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ON A CLASS OF p-VALENT STARLIKE FUNCTIONS OF ORDER α

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ABSTRACT. Let Ω denote the class of functions $w(z)$, $w(o) = o$, $|w(z)| < 1$ analytic in the unit disc $U = \{z: |z| < 1\}$. For arbitrary fixed numbers A, B, $-1 < A \leq 1$, $-1 \leq B < 1$ and $o \leq \alpha < p$, denote by $P(A, B, p, \alpha)$ the class of functions $P(z) = p + \sum_{n=1}^{\infty} b_n z^n$ analytic in U such that $P(z) \in P(A, B, p, \alpha)$ if and only if $P(z) = \frac{p + [p B + (A-B)(p-\alpha)] w(z)}{1 + Bw(z)}$, $w \in \Omega$, $z \in U$. Moreover, let $S(A, B, p, \alpha)$ denote the class of functions $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ analytic in U and satisfying the condition that $f(z) \in S(A, B, p, \alpha)$ if and only if $\frac{zf'(z)}{f(z)} = P(z)$ for some $P(z) \in P(A, B, p, \alpha)$ and all z in U.

In this paper we determine the bounds for $|f(z)|$ and $|\arg \frac{f(z)}{z}|$ in $S(A, B, p, \alpha)$, we investigate the coefficient estimates for functions of the class $S(A, B, p, \alpha)$ and we study some properties of the class $P(A, B, p, \alpha)$.

KEY WORDS AND PHRASES. p- Valent, analytic, bounds, starlike functions of order α.

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1. INTRODUCTION.

Let A_p (p a fixed integer greater than zero) denote the class of functions $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ which are analytic in $U = \{z : |z| < 1\}$. We use Ω to denote the class of bounded analytic functions $w(z)$ in U satisfies the conditions $w(o) = o$ and $|w(z)| \leq |z|$ for $z \in U$.

Let $P(A, B)(-1 \leq B < A \leq 1)$ denote the class of functions having the form

$$P_1(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \tag{1.1}$$

which are analytic in U and such that $P_1(z) \in P(A, B)$ if and only if

$$P_1(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad w \in \Omega,\ z \in U. \tag{1.2}$$

The class P(A, B) was introduced by Janowski [1].

For $-1 \le B < A \le 1$ and $o \le \alpha < p$, denote by $P(A, B, p, \alpha)$ the class of functions

$P(z) = p + \sum_{k=1}^{\infty} c_k z^k$ which are analytic in U and which satisfy that $P(z) \varepsilon P(A, B, p, \alpha)$ if and only if

$$P(z) = (p - \alpha) P_1(z) + \alpha \ , \ P_1(z) \varepsilon P(A, B). \tag{1.3}$$

Using (1.2) in (1.3), one can show that $P(z) \varepsilon P(A, B, P, \alpha)$ if and onlv if

$$P(z) = \frac{p + [pB + (A - B)(p - \alpha)] w(z)}{1 + B w(z)} \ , w \varepsilon \Omega \ . \tag{1.4}$$

It was shown in [1] that

$P_1(z) \varepsilon P(A, B)$ if and only if

$$P\ (z) = \frac{(1 + A)\ p(z) + 1-A}{(1 + B)\ p(z) + 1 - B} \tag{1.5}$$

for some $p(z) \varepsilon P(1, -1) = P$ (the class of functions of form (1.1) which are analytic in U and have a positive real part in U). Thus, from (1.3) and (1.5), we have

$P(z) \varepsilon P(A, B, p, \alpha)$ if and only if

$$P(z) = \frac{[p+pB+(A-B)(p-\alpha)]p(z)+[p-pB-(A-B)(p-\alpha)]}{(1+B)p(z) + 1-B} \ , \ p(z) \varepsilon P. \tag{1.6}$$

Moreover, let $S(A, B, p, \alpha)$ denote the class of functions $f(z) \varepsilon A_p$ which satisfy

$$\frac{zf'(z)}{f(z)} = P(z) \tag{1.7}$$

for some $P(z)$ in $P(A, B, p, \alpha)$ and all z in U.

