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MODIFIED WHYBURN SEMIGROUPS

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ABSTRACT. Let $f: X \to Y$ be a continuous semigroup homomorphism. Conditions are given which will ensure that the semigroup $X \cup Y$ is a topological semigroup, when the modified Whyburn topology is placed on $X \cup Y$.

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1. INTRODUCTION.

Let (X,m_1) and (Y,m_2) be semigroups and let $f: X \to Y$ be a semigroup homomorphism. An associative multiplication m may be defined on the disjoint union of X and Y as follows: m is m_1 on X, m_2 on Y and $m_2(f(x),y)$ if $x \in X$ and $y \in Y$. If we assume that X and Y are Hausdorff semigroups and that f is continuous, then m is continuous in the disjoint union (or direct sum) topology. Let $(X \cup Y,m)$ denote this Hausdorff semigroup.

Let Z denote the disjoint union of X and Y with Whyburn's unified topology [1]; i.e., V is open in Z iff $V \cap X$ and $V \cap Y$ are open in X and Y, respectively, and for any compact K in $V \cap Y$, $f^{-1}(K) - V$ is compact. If X is locally compact, then Z is Hausdorff, and if Y is also locally compact, so is Z. If f is a compact map, then Z and $X \cup Y$ are the same. If X and Y are locally compact, Hausdorff semigroups, (Z,m) is a locally compact Hausdorff semigroup provided m_1 is a compact map [2].

In this paper we consider the modified Whyburn topology which is coarser than the disjoint union topology, but finer than the Whyburn topology and ask what conditions will insure that m will be continuous.

2. MAIN RESULTS.

Let W denote the disjoint union of X and Y with the modified Whyburn topology; V is open in W iff $V \cap X$ and $V \cap Y$ are open in X and Y, respectively, and $f^{-1}(y) - V$ is compact for every y in $V \cap Y$. The following notions and facts are due to Stallings [3]. A subset A of X is fiber compact relative to $f: X \to Y$ iff A is closed in X and $A \cap f^{-1}(x)$ is compact for every $y \in Y$, and X is locally fiber compact iff every point in X has a neighborhood with a fiber compact closure. Fiber compact subsets of X are closed in W and W is Hausdorff if X is locally fiber compact. If Y is first countable, then Z and W

are the same iff f is closed.

The proof given in [2] that m is a continuous operation on Z did not use the assumption that $m_1^{-1}(K)$ is compact for every compact K in X, but used an equivalent condition instead. The appropriate generalization of that condition for W is:

CONDITION 1. For every fiber compact K_1 in X, there is a fiber compact K_2 in X such that for all $x, y \in X$, if $m_1(x,y) \in K_1$, then $x \in K_2$ and $y \in K_2$.

This condition is equivalent to: $\overline{p_i(m_1^{-1}(K))}$, $i = 1,2$, are fiber compact for each fiber compact K in X, where p_1 and p_2 are the projections on $X \times X$.

THEOREM 1. If X is locally fiber compact, Y is regular and m_1 satisfies Condition 1, then m is continuous and hence W is a Hausdorff semigroup.

PROOF. The argument is similar to the one given for Z. We will show continuity at a point (x,y) where $x \in X$ and $y \in Y$. Let $w = m(x,y) = m_2(f(x),y)$. Let V be an open set in W containing w. Since Y is regular, there is a Y-open set U containing y such that $\bar{U} \subset Y \cap V$. Since m_2 is continuous, there are Y-open neighborhoods U_1 and U_2 of $f(x)$ and y, respectively, such that $m_2(U_1 \times U_2) \subset U \subset V$. Then $V_i = f^{-1}(U_i) \cup U_i$, $i = 1,2$, are W-open neighborhoods of x and y, respectively. Since $f^{-1}(\bar{U}) - V$ is fiber compact, Condition 1 guarantees the existence of a fiber compact K in X such that if $m_1(x,y)$ are in $f^{-1}(\bar{U}) - V$, then x and y are in K. Since K is fiber compact, K is closed in W and so $K \times K$ is closed in $W \times W$. Hence $V_1 \times V_2 - K \times K$ is an open set containing (x,y) and a calculation shows that m maps $V_1 \times V_2 - K \times K$ into V.

Let $X = (0,1] \times [0,1]$, $Y = [0,1]$ and $f\colon X \to Y$ by $f(x,y) = y$. If X and Y have the usual multiplications, then Z is $[0,1] \times [0,1]$ with the usual multiplication. However, the multiplication is not continuous on W since $\{(\frac{1}{n},1)\} \to 1$ and $\{(1,1-\frac{1}{n})\} \to (1,1)$ in W but $\{(\frac{1}{n},1-\frac{1}{n})\}$ does not converge since it is a fiber compact set in X and hence closed in W.

If the multiplication on X is changed to be the usual multiplication in the first factor and the zero multiplication in the second and if Y is given the zero multiplication, then the conditions of Theorem 1 are satisfied. Since f is not a closed map, W is not the same as Z. Hence W is a Hausdorff semigroup topologically different from $[0,1] \times [0,1]$.

These examples illustrate how difficult it is to have m continuous on W. In fact, we have:

THEOREM 2. Suppose X is connected and for each y in Y, $f^{-1}(y)$ is not compact. If (W,m) is a first countable, Hausdorff semigroup, then Y has the zero multiplication.

PROOF. Let $t,y \in Y$ and let $z = m_2(t,y)$. Let $A = \{x \in X | m(x,y) = z\}$. Since $f^{-1}(t) \subset A$, $A \neq \phi$. Also A is closed in X since $m(A,y) = z$ implies that $m(\bar{A},y) = z$. Since $f^{-1}(y)$ is not compact, y is a limit point of $f^{-1}(y)$ in W and so there is a sequence $\{y_i\}$ in $f^{-1}(y)$ converging to y in W. Let $x \in A$ and $\{V_i\}$ be a countable neighborhood basis at x. If we assume that no V_i is contained in A, we can find a sequence $\{x_i\}$ which converges to x such that

$m(x_i y) \neq z$. Hence $m_1(x_i,y_i)$ is not in $f^{-1}(z)$ for all i, but $\{m_1(x_i,y_i)\}$ converges to z. Thus the set $B = \{m_1(x_i,y_i)\}$ is closed in X. For any $w \varepsilon Y$, $f^{-1}(w) \cap B$ is finite because otherwise B will have a convergent subsequence in the compact set $\{w\} \cup f^{-1}(w)$. This means that B is fiber compact and $W - B$ is a neighborhood of z which contradicts the fact that $\{m_1(x_i,y_i)\}$ converges to z. Thus A is open and must equal X since X is connected. All of this yields $m_2(Y,y) = z$. Let $t',y' \varepsilon Y$ and let $z' = m_2(t',y')$. The argument above will give that $m_2(t',Y) = z'$. Hence $z = z'$ and Y has the zero multiplication.

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