Internat. J. Math. & Math Sci.
VOL. 11 NO. 1 (1988) 27-36

SUMMABILITY METHODS BASED ON THE RIEMANN ZETA FUNCTION

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(Received September 12, 1986)

ABSTRACT. This paper is a study of summability methods that are based on the Riemann Zeta function. A limitation theorem is proved which gives a necessary condition for a sequence x to be zeta summable. A zeta summability matrix Z_t associated with a real sequence t is introduced; a necessary and sufficient condition on the sequence t such that Z_t maps l_1 to l_1 is established. Results comparing the strength of the zeta method to that of well-known summability methods are also investigated.

KEY WORDS AND PHRASES. Zeta summability method, zeta matrix method, l-l matrix, Cesaro method, Euler-Knopp method.

1980 AMS SUBJECT CLASSIFICATION CODE. 40C05, 40C15, 40D05, 40D25.

1. INTRODUCTION.

Recall that the Riemann zeta function is given by $\zeta(s) = \sum_{k=1}^{\infty}(1/k^s)$ for $s > 1$ (Titschmarch [1]). A number sequence is said to be *zeta summable* to L (or *ζ-summable* to L) provided that

$$\lim_{s \to 1^+} \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{x_n}{n^s} = L.$$

The zeta method is a "sequence-to-function" summability method whose domain consists of those sequence x such that the Dirichlet's series $\sum_{k=1}^{\infty}(x_k/k^s)$ is convergent for $s > 1$.

In the second section it is shown that the zeta summability method is regular and totally regular (preserves finite and infinite limits). A limitation theorem is proved which gives a necessary condition for a sequence x to be zeta summable. In section 3 we introduce a zeta summability matrix Z_t associated with a real sequence t; a necessary and sufficient condition on the sequence t such that Z_t maps l_1 into l_1 is established. The final section contains results comparing the strength of the zeta method to that of well-known summability methods. For example, the zeta method is stronger than the Cesàro method of order 1 but does not include the Cesàro method of order 2; the zeta method does not include and is not included in the Euler-Knopp method of order r for $0 < r < 1$.

2. BASIC THEOREMS

THEOREM 1. The ζ-summability method is totally regular.

Proof. First let x be a sequence satisfying $\lim_k x_k = L$, and suppose $\varepsilon > 0$. Then choose N_1 so that $k > N_1$ implies $|x_k - L| < \varepsilon/2$. Now for any positive integer k and $s > 1$ we see that $\sum_{k=1}^{N_1}(|x_k - L|/k^s)$ is bounded by $\sum_{k=1}^{N_1} |x_k - L| = M$. Since $\sum_{k=1}^{\infty} 1/k = \infty$, we can choose $N_2 > N_1$ so that $\sum_{k=1}^{N_2} 1/k > (2M/\epsilon) + 1$. Now choose δ such that $0 < \delta < \log[1 + (1/N_2)]/\log N_2$. Then for each $k \le N_2$, we have

$$k^{\delta} < k^{\log[1+(1/N_2)]/\log N_2} \le 1 + (1/N_2)\ ;$$

and if $1 < s < 1 + \delta$

$$(1/k) - (1/k^s) < (k^{\delta} - 1)/k^s < k^{\delta} - 1 < 1/N_2\ .$$

Summing from $k = 1$ to N_2, we obtain

$$\sum_{k=1}^{N_2} \left(\frac{1}{k^s}\right) > \sum_{k=1}^{N_2} \left(\frac{1}{k} - \frac{1}{N_2}\right)$$

$$> \left(\frac{2M}{\epsilon}\right) + 1 - 1$$

$$= \frac{2M}{\epsilon}\ .$$

Thus for $1 < s < 1 + \delta$,

$$\zeta(s) > \sum_{k=1}^{N_2} \frac{1}{k^s}$$

$$> \frac{2M}{\epsilon}\ .$$

and

$$\left| \frac{1}{\zeta(s)} \sum_{k=1}^{\infty} \frac{x_k}{k^s} - L \right| \le \frac{1}{\zeta(s)} \sum_{k=1}^{N_1} \frac{1}{k^s} |x_k - L| + \frac{1}{\zeta(s)} \sum_{k>N_1} \frac{1}{k^s} |x_k - L|$$

$$< \frac{\epsilon}{2M} M + \frac{\epsilon}{2}$$
$$= \epsilon\ .$$

Hence,

$$\lim_{s \to 1^+} \frac{1}{\zeta(s)} \sum_{k=1}^{\infty} \left(\frac{x_k}{k^s}\right) = L\ .$$