We note that $S(A, B, 1, o) = S^*(A, B)$, is the class of functions $f_1(z) \varepsilon A_1$ which satisfy

$$\frac{zf'_1(z)}{f_1(z)} = P_1(z), \ P_1 \varepsilon P(A, B). \tag{1.8}$$

The class $S^*(A, B)$ was introduced by Janowski [1]. Also, $S(1, -1, p, \alpha) = S_\alpha(p)$, is the class of p-valent starlike functions of order α, $o \le \alpha < p$, investigated by Goluzine [2].

From (1.3), (1.7) and (1.8), it is easy to show that $f \varepsilon S(A, B, p, \alpha)$ if and only if for $z \varepsilon U$

$$f(z) = z^p \left[\frac{f_1(z)}{z}\right]^{(p - \alpha)} , \ f_1 \varepsilon S^*(A, B). \tag{1.9}$$

2. THE ESTIMATION OF $|f(z)|$ AND $|\arg \frac{f(z)}{z}|$ FOR THE CLASS $S(A, B, p, \alpha)$.

LEMMA 1. Let $P(z) \varepsilon P(A, B, p, \alpha)$. Then, for $|z| \le r$, we have

$$\left| P(z) - \frac{p-[pB+(A-B)(p-\alpha)]Br^2}{1-B^2r^2} \right| \le \frac{(A-B)(p-\alpha)r}{1-B^2r^2} .$$

PROOF. It is easy to see that the transformation

$P_1(z) = \frac{1+Aw(z)}{1+Bw(z)}$ maps $|w(z)| \le r$ onto the circle

$$\left| P_1(z) - \frac{1-ABr^2}{1-B^2r^2} \right| \le \frac{(A-B)r}{1-B^2r^2} . \tag{2.1}$$

Then the result follows from (1.3) and (2.1).

THEOREM 1. If $f(z) \in S(A, B, p, \alpha)$, then for $|z| = r$, $0 \le r < 1$,

$$C(r; -A, -B, p, \alpha) \le |f(z)| \le C(r; A, B, p, \alpha), \tag{2.2}$$

where

$$C(r; A, B, p, \alpha) = \begin{cases} r^p(1 + Br)^{\frac{(A-B)(p-\alpha)}{B}} & \text{for } B \neq 0, \\ r^p . e^{A(p-\alpha)r} & \text{for } B = 0. \end{cases}$$

These bounds are sharp, being attained at the point $z = re^{i\phi}$, $0 \le \phi \le 2\pi$, by

$$f_*(z) = z^p f_0(z; -A, -B, p, \alpha) \tag{2.3}$$

and

$$f^*(z) = z^p f_0(z; A, B, p, \alpha), \tag{2.4}$$

respectively, where

$$f_0(z; A, B, p, \alpha) = \begin{cases} (1 + Be^{-i\phi}z)^{\frac{(A-B)(p-\alpha)}{B}} & \text{for } B \neq 0 \\ e^{A(p-\alpha)e^{-i\phi}z} & \text{for } B = 0 . \end{cases}$$

PROOF. From (1.9), we have $f(z) \in S(A, B, p, \alpha)$ if and only if

$$f(z) = z^p \left[\frac{f_1(z)}{z} \right]^{(p-\alpha)}, \quad f_1 \in S^*(A, B) . \tag{2.5}$$

It was shown by Janowski [1] that for $f_1(z) \in S^*(A, B)$

$$f_1(z) = z \exp\left(\int_0^z \frac{P_1(\zeta) - 1}{\zeta} d\zeta \right), \quad P_1(z) \in P(A, B). \tag{2.6}$$

Thus from (2.5) and (2.6), we have for $f(z) \in S(A, B, p, \alpha)$

$$f(z) = z^p \exp\left(\int_0^z \frac{P(\zeta) - p}{\zeta}\, d\zeta\right),\ P(z) \in P(A, B, p, \alpha).$$

