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THE SEMIGROUP OF NONEMPTY FINITE SUBSETS OF RATIONALS

REUBEN SPAKE

Department of Mathematics
University of California
Davis, California 95616

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ABSTRACT. Let Q be the additive group of rational numbers and let $\mathfrak{R}$ be the additive semigroup of all nonempty finite subsets of Q. For $X \in \mathfrak{R}$, define A_X to be the basis of $\langle X - \min(X)\rangle$ and B_X the basis of $\langle \max(X) - X\rangle$. In the greatest semilattice decomposition of $\mathfrak{R}$, let $\mathcal{A}(X)$ denote the archimedean component containing X. In this paper we examine the structure of $\mathfrak{R}$ and determine its greatest semilattice decomposition. In particular, we show that for $X,Y \in \mathfrak{R}$, $\mathcal{A}(X) = \mathcal{A}(Y)$ if and only if $A_X = A_Y$ and $B_X = B_Y$. Furthermore, if X is a non-singleton, then the idempotent-free $\mathcal{A}(X)$ is isomorphic to the direct product of a power joined subsemigroup and the group Q.

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1. INTRODUCTION.

In [1] we determined the structure of the semigroup of nonempty finite subsets of integers. In this paper we extend the results of [1] for the semigroup of nonempty finite subsets of rationals. In particular, we give a complete description of its greatest semilattice decomposition. We also propose an isomorphism problem. It is assumed the reader is familiar with the basic notions on commutative semigroups and greatest semilattice decompositions; otherwise refer to Clifford and Preston [2] and Petrich [3]. We begin with some notation and several definitions.

Let Q be the group of rational numbers, Z the group of integers, and define $\mathfrak{R}$ to be the semigroup consisting of all nonempty finite subsets of Q with the operation

$$A + B = \{a + b : a \in A, b \in B\}, \quad A, B \in \mathfrak{R}.$$

A singleton element of $\mathfrak{R}$ will be identified with the rational number it contains. The semigroup $\mathfrak{R}$ is a commutative countable semigroup with identity element 0.

Let $X = \{a_1/b_1, a_2/b_2, \ldots, a_n/b_n\} \in \mathfrak{R}$, where $a_1/b_1 < \cdots < a_n/b_n$ and each a_i, b_i are relatively prime integers, (if X contains an integer x, then express x as x/1). Define $\min(X) = a_1/b_1$, $\max(X) = a_n/b_n$, and let $\mathfrak{L}(X)$ denote the least (positive) common multiple of the integers $b_1, b_2, \ldots, b_n$. If X consists only of integers, then define $\gcd(X)$ to be the greatest (non-negative) common divisor of the integers in X, where $\gcd(0) = 0$ and $\gcd(X \cup \{0\}) = \gcd(X)$. Let Z_+ be the set of positive integers and define the integer interval $[a,b] = \{x \in Z : a \leq x \leq b\}$ if $a, b \in Z$ with $a \leq b$. For $U \in \mathfrak{R}$, let $\langle U \rangle$ denote the semigroup generated by the set U, and for $m \in Z_+$, $r \in Q$ define

$$mU = \underbrace{U + \cdots + U}_{m}, \quad r*U = \{ru : u \in U\}, \quad \text{and} \quad Z_m = Z/\langle -m, m \rangle.$$

In the greatest semilattice decomposition of $\mathfrak{R}$, let $\mathcal{A}(X)$ denote the archimedean component containing X. Define the partial order $\leq$ on the (lower) semilattice as: $\mathcal{A}(X) \leq \mathcal{A}(Y)$ if and only if $nX = Y + W$ for some $W \in \mathfrak{R}$ and $n \in Z_+$ (equivalently: $U + V \in \mathcal{A}(X)$ for some (all) $U \in \mathcal{A}(X)$ and $V \in \mathcal{A}(Y)$). Note that $\mathcal{A}(0)$ consists of all the singletons in $\mathfrak{R}$ and $\mathcal{A}(0) \cong Q$. Moreover, since 0 is clearly the only idempotent in $\mathfrak{R}$, evidently $\mathcal{A}(X)$ is idempotent-free if and only if X is a non-singleton. We will show later in Theorem 2.1 that there are in fact infinitely many archimedean components in the greatest semilattice decomposition of $\mathfrak{R}$.