Now assume x is a real number sequence which diverges to ∞. Then for each number $M > 0$ there exists a positive interger N such that $x_k > M + 1$ for all $k > N$. Suppose $s > 1$ and consider

$$\frac{1}{\zeta(s)} \sum_{k=1}^{\infty} \frac{x_k}{k^s} > \frac{1}{\zeta(s)} \sum_{k=1}^{N} \frac{x_k}{k^s} + \frac{M+1}{\zeta(s)} \sum_{k>N} \frac{1}{k^s}$$

$$= \frac{1}{\zeta(s)} \sum_{k=1}^{N} \left(\frac{x_k - M - 1}{k^s}\right) + (M+1)\ .$$

Since $\zeta(s) \to \infty$ as $s \to 1^+$, we see that if s is sufficiently close to 1 on the right, then

$$\left| \frac{1}{\zeta(s)} \sum_{k=1}^{N} \left(\frac{x_k - M - 1}{k^s} \right) \right| < 1 \ ;$$

this implies that

$$\frac{1}{\zeta(s)} \sum_{k=1}^{\infty} \frac{x_k}{k} > \frac{1}{\zeta(s)} \sum_{k=1}^{N} \left(\frac{x_k - M - 1}{k^s} \right) + M + 1$$

$$> -1 + M + 1$$
$$= M \ .$$

Since M > 0 was chosen arbitrarily, we conclude that

$$\lim_{s \to 1^+} \frac{1}{\zeta(s)} \sum_{k=1}^{\infty} \frac{x_k}{k^s} = \infty \ .$$

A previous definition of "zeta summability" was given in Diaconis [2]. In that paper the bounded sequence x is said to be zeta summable to L if

$$\lim_{s \to 1^+} (s-1) \sum_{i=1}^{\infty} \frac{x_i}{i^s} = L \ .$$

This is equivalent to the difinition of the zeta method introduced in this paper. There equivalence is an immediate consequence of the fact that $\lim_{s \to 1+} \zeta(s)\,(s - 1) = 1$.

Recall that a Stoltz domain of angle , where $0 < \alpha < \pi/2$, is a complex number set of the form

$$S(\alpha) = \{ w : | \operatorname{Arg}(w-1) | < \alpha, \text{ and } | w | < 1 \} \ .$$

(Powell et al [3]).

We shall use a variant of this concept, which we shall call a "reflected Stoltz domain of angle α "

$$S^*(\alpha) = \{ w : | \operatorname{Arg}(w-1) | < \alpha \text{ and } R_e(w) > 1 \} \ .$$

This concept is now used to extend the zeta method to one using a complex-valued function limit, and we establish the regularity of this extension.

THEOREM 2. Let $S^*(\alpha)$ be a reflected Stoltz domain of angle α; if the sequence x converges to L then

$$\lim_{w \to 1, w \in S^*(\alpha)} \frac{1}{\zeta(w)} \sum_{k=1}^{\infty} \left(\frac{x_k}{k^w} \right) = L \ .$$

The proof of Theorem 2 that we shall give needs the following preliminary result.