Therefore

$$|f(z)| = |z|^p \exp\left(\mathrm{Re} \int_0^z \frac{P(\zeta) - p}{\zeta}\, d\zeta\right).$$

substituting $\zeta = zt$, we obtain

$$|f(z)| = |z|^p \exp\left(\mathrm{Re} \int_0^1 \frac{P(zt) - p}{t}\, dt\right).$$

Hence

$$|f(z)| \le |z|^p \exp\left(\int_0^1 \max_{|zt| = rt} \left(\mathrm{Re} \frac{P(zt) - p}{t}\right) dt\right).$$

From Lemma 1, it follows that

$$\max_{|zt| = rt} \frac{P(zt) - p}{t} = \frac{(A-B)(p - \alpha)r}{1+Brt};$$

then, after integration, we obtain the upper bounds in (2.2). Similarly, we obtain the bounds on the left-hand side of (2.2) which ends the proof.

REMARKS ON THEOREM 1.

1. Choosing $A = 1$ and $B = -1$ in Theorem 1, we get the result due to Goluzina [2].
2. Choosing $P = 1$ and $\alpha = o$ in Theorem 1, we get the result due to Janowski [1].
3. Choosing $p = 1$, $A = 1$ and $B = -1$ in Theorem 1, we get the result due to Robertson [3].
4. Choosing $p = 1$, $A = 1$ and $B = \alpha = o$ in Theorem 1, we get the result due to Singh [4].

THEOREM 2. If $f(z) \in S(A, B, p, \alpha)$, then for $|z| = r < 1$

$$\left| \arg \left(\frac{f(z)}{z^p}\right) \right| \le \frac{(A - B)(p - \alpha)}{B} \sin^{-1}(Br),\ B \ne o, \tag{2.7}$$

$$\left| \arg \left(\frac{f(z)}{z^p}\right) \right| \le A(p - \alpha)r,\ B = o. \tag{2.8}$$

These bounds are sharp, being attained by the function $f_o(z)$ defined by

$$f_o(z) = \begin{cases} z^p(1 + B\delta z)^{\frac{(A - B)(p - \alpha)}{B}}, & B \ne o, \\ z^p \exp(A(p - \alpha)\delta z), & B = o,\ |\delta| = 1. \end{cases} \tag{2.9}$$

PROOF. It was shown by Goel and Mehrok [5] that for $f_1 \in S^*(A, B)$

$$\left| \arg \frac{f_1(z)}{z} \right| \le \frac{(A - B)}{B} \sin^{-1} (Br), \ B \ne o, \tag{2.10}$$

$$\left| \arg \frac{f_1(z)}{z} \right| \le Ar , \quad B = o. \tag{2.11}$$

Therefore, the proof of Theorem 2 is an immediate consequence of (1.9), (2.10) and (2.11).

REMARK ON THEOREM 2.

Choosing $p = 1$, $A = 1$ and $B = -1$ in Theorem 2, we get the result due to Pinchuk [6].

3. COEFFICIENT ESTIMATES FOR THE CLASS S(A, B, P, α).

LEMMA 2. If integers p and m are greater than zero; $o \le \alpha < p$ and $-1 \le B < A \le 1$. Then

$$\prod_{j=o}^{m-1} \frac{|(B - A)(p - \alpha) + Bj|^2}{(j + 1)^2} = \frac{1}{m^2} \{ (B - A)^2(p - \alpha)^2 +$$

$$\sum_{k=1}^{m-1} [k^2(B^2 -1) + (B - A)^2(p - \alpha)^2 + 2kB(B - A)(p - \alpha)] \times$$

$$\prod_{j=o}^{k-1} \frac{|(B - A)(p - \alpha) + Bj|^2}{(j + 1)^2} \} . \tag{3.1}$$

