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# TWO PROPERTIES OF THE POWER SERIES RING

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ABSTRACT. For a commutative ring with unity, A, it is proved that the power series ring $A[[X]]$ is a PF-ring if and only if for any two countable subsets S and T of A such that $S \subseteq \operatorname{ann}_A(T)$, there exists $c \in \operatorname{ann}_A(T)$ such that $bc = b$ for all $b \in S$. Also it is proved that a power series ring $A[[X]]$ is a PP-ring if and only if A is a PP-ring in which every increasing chain of idempotents in A has a supremum which is an idempotent.

KEY WORDS AND PHRASES. Power series ring, PP-ring, PF-ring, flat, projective, annihilator ideal and idempotent element.
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## 1. INTRODUCTION.

Rings considered in this paper are all commutative with unity. Let $A[[X]]$ be the power series ring over the ring A. Recall that a ring A is called a PF-ring if every principal ideal is a flat A-module. Also a ring A is called a PP-ring if every principal ideal is a projective A-module.

It is proved in Al-Ezeh [1] that a ring A is a PF-ring if and only if the annihilator of each element $a \in A$, $\operatorname{ann}_A(a)$, is a pure ideal, that is for all $b \in \operatorname{ann}_A(a)$ there exists $c \in \operatorname{ann}_A(a)$ such that $bc = b$. A ring A is a PP-ring if and only if for each $a \in A$, $\operatorname{ann}_A(a)$ is generated by an idempotent, see Evans [2]. In Brewer [3], semihereditary power series rings over von Neumann regular rings are characterized. In this paper we characterize PF- power series rings and PP- power series rings over arbitrary rings.

For any reduced ring A (i.e. a ring with no nonzero nilpotent elements), it was proved in Brewer et al. [4] that

$$\operatorname{ann}_{A[[X]]}(a_0 + a_1X + \dots) = N[[X]]$$

where N is the annihilator of the ideal generated by the coefficients $a_0, a_1, \dots$

## 2. MAIN RESULTS.

LEMMA 1. Any PF-ring A is a reduced ring.

PROOF. Assume that there is a nonzero nilpotent element in A. Let n be the least positive integer greater than 1 such that $a^n = 0$. So $a \in ann(a^{n-1})$. Because A is a PF-ring there exists $b \in \underset{A}{ann}(a^{n-1})$ such that $ab = a$. Thus $a^{n-1} \overset{A}{=} (ab)^{n-1} = a^{n-1}b^{n-1} = 0$ since $ba^{n-1} = 0$.

Contradiction. So any PP-ring is a reduced ring.

THEOREM 2. The power series ring $A[[X]]$ is a PF-ring if and only if for any two countable sets $S = \{b_0, b_1, b_2, \ldots\}$ and $T = \{a_0, a_1, \ldots\}$ such that $S \subseteq \underset{A}{ann}(T)$, there exists $c \in \underset{A}{ann}(T)$ such that $b_i c = b_i$ for $i = 0, 1, 2, \ldots$

PROOF. First, we prove that $A[[X]]$ is a PF-ring.

Let $g(X) = b_0 + b_2X + \ldots,$ and

$f(X) = a_0 + a_1X + \ldots,$ and let

$g(X) \in \underset{A[[X]]}{ann}(f(X))$. Then $g(X)\,f(X) = 0$.

The ring A is inparticular a PF-ring because for all $b \in \underset{A}{ann}(a)$, there exists $c \in \underset{A}{ann}(a)$ such that $bc = b$. So by Lemma 1, A is a reduced ring. Thus

$$b_i a_j = \text{ for all } i = 0, 1, \ldots;\ j = 0, 1, 2, \ldots.$$

So

$$\{b_0, b_1, \ldots\} \subseteq \underset{A}{ann}(a_0, a_1, \ldots).$$ So by assumption, there exists $c \in \underset{A}{ann}(a_0, a_1, \ldots)$ such that $b_i c = b_i$ for all $i = 0, 1, \ldots.$ Hence $g(X)c = g(X)$ and $c \in \underset{A[[X]]}{ann}(f(X))$. Consequently, the ring $A[[X]]$ is a PF-ring. Conversely, assume $A[[X]]$ is a PF-ring.