LEMMA 1. For $w = \sigma + i\, t$, $w \in S^*(\alpha)$, and w sufficiently close to 1, we have

$$\frac{1}{|\zeta(w)|} \sum_{k=1}^{\infty} \left| \frac{1}{k^w} \right| \le 2 \sec \alpha \ .$$

Proof. Since $\zeta(w)$ can be expanded in the form $(w - 1)^{-1} + P(w - 1)$, where $P(w - 1)$ is a power series in $(w - 1)$, (Hardy [4], p. 333), we have

$$\frac{1}{|\zeta(s)|}\sum_{k=1}^{\infty}\frac{1}{k^{w}}=\frac{|\zeta(\sigma)|}{|\zeta(w)|}$$

$$=\frac{\left|\frac{1}{\sigma-1}+P(\sigma-1)\right|}{\left|\frac{1}{w-1}+P(w-1)\right|}$$

$$\rightarrow\left|\frac{w-1}{\sigma-1}\right| \qquad \text{as } w\rightarrow 1\ .$$

Since the limit value $|w-1|/|\sigma-1| \leq \sec\alpha$ for $w \in S^*(\alpha)$, this proves the assertion.

Now we prove Theorem 2.

Proof (of Theorem 2). Let $\varepsilon > 0$. Since x converges to L, we can choose $N_1 \ni |x_k - L| < (\varepsilon/4)\cos\alpha$ for $k \geq N_1$. Let $\sum_{k=1}^{N_1}|x_k - L| = M$. Since $\zeta(w) \rightarrow \infty$ as $w \rightarrow 1$, we have $1/\zeta(w) < \varepsilon/2M$ for w sufficiently close to 1.

Now for $w \in S^*(\alpha)$, we have

$$\left|\frac{1}{\zeta(w)}\sum_{k=1}^{\infty}\frac{x_k}{k^w}-L\right| \leq \frac{1}{|\zeta(w)|}\sum_{k=1}^{\infty}\frac{1}{|k^w|}\cdot|x_k-L|$$

$$=\frac{1}{|\zeta(w)|}\left[\sum_{k=1}^{N_1}\frac{1}{|k^w|}|x_k-L|+\sum_{k>N_1}\frac{1}{|k^w|}|x_k-L|\right]$$

$$<\frac{M}{|\zeta(w)|}+\frac{1}{|\zeta(w)|}\cdot\frac{\epsilon}{4}\cos\alpha\sum_{k=1}^{\infty}\frac{1}{|k^w|}$$

$$<\frac{\epsilon}{2}+\frac{\epsilon}{4}(\cos\alpha)\,2\sec\alpha$$

$$=\epsilon\ .$$

Next we prove a limitation theorem which asserts that the ζ-summability method cannot sum a sequence that diverges too rapidly.

THEOREM 3. If a complex number sequence x is ζ-summable, then for each $s > 1$, $x_n = o(n^s)$. Moreover, the term $o(n^s)$ is the best possible in the sense that the conclusion fails if n^s is replaced by any real sequence to such that t_n/n^s decreases to zero.

Proof. For x to be ζ-summable, x must be in the domain of the ζ-summability method. Therefore $\sum_{n=1}^{\infty}(x_n/n^s)$ converges for all $s > 1$, which implies that $\lim_n(x_n/n^s) = 0$. If n^s is replaced by t_n, where t_n/n^s decreased to 0, then we assert that it will not be true that $x_n = o(t_n)$ whenever x is ζ-summable. This is equivalent to showing that there is a sequence x such that x is ζ-summable and $x_n \neq o(t_n)$. Define the sequence x by $x_n = (-1)^{n+1}t_n$, so that

$$\sum_{n=1}^{\infty}\frac{x_n}{n^s}=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{t_n}{n^s}\ .$$

This is a convergent alternating series, and its (positive) sum is bounded by its first term t_1.

Hence,

$$\lim_{s \to 1^+} \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{x_n}{n^s} = 0,$$

i.e., x is ζ-summable to 0. But $x_n \neq o(t_n)$ because for each n, $|x_n/t_n| = 1$.