PROOF. We prove the lemma by induction on m. For $m = 1$, the lemma is obvious. Next, suppose that the result is true for $m = q-1$. We have

$$\frac{1}{q^2} \{ (B - A)^2(p - \alpha)^2 + \sum_{k=1}^{q-1} [k^2(B^2 - 1) +$$

$$(B - A)^2(p - \alpha)^2 + 2kB(B - A)(p - \alpha)] \times \prod_{j=o}^{k-1} \frac{|(B - A)(p - \alpha) + Bj|^2}{(j + 1)^2} \}$$

$$= \frac{1}{q^2} \{ (B - A)^2(p - \alpha)^2 + \sum_{k=1}^{q-2} [k^2(B^2 - 1) +$$

$$(B - A)^2(p - \alpha)^2 + 2kB(B - A)(p - \alpha)] \times \prod_{j=o}^{k-1} \frac{|(B - A)(p - \alpha) + Bj|^2}{(j + 1)^2}$$

$$+ [(q - 1)^2 (B^2 - 1) + (B - A)^2 (p - \alpha)^2 + 2(q - 1)B(B - A)(p - \alpha)] \times$$

$$\prod_{j=o}^{q-2} \frac{|(B - A)(p - \alpha) + Bj|^2}{(j + 1)^2} \}$$

$$= \prod_{j=o}^{q-2} \frac{|(B - A)(p - \alpha) + Bj|^2}{(j + 1)^2} \times \left\{ \frac{(q - 1)^2B^2 + (B - A)^2 (p - \alpha)^2 + 2(q - 1)B(B - A)(p - \alpha)}{q^2} \right\}$$

$$= \prod_{j=o}^{q-1} \frac{|(B - A)(p - \alpha) + Bj|^2}{(j + 1)^2} .$$

Showing that the result is valid for $m = q$. This proves the lemma.

THEOREM 3. If $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \in S(A, B, p, \alpha)$, then

$$|a_n| \leq \prod_{k=o}^{n-(p+1)} \frac{|(B - A)(p - \alpha) + Bk|}{k + 1} \tag{3.2}$$

for $n \geq p + 1$, and these bounds are sharp for all admissible A, B and α and for each n.

PROOF. As $f \in S(A, B, p, \alpha)$, from (1.4) and (1.7), we have

$$\frac{zf'(z)}{f(z)} = \frac{p + [pB + (A - B)(p - \alpha)]w(z)}{1 + Bw(z)} , \quad w \in \Omega .$$

This may be written as

$$\{ Bzf'(z) + [-pB + (B - A)(p - \alpha)]f(z) \} w(z) = pf(z) - zf'(z).$$

Hence

$$[B \{ pz^p + \sum_{k=1}^{\infty} (p + k)a_{p+k} z^{p+k}\} + [-pB + (B - A)(p - \alpha)] \{ z^p +$$

$$\sum_{k=1}^{\infty} a_{p+k} z^{p+k}\}]w(z) = p \{ z^p + \sum_{k=1}^{\infty} (p + k)a_{p+k} z^{p+k}\} -$$

$$\{ pz^p + \sum_{k=1}^{\infty} (p + k)a_{p+k} z^{p+k} \}$$

or

$$[pB + [-pB + (B - A)(p - \alpha)] + \sum_{k=1}^{\infty} \{ (p + k)B +$$

$$[-pB + (B - A)(p - \alpha)] \} a_{p+k} z^k] w(z) = (P - p) +$$

$$\sum_{k=1}^{\infty} \{ p - (p + k) \} a_{p+k} z^k$$

which may be written as

$$\sum_{k=o}^{\infty} [\{(p+k)B+[-pB+(B-A)(p-\alpha)]\}\, a_{p+k}\, z^{k}]\, w(z)$$

$$\sum_{k=o}^{\infty} \{-k\}\, a_{p+k}\, z^{k} \tag{3.3}$$

where $a_p = 1$ and $w(z) = \sum_{k=o}^{\infty} b_{k+1}\, z^{k+1}$.