Let $\{b_0, b_1, \ldots\} \subseteq \underset{A}{ann}(a_0, a_1, \ldots)$. Let $g(X) = b_0 + b_1X + \ldots,$ and $f(X) = a_0 + a_1 + \ldots$ Then $g(X)\,f(X) = 0$. Therefore $g(X) \in \underset{A[[X]]}{ann}(f(X))$. Thus there exists $h(X) = c_0 + c_1X + \ldots$ in $\underset{A[[X]]}{ann}(f(X))$ such that $g(X)\,h(X) = g(X)$.

Consequently, $h(X)\,f(X) = 0$ and $g(X)\,(h(X) - 1) = 0$. Since A is reduced,

$c_i a_j = 0$ for all $i = 0, 1, \ldots,\ j = 0, 1, 2, \ldots$ and $b_i(c_0 - 1) = 0$ for all i and $b_i c_j = 0$ for all $j \geq 1$. Hence $\{c_0, c_1, \ldots\} \subseteq \underset{A}{ann}(a_0, a_1, \ldots)$ and $b_i(c_0 - 1) = 0$. So $c_0 \in ann(a_0, a_1, \ldots)$ and $b_i c_0 = b_i$ for all $i = 0, 1, \ldots$. Therefore the above condition holds.

Because any PP-ring is a PF-ring, every PP-ring is a reduced ring. On a reduced ring A, a partial order relation can be defined by $a \leq b$ if $ab = a^2$. The following lemma is given in Brewer[3] and Brewer et al.[4].

LEMMA 3. The relation $\leq$ defined above on a reduced ring A is a partial order.

PROOF. Clearly the relation $\leq$ is reflexive. Now assume $a \leq b$ and $b \leq a$. Then $ab = a^2$ and $ba = b^2$. So, $(a-b)^2 = a^2 - 2ab + b^2 = 0$. Because A is reduced $a - b = 0$, or $a = b$. To prove transitivity of $\leq$, assume $a \leq b$ and $b \leq c$. So $ab = a^2$ and $bc = b^2$. Consider

$$\begin{aligned}(ac - ab)^2 &= a^2(c^2 - 2cb + b^2)\\ &= a^2(c^2 - b^2)\\ &= ab(c - b)(c + b)\\ &= 0\end{aligned}$$

because $b(c - b) = 0$. Since A is reduced, $ac - ab = 0$ or $ac = ab = a^2$. Therefore $a \leq b$.

THEOREM 4. The power series ring $A[[X]]$ is a PP-ring if and only if A is a PP-ring in which every increasing chain of idempotents of A with respect to $\leq$ has a supremum which is an idempotent element in A.

PROOF. Assume $A[[X]]$ is a PP-ring. Let $a \in A$. Since $A[[X]]$ is a PP-ring and idempotents in $A[[X]]$ are in A, $\operatorname{ann}_{A[[X]]}(a) = eA[[X]]$. We claim $\operatorname{ann}_A(a) = eA$. Because $ea = 0$, $rea = 0$ for all $r \in A$. Hence $eA \subseteq \operatorname{ann}_A(a)$. Now let $b \in \operatorname{ann}_A(a)$. Hence $b \in \operatorname{ann}_{A[[X]]}(a)$. Thus $b = eg(X)$ for some $g(X) = b_0 + b_1X + \ldots$ . Consequently, $b = eb_0$. That is $b \in eA$. Whence A is a PP-ring.