3. ZETA SUMMABILITY MATRICES

Definition. Let t be a sequence of real numbers such that $t(n) > 1$ for every n and $\lim_n t(n) = 1$. Then the *zeta matrix* $Z_t = [z_{nk}]$ *associated with the sequence t* is defined by

$$z_{nk} = \frac{1}{\zeta(t(n))k^{t(n)}} \quad \text{for } n,k = 1,2,3,\ldots .$$

In this section we make use of two well-known theorems in summability theory, which we shall subsequently cite by name only; they are Silverman-Toeplitz Theorem ([5]and [6]) and the Knopp-Lorentz Theorem [7]. It is an easy calculation to show that Z_1 satisfies the conditious of the Silverman-Toeplitz Theorem for regularity. Moreover, Z_t is totally regular because all of its entries are positive real numbers ([3] p. 35). We summarize these observations in the following theorem.

THEOREM 4. The zeta matrix Z_t associated with the sequence t is totally regular.

The next result is a characterization of those sequences t for which Z_t is an *l-l* matrix, i.e., Z_t maps l_1 into l_1.

THEOREM 5. The matrix Z_t is an *l-l* matrix if and only if t - I is in l_1.

Proof. Since each row sequence of the matrix Z_t is decreasing, the set of the sums of column sequences of the matrix Z_t is bounded by the sum of its first column entries. Therefore by the Knopp-Lorentz Theorem, it is enough to show that the first column sum is finite whenever $\sum_{n=1}^{\infty}$ (t(n) - 1) is convergent. This is a consequence of the inequality

$$\sum_{n=1}^{\infty} \frac{1}{\zeta(t(n))} \leq \sum_{n=1}^{\infty} (t(n)-1),$$

which follows immediately from the fact that for s > 1,

$$\frac{s-1}{s} \leq \frac{1}{\zeta(s)}$$

$$\leq s-1 \qquad (*)$$

Hence Z_t is an *l-l* matrix.

Conversely, assume Z_t maps l_1 to l_1. Since $t(n) > 1$ and $\lim_n t(n) = 1$ for every n, we can choose a positive integer N such that $0 < t(n) - 1 < 1$ for $n \geq N$. Suppose t - 1 is not in l_1; then

$$\sum_{n=N}^{\infty} \left(\frac{1}{t(n)}\right) = \sum_{n=N}^{\infty} \left(\frac{t(n)-1}{t(n)}\right)$$

$$> \sum_{n=N}^{\infty} \left(\frac{t(n)-1}{2}\right)$$

$$= \infty .$$

Now $\sum_{n=1}^{\infty}(1/\zeta(t(n)))$ diverges to infinity because of the inequality $1/\zeta(t(n)) \geq (1 - 1/t(n))$ as in (*). Therefore, by the Knopp-Lorentz Theorem, Z_t is not an *l-l* matrix. This completes the proof of the theorem.

4. INCLUSION THEOREMS.

In this section we compare the strength of the zeta method and the zeta matrix methods to several well-known summability methods. Throughout this section C_α denotes the Cesaro summability matrix of order α and E_r the Euler-Knopp summability matrix of order r.

LEMMA 2. If x is a sequence that is C_1-summable, then x is in the domain of the ζ-summability method, and hence, x is in the domain of every Z_t method.

Proof. Assume that x is C_1-summable to L: $\lim_n (x_1 + \ldots + x_n)/n = L$. To get the conclusion it is enough to show that the abscissa of convergence σ_o of the Direchlet series $\sum_{n=1}^{\infty} x_n/n^s$ is less than or equal to 1, where σ_o is given by

$$\sigma_o = \limsup_{n\to\infty} \frac{\log\left|\sum_{k=1}^{n} x_k\right|}{\log n}.$$

(Hardy et al [8] or Titschmarch [9]). Since x is c_1-summable to L, there exists a positive integer N such that if $n \geq N$, then

$$\frac{\left|\sum_{k=1}^{n} x_k\right|}{n} \leq |L| + 1.$$

This implies that $\left|\sum_{k=1}^{n} x_k\right| \leq n(|L| + 1)$, so

$$\log\left|\sum_{k=1}^{n} x_k\right| \leq \log[n(|L| + 1)].$$

Therefore

$$\begin{aligned}\sigma_o &= \limsup_{n\to\infty} \frac{\log\left|\sum_{k=1}^{n} x_k\right|}{\log n} \\ &\leq \limsup_{n\to\infty} \left\{\frac{\log n(|L| + 1)}{\log n}\right\} \\ &= 1.\end{aligned}$$

THEOREM 6. The Z_t method includes the C_1 method.