Equating coefficients of z^m on both sides of (3.3), we obtain

$$\sum_{k=o}^{m-1} \{(p+k)B+[-pB+(B-A)(p-\alpha)]\}\, a_{p+k}\, b_{m-k} =$$

$$\{-m\}\, a_{p+m} ;$$

which shows that a_{p+m} on right-hand side depends only on

$$a_p,\ a_{p+1},\ \cdots,\ a_{p+(m-1)}$$

of left-hand side. Hence we can write

$$\sum_{k=o}^{m-1} [\{(p+k)B+[-pB+(B-A)(p-\alpha)]\}\, a_{p+k}\, z^{k}]\, w(z)$$

$$= \sum_{k=o}^{m} \{-k\}\, a_{p+k}\, z^{k} + \sum_{k=m+1}^{\infty} A_k z^k$$

for $m = 1, 2, 3, \ldots$, and a proper choise of $A_k (k \geq o)$.

Let $z = re^{i\theta}$, $o < r < 1$, $o \leq \theta \leq 2\pi$, then

$$\sum_{k=o}^{m-1} |(p+k)B+[-pB+(B-A)(p-\alpha)]|^2 |a_{p+k}|^2 r^{2k}$$

$$= \frac{1}{2\pi}\int_o^{2\pi} \left|\sum_{k=o}^{m-1} \{(p+k)B+[-pB+(B-A)(p-\alpha)]\}\, a_{p+k}\, r^k e^{i\theta k}\right|^2 d\theta$$

$$\geq \frac{1}{2\pi}\int_o^{2\pi} \left|\sum_{k=o}^{m-1} \{(p+k)B+[-pB+(B-A)(p-\alpha)]\}\, a_{p+k}\, r^k e^{i\theta k}\right|^2 |w(re^{i\theta})|^2 d\theta$$

$$\geq \frac{1}{2\pi}\int_o^{2\pi} \left|\sum_{k=o}^{m} \{-k\}\, a_{p+k}\, r^k e^{i\theta k} + \sum_{k=m+1}^{\infty} A_k\, r^k e^{i\theta k}\right|^2 d\theta$$

$$\geq \sum_{k=o}^{m} |-k|^2 \left|a_{p+k}\right|^2 r^{2k} + \sum_{k=m+1}^{\infty} |A_k|^2 r^{2k}$$

$$\geq \sum_{k=o}^{m} k^2 \left|a_{p+k}\right|^2 r^{2k} . \tag{3.4}$$

Setting $r \to 1$ in (3.4), the inequality (3.4) may be written as

$$\sum_{k=o}^{m-1} \{ |(p+k) B + [-pB + (B - A)(p - \alpha)] |^2 - k^2\} |a_{p+k}|^2$$

$$\geq m^2 \ |a_{p+m}|^2 \ . \tag{3.5}$$

Simplification of (3.5) leads to

$$|a_{p+m}|^2 \leq \frac{1}{m^2} \sum_{k=o}^{m-1} \{ k^2(B^2 - 1) + (B - A)(p - \alpha)[(B - A)(p - \alpha) + 2kB]\} |a_{p+k}|^2 . \tag{3.6}$$

Replacing $p + m$ by n in (3.6), we are led to

$$|a_n|^2 \leq \frac{1}{(n-p)^2} \cdot \sum_{k=o}^{n-(p+1)} \{ k^2(B^2 - 1) + (B - A)(p - \alpha)x$$

$$[B - A)(p - \alpha) + 2kB] \} \ |a_{p+k}|^2 , \tag{3.7}$$

where $n \geq p + 1$.
For $n = p + 1$, (3.7) reduces to

$$|a_{p+1}|^2 \leq (B - A)^2(p - \alpha)^2$$

or

$$|a_{p+1}| \leq (A - B)(p - \alpha) \tag{3.8}$$

which is equivalent to (3.2).

To establish (3.2) for $n > p + 1$, we will apply induction argument.