For $X \in \mathfrak{R}$, define A_X to be the basis of $\langle X - \min(X) \rangle$ and B_X the basis of $\langle \max(X) - X \rangle$. Also, if X is a non-singleton define $\mathrm{id}(X) = \min(A_X \setminus \{0\})$ and $\mathrm{fd}(X) = \min(B_X \setminus \{0\})$. Note that $\Omega(A_X) = \Omega(X - x) = \Omega(B_X)$ for all $x \in X$. When X is a non-singleton, A_X and B_X have at most $1 + \Omega(A_X)\mathrm{id}(X)$ and $1 + \Omega(B_X)\mathrm{fd}(X)$ elements, respectively (if X is a singleton then $A_X = B_X = \{0\}$). We close this introduction with an example. Let $X = \{-3/10, -1/5, 4/5, 11/6, 2\}$. We wish to determine A_X and B_X. First, $\Omega(X) = 30$, so $X = 1/30 * \{-9, -6, 24, 55, 60\}$. Thus

$$X - \min(X) = 1/30 * \{0, 3, 33, 64, 69\} \text{ and}$$
$$\max(X) - X = 1/30 * \{0, 5, 36, 66, 69\}.$$

Consequently, $A_X = 1/30 * \{0, 3, 64\} = \{0, 1/10, 32/15\}$ and $B_X = 1/30 * \{0, 5, 36, 69\} = \{0, 1/6, 6/5, 23/10\}$.

2. STRUCTURE OF $\mathfrak{R}$.

In this section we examine the structure of $\mathfrak{R}$ by determining its greatest semilattice decomposition and describing the structure of its archimedean components. The first result gives a necessary and sufficient condition for two elements of $\mathfrak{R}$ to be in the same archimedean component.

THEOREM 2.1. <u>For</u> $X, Y \in \mathfrak{R}$, $\mathcal{A}(X) = \mathcal{A}(Y)$ <u>if and only if</u> $A_X = A_Y$ <u>and</u> $B_X = B_Y$.

<u>PROOF</u>. Let $X, Y \in \mathfrak{R}$ and without loss of generality assume $\min(X) = \min(Y) = 0$. Let U and V be such that $U = \Omega(X)*X$ and $V = \Omega(Y)*Y$. Note that U and V are finite sets of integers. Suppose $A_X = A_Y$ and $B_X = B_Y$. Since $\min(X) = \min(Y) = 0$, this implies $\Omega(X) = \Omega(A_X) = \Omega(A_Y) = \Omega(Y)$. Hence $A_U = A_V$ and $B_U = B_V$. By [1], $\mathcal{A}(U) = \mathcal{A}(V)$ and therefore it follows that $\mathcal{A}(X) = \mathcal{A}(Y)$.

Conversely, suppose $\mathcal{A}(X) = \mathcal{A}(Y)$. There exist $n, m \in Z_+$ and $S, T \in \mathfrak{R}$ such that

$$nX = Y + S \text{ and } mY = X + T .$$

Since necessarily $\min(S) = \min(T) = 0$, evidently

$$A_Y \subseteq Y \subseteq Y + S \subseteq \langle A_X \rangle$$

and likewise $A_X \subseteq \langle A_Y \rangle$. Consequently, $\langle A_X \rangle = \langle A_Y \rangle$ and by definition this implies $A_X = A_Y$. Similarly $B_X = B_Y$ and this completes the proof.

Using the above theorem we can determine when two archimedean components are related with respect to the order on the semilattice.

THEOREM 2.2. <u>The following are equivalent</u>.

(i) $\mathcal{A}(X) \leq \mathcal{A}(Y)$.

(ii) $A_Y \subseteq \langle A_X \rangle$ <u>and</u> $B_Y \subseteq \langle B_X \rangle$.

(iii) $A_{X+Y} = A_X$ <u>and</u> $B_{X+Y} = B_X$.