Proof. This inclusion is equivalent to the regularity of the matrix $Z_tC_1^{-1}$, which can be verified by direct calculation using the Silverman-Toeplitz Theorem.

The following example shows that the C_1 method does not include the Z_t method.

EXAMPLE.

$$\text{Let } x = \{(-1)^k k\}\ ;\ \text{then}$$

$$(Z_t x)_n = \sum_{k=1}^{\infty} \frac{(-1)^k k}{\zeta(t(n))k^{t(n)}}$$

$$= \frac{1}{\zeta(t(n))} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{t(n)-1}}$$

Since

$$-1 \leq \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{t(n)-1}}$$

$$\leq 0 \qquad \text{for } t^{(n)}>1$$

and $\lim_n \zeta(t(n)) = \infty$, it is easy to see that $\lim_n(Z_t x)_n = 0$. On the other hand, we have

$$(C_1 x)_n = \frac{1}{n}\sum_{k=1}^{n}(-1)^k k$$

$$= \begin{cases} \frac{1}{2}\ , \text{if } n \text{ is even} \\ \frac{-1}{2}\ , \text{if } n \text{ is odd.} \end{cases}$$

Thus $\lim_n(c_1 x)_n$ does not exist, so x is not C_1-summable.

By a "continuous parameter sequence-to-function transformation", we mean a summablility method F that is determined as follows by a fuction sequence $\{f_k(z)\}_{k=1}^{\infty}$: for a given sequence x form the function

$$F_x(z) = \sum_{k=1}^{\infty} f_k(z)x_k\ ; \qquad (**)$$

if $\lim_{z \to a} F_x(z) = L$, then we say that "x is F-summable to L". For a given function sequence $\{f_k(z)\}_{k=1}^{\infty}$ and a given number sequence t, we can also form an associated matrix F_t, which is given by

$$F_t[n,k] = f_k(t(n))\ .$$

The next lemma, which will be used to compare the C_1 method and the ζ method, is a comparision of the method if and the associated matrix method F_t.

LEMMA 3. Let F be a continuous parameter sequence-to-function transformation as in (**) and define the sequence sets

$$S_F = \{x : \lim_{z \to a} F_x(z) \text{ exists}\}\ ,$$

$$S_{F_t} = \{x : F_t x \text{ is convergent }\}\ ,$$

and

$$T = \{t : \lim_n t(n) = a\}\ ;$$

then

$$S_F = \bigcap_{t\epsilon T} S_{F_t}.$$

Proof. We show that each of S_F and $\bigcap_{t\epsilon T} S_{F_t}$ contains the other. Since F_t includes F for t in T, we have

$$S_F \subseteq \bigcap_{t\epsilon T} S_{F_t}.$$

To prove the reverse inclusion, we consider a sequence x which is not in S_F. It follows that $\lim_{z \to a} F_x(z)$ does not exist. By the sequential criterion for function limits (Almsted [10], p. 73), there is a sequence t\ in T such that $\lim_n (F_t x)_n$ does not exist. This implies that x is not in the set S_{F_t}. Hence x is not in the set $\bigcap_{t\epsilon T} S_{F_t}$.

THEOREM 7. The ζ-summability method is stronger than the C_1 method.

Proof. By Lemma 3, we have $S_\zeta = \bigcap_{t\epsilon T} S_{z_t}$. Since the Z_t method includes the C_1 method for all t in T, we have $S_{C_1} \subseteq \bigcap_{t\epsilon T} S_{z_t} = S_\zeta$. Now if x is a sequence that is C_1-summable to L, then x is Z_t summable to L for all t in T. Therefore the sequential criterion for function limits ensures that x is ζ-summable to L. Hence, the ζ method includes the C_1 method. It is easy to see that the C_1 method does not include the ζ method because C_1 method does not include the Z_t method.

