrix obtained from the identity matrix I by interchanging (m_k+j)-th column and $(m_k + n_k + j)$-th column for all k and $j (1 \le j \le \beta)$. Then P is an isometry in X since $n_k \in \beta N$.

CLAIM. If $B \in M$, then $M = B(X)$.

Choose $\Phi \in c_o^{\perp} \subseteq \ell_\infty^*$ so that $||\Phi|| = 1 = \Phi((1,1,1,1,\ldots))$. Define norm one functional $\gamma \in B(X)^*$ by $\gamma(G) = \Phi((g_{m_k+n_k+1,\, m_k+1})_{k\geq 1})$ where $G = \Sigma g_{ij}e_j \otimes e_i$. Then $\gamma \notin M^{\perp}$. In fact, if $\gamma \in M^{\perp}$, then $\gamma_1 \in B(X)^*$ defined by $\gamma_1(G) = \Phi((DG)_{m_k+1,\, m_k+1})$ has norm one and attains its norm at $B \in M$. Hence $\gamma_1 \in \tilde{M}$ and $||\gamma + \gamma_1|| = 2$. But for any norm one $G \in B(X)$, we have

$$|(\gamma + \gamma_1)(G)| = |\Phi(g_{m_k+n_k+1,m_k+1} + \sum_{j\in\sigma_k} z_j^k g_{j,m_k+1})_{k\geq 1}|$$

$$\le \sup_k ||z_k + e_{m_k+n_k+1}||(z_k + e_{m_k+n_k+1} \in X^*, ||G|| = 1)$$

$$= 2^{1/p'} \text{ where } \frac{1}{p} + \frac{1}{p'} = 1.$$

so $\|\gamma + \gamma_1\| \leq 2^{1/p'}$ contradiction! Thus $\gamma \notin M^{\perp}$. Since $\gamma \notin M^{\perp}$, there is $G \in M$ s.t. $\gamma(G) \neq 0$. So $(g_{m_k+n_k+1, m_k+1})_{k\geq 1} \in \ell_\infty \setminus c_o$. The sequence of the diagonal entries of P(G) belongs to $\ell_\infty \setminus c_o$. Thus by corollary 2, M = B(X). This proves the claim.

Next $\Psi \in B(X)^*$ defined by $\Psi(G) = \Phi(((CG)_{m_k+1, m_k+n_k+1})_{k\geq 1})$ is not in $M^{\perp}$. Indeed, if $\Psi \in M^{\perp}$, then since $\Psi_1 \in B(X)^*$ defined by $\Psi_1(G) = \Phi(((CGB)_{m_k+1, m_k+1})_{k\geq 1})$ has norm one and attains its norm at $T \in M$, $\Psi_1 \in \tilde{M}$ and so $\|\Psi + \Psi_1\| = 2$. But for any norm one $G \in B(X)$, we have

$$|(\Psi + \Psi_1)(G)| \leq \sup_k |(CG)_{m_k+1, m_k+n_k+1} + \sum_{j\varepsilon\sigma_k} (CG)_{m_k+1, j}\, x_j^k|$$

$$\leq \sup_K \|x^k + e_{m_k+n_k+1}\| \quad \text{since } CG \in B(X),\ \|CG\| = 1$$

$$= 2^{1/p}, \text{ contradiction!}$$

Thus $\Psi \notin M^{\perp}$. So there is $G = \Sigma g_{ij} e_j \otimes e_i \in M$ such that $((CG)_{m_{k+1}, m_k+n_k+1})_{k\geq 1} \in \ell_\infty \setminus c_o$. There is $\varepsilon > 0$ such that $\|G_k\| > \varepsilon$ for infinitely many k, where $G_k = \sum_{j\varepsilon\sigma_k} g_{j, m_k+n_k+1} e_{m_k+n_k+1} \quad e_j$. We can choose diagonal matrices D_1 and D_2 in B(X) so that D_1GD_2 has the same form as B in the claim above. Since $D_1GD_2 \in M$, M = B(X).

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