To complete the proof of this direction, let $e_0 \leq e_1 \leq e_2 \ldots$ be an increasing chain of idempotents in A. Because $A[[X]]$ is a PP-ring and since idempotents of $A[[X]]$ are in A, $\operatorname{ann}_{A[[X]]}(e_0 + e_1X + \ldots) = eA[[X]]$. Now we claim $1 - e = \sup\{e_0, e_1, \ldots\}$. Since $ee_i = 0$, $e_i(1 - e) = e_i = e_i^2$, $i = 0, 1, \ldots$ .

So $e_i \leq 1 - e$ for all $i = 0, 1, \ldots$ . Let y be an upper bound of $\{e_0, e_1, \ldots\}$. So $e_i \leq y$ for $i = 0, 1, \ldots$ .

Hence $1 - y \in \operatorname{ann}_{A[[X]]}(e_0 + e_1X + \ldots)$.

Thus $1 - y = ec$ for some $c \in A$. Consequently,

$$\begin{aligned}y(1 - e) &= (1 - ce)(1 - e)\\ &= 1 - ec - e + ec\\ &= 1 - e\end{aligned}$$

So $1 - e \leq y$. Therefore $1 - e = \sup\{e_0, e_1 \ldots\}$.

To prove the other way around, consider $\operatorname{ann}_{A[[X]]}(f(X))$ where $f(X) = a_0 + a_1X + \ldots$ .

Hence

$$\operatorname{ann}_{A[[X]]}(f(X)) = \operatorname{ann}_A(a_0, a_1, \ldots)[[X]]$$

$$\operatorname{ann}_A(a_0, a_1, \ldots) = \bigcap_{i=0}^{\infty} \operatorname{ann}(a_i)$$

$$= \bigcap_{i=0}^{\infty} e_i A, \; e_i^2 = e_i$$

because A is a PP-ring.

Let $d_0 = e_0$, $d_1 = e_0 e_1$, ..., $d_n = d_{n-1} e_n$, ...

One can easily check that

$$\bigcap_{i=0}^{\infty} e_i A = \bigcap_{i=0}^{\infty} d_i A$$

Also it is clear that

$$d_0 \geq d_1 \geq d_2 \; \cdots$$

Therefore

$$1 - d_0 \leq 1 - d_1 \leq 1 - d_2 \cdots$$

By assumption, this increasing chain of idempotents has a supremum which is an idempotent. Let

$$\operatorname{Sup}\{1 - d_0, \; 1 - d_1, \; 1 - d_2, \ldots\} = d. \quad \text{So}$$

$$(1 - d_i)\, d = 1 - d_i \quad \text{for all } i = 0, 1, \ldots .$$

We claim that

$$\bigcap_{i=0}^{\infty} d_i A = (1 - d)A.$$

Now $1 - d \geq d_i$. So $(1 - d)d_i = 1 - d$. Hence

$$(1 - d)A \subseteq d_i A \quad \text{for all } i = 0, 1, \ldots .$$

Thus $(1 - d)A \subseteq \bigcap_{i=0}^{\infty} d_i A.$

Let $y \in \bigcap_{i=0}^{\infty} d_i A$. Then $y = d_i y_i$, $i, 0, 1, \ldots$ .

Consequently

$$\begin{aligned}(1 - d_i)(1 - y) &= 1 + d_i y - d_i - y \\ &= 1 - d_i\end{aligned}$$

Because $y d_i = d_i^2 = d_i^2 y_i = d_i y_i = y.$

Therefore $1 - d_i \leq 1 - y$ for all $i = 0, 1, \ldots$ .

Because $d = \text{Sup}\{1 - d_0, 1 - d_1, 1 - d_2, \ldots\}$,
$d \leq 1 - y$. So $d = d(1 - y) = d - dy$

Hence $dy = 0$. Thus $y(1 - d) = y - yd = y$

That is $y \in (1 - d)A$. Therefore $\bigcap_{i=0}^{\infty} d_i A = (1 - d)A$.

Consequently,

$$\text{ann}_{A[[X]]} (f(X)) = (1 - d)\, A[[X]]$$

Therefore A[[X]] is a PP-ring.

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