<u>PROOF</u>. Suppose $\mathcal{A}(X) \leq \mathcal{A}(Y)$. There exist $U \in \mathfrak{R}$ and $n \in Z_+$ such that

$$n(X - \min(X)) = Y - \min(Y) + U.$$

Since $\min(U) = 0$,

$$A_Y \subseteq Y - \min(Y) \subseteq Y - \min(Y) + U \subseteq \langle A_X \rangle .$$

Similarly $B_Y \subseteq \langle B_X \rangle$. Suppose next that assertion (ii) holds. Then

$$Y - \min(Y) \subseteq \langle A_Y \rangle \subseteq \langle A_X \rangle$$

and thus

$$A_X \subseteq X + Y - \min(X + Y) \subseteq \langle A_X \rangle.$$

Hence $A_{X+Y} = A_X$. Likewise $B_{X+Y} = B_X$. Finally, if (iii) holds, then by Theorem 2.1 $X + Y \in \mathcal{A}(X)$; that is, $\mathcal{A}(X) \leq \mathcal{A}(Y)$ and the proof is complete.

Since A_Y and B_Y are finite sets, it is relatively easy to determine when $\mathcal{A}(X) \leq \mathcal{A}(Y)$ using Theorem 2.2(ii). For example, let $W = \{-10/7, -8/7, 22/7, 33/7, 5\}$, $X = \{1/7, 5/21, 29/21, 68/21, 23/7\}$, and $Y = \{-15, -13, 8, 28, 30\}$. Then $\mathfrak{L}(W) = 7$, $\mathfrak{L}(X) = 21$, and $\mathfrak{L}(Y) = 1$. Thus

$W - \min(W) = 1/7 * \{0,2,32,43,45\}$, $\max(W) - W = 1/7 * \{0,2,13,43,45\}$,

$X - \min(X) = 1/21 * \{0,2,26,65,66\}$, $\max(X) - X = 1/21 * \{0,1,40,64,66\}$,

$Y - \min(Y) = \{0,2,23,43,45\}$, and $\max(Y) - Y = \{0,2,22,43,45\}$.

Hence, $A_W = \{0, 2/7, 43/7\}$, $A_X = \{0, 2/21, 65/21\}$, $A_Y = \{0,2,23\}$, $B_W = \{0,2/7,13/7\}$, $B_X = \{0, 1/21\}$, and $B_Y = \{0,2,43\}$. Therefore, it follows that $\mathcal{A}(X) \leq \mathcal{A}(W) \leq \mathcal{A}(Y)$ with $A(X) \neq \mathcal{A}(W)$ and $\mathcal{A}(W) \neq \mathcal{A}(Y)$.

Next, for $X \in \mathfrak{R}$ define $\mathcal{A}_0(X) = \{Y \in \mathcal{A}(X) : \min(Y) = 0\}$. It is clear that $\mathcal{A}_0(X)$ is a subsemigroup of $\mathcal{A}(X)$. In general, elements of $\mathcal{A}(X)$ can be uniquely expressed in the form $U + q$, where $U \in \mathcal{A}_0(X)$ and $q \in Q$. Hence it follows that $\mathcal{A}(X) \cong \mathcal{A}_0(X) \oplus Q$. Moreover, we have the following

THEOREM 2.3. <u>The idempotent-free archimedean component</u> $\mathscr{A}(X)$, <u>where</u> X <u>is a non-singleton, is isomorphic to the direct product of the idempotent-free power joined subsemigroup</u> $\mathscr{A}_0(X)$ <u>and the group</u> Q.

<u>PROOF</u>. Let X be a non-singleton with $\min(X) = 0$ and let q be a non-zero rational number. We will first show that $\mathscr{A}_0(X) \cong \mathscr{A}_0(q*X)$ under the isomorphism which maps U to $q*U$. First, if $U \in \mathscr{A}_0(X)$, then there exist U_1, $X_1 \in \mathfrak{R}$ and n, $m \in Z_+$ such that

$$nX = U + U_1 \quad \text{and} \quad mU = X + X_1 .$$

Hence

$$n(q*X) = q*U + q*U_1 \quad \text{and}$$
$$m(q*U) = q*X + q*X_1$$

giving $q*U \in \mathscr{A}_0(q*X)$. It suffices therefore to show that for each $V \in \mathscr{A}_0(q*X)$ there exists $V_1 \in \mathscr{A}_0(X)$ such that $V = q*V_1$. Let $V \in \mathscr{A}_0(q*X)$. Then there exist S, $T \in \mathfrak{R}$ and s, $t \in Z_+$ such that