Fix n, $n \geq p + 1$, and suppose (3.2) holds for $k = 1, 2, \ldots , \ldots\ldots\ldots, \ldots$ $n-(p+1)$. Then

$$|a_n|^2 \leq \frac{1}{(n-p)^2} \{ (B - A)^2(p - \alpha)^2 + \sum_{k=1}^{n-(p+1)} \{ k^2(B^2 - 1) +$$

$$(B - A)(p - \alpha)[(B - A)(p - \alpha) + 2kB] \} \prod_{j=o}^{k-1} \frac{|(B-A)(p-\alpha) + Bj|^2}{(j + 1)^2} \} . \tag{3.9}$$

Thus, from (3.7), (3.9) and Lemma 2 with $m = n - p$, we obtain

$$|a_n|^2 \leq \prod_{j=o}^{n-(p+1)} \frac{|(B - A)(p - \alpha) + Bj|^2}{(j + 1)^2} .$$

This completes the proof of (3.2). This proof is based on a technique found in Clunie [7].

For sharpness of (3.2) consider

$$f(z) = \frac{z^p}{(1 - B\delta z)^{\frac{(B-A)(p-\alpha)}{B}}}, \quad |\delta| = 1, \ B \neq 0.$$

REMARK ON THEOREM 3.

Choosing p = 1, A = 1, and B = -1 in Theorem 3, we get the result due to Robertson [3] and Schild [8].

4. DISTORTION AND COEFFICIENT BOUNDS FOR FUNCTIONS IN P(A, B, p, α).

THEOREM 4. If $P(z) \in P(A, B, p, \alpha)$, then

$$|\arg P(z)| \le \sin^{-1} \frac{(A-B)(p-\alpha)r}{p-[pB + (A-B)(p-\alpha)]Br^2}, \quad |z| = r.$$

The bound is sharp.

PROOF. The proof follows from Lemma 1. To see that the result is sharp, let

$$P(z) = p \left\{ \frac{1 + [B + (A-B)(1 - \frac{\alpha}{p})]\delta_1 z}{1 + \delta_1 Bz} \right\}, \quad |\delta_1| = 1. \tag{4.1}$$

Putting

$$\delta_1 = \frac{r}{z} \left\{ \frac{-([B + (A-B)(1 - \frac{\alpha}{p})] + B)r}{1 + [B + (A-B)(1 - \frac{\alpha}{p})]Br^2} + \frac{i\sqrt{1 - [B + (A-B)(1 - \frac{\alpha}{p})]^2 r^2}\sqrt{1 - B^2r^2}}{1 + [B + (A-B)(1 - \frac{\alpha}{p})]Br^2} \right., \quad r = |z|,$$

in (4.1), we have

$$\arg P(z) = \sin^{-1} \frac{(A - B)(p - \alpha)r}{p - [pB + (A - B)(p - \alpha)]Br^2}.$$

An immediate consequence of Lemma 1 is

COROLLARY 1. If P(z) is in P(A, B, p, α), then for $|z| \le r < 1$

$$\frac{p - (A-B)(p - \alpha)r-[pB + (A-B)(p - \alpha)]Br^2}{1 - B^2r^2} \le |P(z)| \le \frac{p + (A-B)(p - \alpha)r-[pB + (A-B)(p - \alpha)]Br^2}{1 - B^2r^2},$$

and

$$\frac{p - (A-B)(p - \alpha)r-[pB + (A-B)(p - \alpha)]Br^2}{1 - B^2r^2} \le \text{Re}\,\{P(z)\} \le \frac{p + (A-B)(p - \alpha)r-[pB + (A-B)(p - \alpha)]Br^2}{1 - B^2r^2}.$$

REMARK ON COROLLARY 1.

Choosing $p = 1$, $A = 1$ and $B = -1$ in the above corollary we get the following distortion bounds studied by Libera and Livingston [9] stated in the following corollary.