As a consequence of Theorem 6, we can infer that Z_t includes any method that is included by C_1. For example, Z_t includes the divisor method D_r for $r > 0$. (Fridy [11]).

Let H_2 denote the Holder method of order 2. By arguing as in the proof of Theorem 6, we can prove

THEOREM 8. If the sequence x is H_2-summable to L and x is in the domain of the Z_t method, then x is Z_t summable to L.

COROLLARY. If the sequence x is H_2-summable to L and x is in the domain of the ζ-summability method, then x is ζ-summable to L.

The conclusion of the preceding Corollary does not hold if x is not in the domain of the ζ method. This is shown by the following example.

EXAMPLE. Let x be the sequence defined by

$$x_n = \begin{cases} (-1)^k k^{\frac{3}{2}}, & \text{if } n=2k,\ k=1,2,\ldots \\ (-1)^{k+1} k^{\frac{3}{2}}, & \text{if } n=2k-1,\ k=1,2,\ldots \end{cases},$$

If $x \leq 3/2$, then the series $\sum_{n=1}^{\infty} (x_n/n^s)$ is divergent because its n^{th} term does not approach 0. Therefore x is not in the domain of the ζ method, and hence, x is not ζ-summable. Now we show that x is H_2-summable to zero. Since $(C_1x)_{2k-1} = (-1)^{k+1}k^{3/2}/(2k-1)$ and $(C_1x)_{2k} = 0$, we see that the (odd) partial sums alternate in sign after k = 3; thus the partial sum is not greater than the last term, which is $0(k^{1/2})$.

Therefore, upon dividing by 2k-1 to form $C_1(C_1x)_{2k-1}$, we have

$$
\begin{aligned}
(H_2x)_{2k-1} &= \left(\frac{1}{2k-1}\right) O\left(k^{\frac{1}{2}}\right) \\
&= O\left(k^{-\frac{1}{2}}\right) \\
&= o(1),
\end{aligned}
$$

which proves that x is H_2-summable to zero.

Since the Holder method of order 2 is equivalent to the Cesaro method of order 2 (Hardy [4], p. 103), we can immediately get the following theorem.

THEOREM 9. If x is a sequence which is C_2-summable to L and x is in the domain of the summability method, then x is ζ-summable to L.

It is well known that for each number r satisfying $0 < r < 1$ and any nonzero real number α, E_r. By using these facts, we have the following result.

THEOREM 10. The ζ method is not included in E_r for $0 < r < 1$.

The following example shows that the ζ method does not include E_r for $0 < r < 1$.

EXAMPLE. Given r between 0 and 1 choose $\varepsilon > 0$ satisfying $r < 2/(2+\varepsilon)$. Next define $x_k = (-1-\varepsilon)^k$. Then

$$
\begin{aligned}
(E_rx)_n &= \sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k} (-1-\epsilon)^k \\
&= [(-1-\epsilon)r + (1-r)]^n \\
&= [(-2-\epsilon)r + 1]^n .
\end{aligned}
$$

Since $0 < r < 2/(2+\varepsilon)$, we have $-1 < (-2-\varepsilon)r + 1 < 1$. This implies that

$$
\begin{aligned}
\lim_n (E_rx)_n &= \lim_n [(-2-\epsilon)r + 1]^n \\
&= 0,
\end{aligned}
$$

i.e., x is E_r-summable to 0. But x is not in the domain of the ζ method because the series

$$
\sum_{k=1}^{\infty} \frac{(-1-\epsilon)^k}{k^s}
$$

is not convergent for any s, whence x is not in the domain of the ζ method.

ACKNOWLEDGEMENT. This work is a portion of the author's doctoral dissertation, written under the supervision of Professor J. A. Fridy at Kent State University, Kent, OH, 1985.

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