$$q*(sX) = V + S \quad \text{and} \quad tV = q*X + T.$$

Let V_1, S_1, and T_1 be such that $V = q*V_1$, $S = q*S_1$, and $T = q*T_1$. Then

$$sX = V_1 + S_1 \quad \text{and} \quad tV_1 = X + T_1.$$

Hence $V_1 \in \mathscr{A}_0(X)$ and consequently $\mathscr{A}_0(X) \cong \mathscr{A}_0(q*X)$ for each non-zero rational q. In particular, $\mathscr{A}_0(X) \cong \mathscr{A}_0(\mathfrak{L}(X)*X)$. Since $\mathfrak{L}(X)*X$ is a set of integers, by [1] $\mathscr{A}_0(\mathfrak{L}(X)*X)$ is power joined. Therefore $\mathscr{A}_0(X)$ is power joined and this completes the proof.

COROLLARY 2.4. <u>For</u> $X \in \mathfrak{R}$, $\mathscr{A}(X) \cong \mathscr{A}(q*X)$ <u>for each non-zero rational number</u> q.

The following equivalence relation on $\mathfrak{R}$ is called the $\mathscr{J}$-relation on $\mathfrak{R}$ (see [2] and [3] for more on the $\mathscr{J}$-relation):

$$X \mathscr{J} Y \text{ if and only if } X = Y + U \text{ and } Y = X + V \text{ for some } U, V \in \mathfrak{R}.$$

However, observe that if $X = Y + U$ and $Y = X + V$, then X and Y must necessarily be of the same cardinality since $X,Y \in \mathfrak{R}$; that is, evidently U and V are singletons. Hence

$$X \mathscr{J} Y \text{ if and only if } X = Y + q \text{ for some } q \in Q.$$

Therefore, in $\mathfrak{R}$ the $\mathscr{J}$-class of X is the set of all rational translates of X (i.e. elements of the form $X + q$, $q \in Q$).

Let ρ_0 denote the least semilattice congruence on $\mathfrak{R}$. Define an equivalence relation π on $\mathfrak{R}$ by

$$X \pi Y \text{ if and only if } nX = mY \text{ for some } n, m \in Z_+.$$

Using Theorem 2.3 we immediately have

THEOREM 2.5. *The least semilattice congruence on* $\mathfrak{R}$ *is* $\rho_0 = \mathcal{J} \pi \mathcal{J}$. *That is,* $X \rho_0 Y$ *if and only if* $X \mathcal{J} X_0 \pi Y_0 \mathcal{J} Y$ *for some* $X_0, Y_0 \in \mathfrak{R}$.

Next we look more deeply into the structure of $\mathcal{A}(X)$. The structure of $\mathcal{A}(0)$ is clear since $\mathcal{A}(0) \cong Q$. Using the above results, evidently $Y \in \mathcal{A}(X)$ if and only if $Y\text{-min}(Y) = \mathcal{A}_X \cup Y_1$ where $Y_1 \subseteq \langle \mathcal{A}_X \rangle$ and $\max(Y)-Y = B_X \cup Y_2$ where $Y_2 \subseteq \langle B_X \rangle$. More precisely we have the following direct consequence of Theorem 3.2 from [1].

THEOREM 2.6. *Let* X *be a non-singleton and* U *be such that* $X\text{-min}(X) = g*U$, *where* $g = \gcd(\Omega(A_X)*A_X)/\Omega(A_X)$. *Define* $A_i = \{x \in \langle A_U \rangle : x \equiv i \pmod a\}$ *and* $B_j = \{x \in \langle B_U \rangle : x \equiv j \pmod b\}$ *for* $i \in [0,a-1]$, $j \in [0, b-1]$, *where* $a = id(U)$ *and* $b = fd(U)$. *Let* $c = \max\{\min(A_i) : i \in [0,a-1]\}$ *and* $d = \max\{\min(B_i) : i \in [0,b-1]\}$. *Then* $Y \in \mathcal{A}(X)$ *if and only if there exist* $V \in \mathfrak{R}$ *and* $n_0 \in Z_+$ *such that* $Y\text{-min}(Y) = g*V$ *and for all integers* $n \geq n_0$