COROLLARY 2. If $P(z)$ is in $P(1, -1, \alpha) = P(\alpha)$, $0 \le \alpha < 1$ (the class of functions $P(z)$ with positive real part of order α, $0 \le \alpha < 1$), then for $|z| \le r < 1$

$$\frac{1 - 2(1-\alpha)r+(1-2\alpha)r^2}{1-r^2} \le |P(z)| \le \frac{1 + 2(1-\alpha)r+(1-2\alpha)r^2}{1-r^2}.$$

and

$$\frac{1 - 2(1-\alpha)r+(1-2\alpha)r^2}{1-r^2} \le \operatorname{Re}\{P(z)\} \le \frac{1 + 2(1-\alpha)r+(1-2\alpha)r^2}{1-r^2} .$$

The coefficient bounds which follow are derived by using the method of Clunie [7].

THEOREM 5. If $P(z) = p + \sum_{k=1}^{\infty} b_k z^k$ is in $P(A, B, p, \alpha)$, then

$$|b_n| \le (A - B)(p - \alpha), \; n = 1, 2, \dots ; \tag{4.2}$$

these bounds are sharp.

PROOF. The representation $P(z)$ in (1.4) is equivalent to

$$[B + (A - B)(p - \alpha) - B\, P(z)]w(z) = P(z) - p, \; w \in \Omega . \tag{4.3}$$

or

$$[B + (A - B)(p - \alpha) - B \sum_{k=0}^{\infty} b_k z^k]w(z) = \sum_{k=1}^{\infty} b_k z^k, \; b_0 = p. \tag{4.4}$$

This can be written as

$$[(A - B)(p - \alpha) - B \sum_{k=1}^{n-1} b_k z^k]w(z) = \sum_{k=1}^{n} b_k z^k + \sum_{k=n+1}^{\infty} q_k z^k , \tag{4.5}$$

the last term also being absolutely and uniformly convergent in compacta on U. Writing $z = re^{i\theta}$, performing the indicated integration and making use of the bound $|w(z)| \le |z| < 1$ for z in U gives

$$(A - B)^2(p - \alpha)^2 + B^2 \sum_{k=1}^{n-1} |b_k|^2 r^{2k} =$$

$$\frac{1}{2\pi}\int_0^{2\pi} \left|(A - B)(p - \alpha) + B \sum_{k=1}^{n-1} b_k r^k e^{ik\theta}\right|^2 d\theta$$

$$\geq \frac{1}{2\pi}\int_0^{2\pi} \left|\{(A - B)(p - \alpha) + B \sum_{k=1}^{n-1} b_k r^k e^{i\theta k}\} w(re^{i\theta})\right|^2 d\theta$$

$$\geq \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{k=1}^{n} b_k r^k e^{ik\theta} + \sum_{k=n+1}^{\infty} q_k r^k e^{ik\theta}\right|^2 d\theta$$

$$\geq \sum_{k=1}^{n} |b_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |q_k|^2 r^{2k} .$$

The last term is non-negative and $r < 1$, therefore

$$(A - B)^2(p - \alpha)^2 + B^2 \sum_{k=1}^{n-1} |b_k|^2 \geq \sum_{k=1}^{n} |b_k|^2 , \tag{4.6}$$

or

$$|b_n|^2 \leq (A - B)^2(p - \alpha)^2 + (B^2 - 1) \sum_{k=1}^{n-1} |b_k|^2 . \tag{4.7}$$

Since $-1 \leq B < 1$, we have $B^2 - 1 \leq 0$. Hence

$$|b_n| \leq (A - B)(p - \alpha) , \tag{4.8}$$

and this is equivalent to (4.2). If $w(z) = z^n$, then

$$P(z) = p + (A - B)(p - \alpha)z^n + \dots ,$$

which makes (4.2) sharp.

REMARK ON THEOREM 5.

Choosing $p = 1$, $A = 1$, and $B = -1$ in Theorem 5, we get the result due to Libera [10] stated in the following corollary.

COROLLARY 3. If $P(z) = 1 + \sum_{k=1}^{\infty} b_k z^k \in P(1, -1, 1, \alpha) = P(\alpha)$, $0 \leq \alpha < 1$, then

$$|b_n| \leq 2(1 - \alpha), \; n = 1, 2, 3, \dots ;$$

these bounds are sharp.

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