$$nV = \bigcup_{i=0}^{a-1} \{x \in A_i : x < c-a\} \cup [c-a+1, n\max(V) + b-d-1]$$

$$\cup \bigcup_{i=0}^{b-1} \{n\max(V) - x : x \in B_i, x < d-b\}$$

$$= \langle A_U \rangle \cap (n\max(V) - \langle B_U \rangle) .$$

Next we reproduce several definitions and facts from Tamura [4] that we will use in the next theorem. Let T be an additively denoted idempotent-free commutative archimedean semigroup. For fixed $b \in T$, define a congruence ρ_b on T by

$$x \rho_b y \text{ if and only if } nb + x = mb + y \text{ for some } n,m \in Z_+.$$

Then $T/\rho_b = G_b$ is a group called the structure group of T determined by the standard element b. Also, define a compatible partial order $\underset{b}{<}$ on T by

$$x \underset{b}{<} y \text{ if and only if } x = nb + y \text{ for some } n \in Z_+ .$$

Then $T = \bigcup_{\lambda \in G_b} T_\lambda$, equivalently $T/\rho_b = \{T_\lambda\}$, $\lambda \in G_b$, where each T_λ is a discrete tree without smallest element with respect to $\underset{b}{<}$, (a discrete tree, with respect to $\underset{b}{<}$, is a lower semilattice such that for any $c \underset{b}{<} d$ the set $\{x : c \underset{b}{<} x \underset{b}{<} d\}$ is a finite chain). Finally, define a relation η on T as follows:

$$x \eta y \text{ if and only if } nb + x = nb + y \text{ for some } n \in Z_+.$$

The relation η is the least cancellative congruence on T. Let Q_+ denote the set of positive rational numbers.

THEOREM 2.7. *Let* $A \in \mathfrak{R}$ *be a non-singleton with* $\min(A) = 0$ *and* $g = \gcd(\Omega(A)*A)/\Omega(A)$. *The structure group of* $\mathcal{A}_0(A)$ *determined by the standard* *element* A *is* Z_m, *where* $m = \max(A)/g$. *Moreover*, $\mathscr{A}_0(A) = \bigcup_{i=0}^{m-1} \mathscr{A}_i$ *where* $\mathscr{A}_i = \{X \in \mathscr{A}_0(A) : \max(X)/g \equiv i (\mathrm{mod}\ m)\}$ *is a discrete tree without smallest element with respect to* $<_A$. *Furthermore, the structure group of* $\mathscr{A}(A)$ *determined by the standard element* A *is* $Q \oplus Z_m$.

PROOF. This follows from [1] since $\mathscr{A}_0(A) \cong \mathscr{A}_0(1/g*A)$.

Using Theorem 2.7 we have the immediate

PROPOSITION 2.8. *Let* A *be a non-singleton. The homomorphism* $h : \mathscr{A}_0(A) \to Q_+$ *defined by* $h(X) = \max(X)$ *is the greatest cancellative homomorphism. That is, the relation* η *on* $\mathscr{A}_0(A)$ *defined by*

$$X \ \eta \ Y \ \text{ if and only if } \ \max(X) = \max(Y)$$

is the least cancellative congruence. Moreover, the relation σ *on* $\mathscr{A}(A)$ *defined by*

$$X \ \sigma \ Y \ \text{ if and only if } \ \min(X) = \min(Y) \text{ and } \max(X) = \max(Y)$$

is the least cancellative congruence. The semigroups $\mathscr{A}_0(A)/\eta$ *and* $\mathscr{A}(A)/\sigma$ *are idempotent-free commutative archimedean cancellative semigroups.*

For a description of the greatest cancellative homomorphic image of $\mathscr{A}_0(A)$ we direct the reader to [1]. We close this report with an open isomorphism problem. Any partial solutions would be appreciated.

PROBLEM. For $X, Y \in \mathscr{R}$, under what conditions will $\mathscr{A}(X)$ be isomorphic to $\mathscr{A}(Y)$? See Theorem 5.5 of [5] for some related results and also recall Corollary 2.4.

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