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A VARIATIONAL PRINCIPLE FOR COMPLEX BOUNDARY VALUE PROBLEMS

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ABSTRACT This paper provides a variational formalism for boundary value problems which arise in certain feilds of research such as that of electricity, where the associated boundary conditions contain complex periodic conditions. A functional is provided which embodies the boundary conditions of the problem and hence the expansion (trial) functions need not satisfy any of them.

KEY WORDS AND PHRASES Variational principle, functional, stationary, boundary conditions complex functions, line integral.
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1. INTRODUCTION.

Motivated by complex periodic boundary conditions which arise in certain problems such as those of modelling the stator of a turbogenerator (see next section for detail), we give in this paper a variational formalism which takes into consideration such boundary conditions. We produce a functional which is stationary at the solution of a given boundary value problem for a class of expansion functions which do not satisfy any of the boundary conditions; these are satisfied only at the solution point. Three types of conditions are considered: 1) Dirichlet conditions, 2) Neumann or mixed conditions and 3) periodic conditions on parallel segments of the boundary.

Let R be a given complex domain with boundary Γ. Following the work of Delves and Hall [1], we split the boundary into four non-overlapping segments Γ_i $i = 1,2,3,4$ and assume that periodicity conditions are imposed on the segments Γ_3 and Γ_4 such that for some fixed $\underline{a}$, $\Gamma_4 = \{\underline{y} = \underline{x} + \underline{a} \mid \underline{x} \in \Gamma_3\}$. In this case we have the relations:

$$\underline{n}_4(\underline{x} + \underline{a}) = (\underline{n}_3 \cdot \underline{n}_4) \cdot \underline{n}_3(\underline{x})$$

and (1.1)

$$\int_{\Gamma_4} I(\underline{y})\, ds = \int_{\Gamma_3} I(\underline{x} + \underline{a})\, ds$$

where $\underline{n}_3$ and $\underline{n}_4$ are the unit outward normals to Γ_3 and Γ_4 respectively and $\int ds$ is a line integral along the boundary with positive direction taken counterclockwise.

2. THE PROBLEM

Let the problem whose solution is sought be of the following form:

$$-\nabla^2 u + B(\underline{x})u = g(\underline{x}),\ \underline{x} \in R \tag{2.1.a}$$

with the prescribed boundary conditions:

$$\begin{aligned} u(\underline{x}) &= g_1(\underline{x}),\ \underline{x} \in \Gamma_1 \\ \nabla u.\underline{n}(\underline{x}) &= qu(\underline{x}) + g_2(\underline{x}),\ \underline{x} \in \Gamma_2 \\ u(\underline{x}) &= e^{i\theta} u(\underline{x} + \underline{a}),\ \underline{x} \in \Gamma_3 \\ \nabla u.\underline{n}(\underline{x}) &= -e^{-i\theta} \nabla u.\underline{n}(\underline{x} + \underline{a}),\ \underline{x} \in \Gamma_3 \end{aligned} \tag{2.1.b}$$

where Γ_2 and/or Γ_3 may be void.

In modelling the stator of a turbogenerator where the rotor rotates at angular frequency and is effectively a bar magnet generating a rotating magnetic field, periodic boundary conditions of the form:

$$u(\underline{x}) = e^{i\theta} u(\underline{x} + \underline{a})$$

arise for the first harmonic component; and the normal gradient condition has:

$$\nabla u.\underline{n}(\underline{x}) = -e^{-i\theta} \nabla u.\underline{n}(\underline{x} + \underline{a})$$

where θ is the sector angle. These two conditions are exactly the last two conditions of (2.1.b).

3. A FUNCTIONAL EMBODYING THE BOUNDARY CONDITIONS.

In this section we produce a functional which is stationary at the solution of (2.1) for a class of functions which do not satisfy any of the boundary conditions since these conditions are incorporated via suitable terms in the functional J given as:

$$\begin{aligned} J(V) = &\int_R [\nabla^2 V + BV^2 - 2gV]\, d\underline{x} \\ &+ 2\int_{\Gamma_1} (g_1 - V)(\nabla V.\underline{n})\, ds \\ &- 2\int_{\Gamma_2} [q/2\, V^2 + g_2 V]\, ds \\ &- \int_{\Gamma_3} [V(\underline{x}) - e^{i\theta} V(\underline{x} + \underline{a})][\nabla V(\underline{x}) - (\underline{n}_3.\underline{n}_4)e^{-i\theta} \nabla V(\underline{x} + \underline{a})].\underline{n}\, ds \end{aligned} \tag{3.1}$$

Next, it will be shown that if we expand the trial function V about the true solution u, of (2.1): $V = u + \varepsilon w$, where ε is a scalar and w is an arbitrary variation, then J(V) is stationary.

Define

$G(\varepsilon) = J(u + \varepsilon w)$, then

$$\begin{aligned} \frac{dG(0)}{d\varepsilon} = &\ 2\int_R [\nabla w . \nabla u + Bwu - gw].\underline{n}\, d\underline{x} \\ &+ 2\int_{\Gamma_1} [(g_1 - u)\nabla w - w\nabla u].\underline{n}\, ds \\ &- 2\int_{\Gamma_2} (qu + g_2)w\, ds \\ &- \int_{\Gamma_3} [u(\underline{x}) - e^{i\theta} u(\underline{x} + \underline{a})][\nabla w(\underline{x}) - (\underline{n}_3.\underline{n}_4)e^{-i\theta} \nabla w(\underline{x} + \underline{a})].\underline{n}\, ds \\ &- \int_{\Gamma_3} [w(\underline{x}) - e^{i\theta} w(\underline{x} + \underline{a})][\nabla u(\underline{x}) - (\underline{n}_3.\underline{n}_4)e^{-i\theta} \nabla u(\underline{x} + \underline{a}).\underline{n}\, ds \end{aligned} \tag{3.2}$$

The first line integral in (3.2) reduces by Green's theorem and (2.1.a) to:

$$2\int_R [\nabla w.\nabla u + Buw - gw]\, d\underline{x} = 2\int_\Gamma w\nabla u.\underline{n}\, ds \tag{3.3}$$

$$= \left(\int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4}\right)(2w\nabla u).\underline{n}\, ds \tag{3.4}$$

where we have written the line integral of (3.3) as the sum of four line integrals along the boundaries into which Γ has been decomposed. The integrals over Γ_1 and Γ_2 of (3.4) cancel the corresponding integrals over Γ_1 and Γ_2 in (3.2) taking into consideration the boundary conditions in (2.1.b). Also from (2.1.b), it is obvious that the first of the two line integrals over Γ_3 in (3.2) is equal to zero. What is left is to show that the last integral in (3.2)(hereafter referred to as LI) cancels the line integrals over Γ_3 and Γ_4 in (3.4). But

$$\begin{aligned} LI = &-\int_{\Gamma_3} w(\underline{x})\, \nabla u(\underline{x}).\underline{n}\, ds \\ &-\int_{\Gamma_3} w(\underline{x})[-e^{-i\theta}\, \nabla u(\underline{x}+\underline{a})](\underline{n}_3.\underline{n}_4).\underline{n}_3\, ds \\ &-\int_{\Gamma_3} w(\underline{x}+\underline{a})[-e^{i\theta}\, \nabla u(\underline{x}).\underline{n}]\, ds \\ &-\int_{\Gamma_3} w(\underline{x}+\underline{a})[\nabla u(\underline{x}+\underline{a})](\underline{n}_3.\underline{n}_4).\underline{n}_3\, ds \end{aligned}$$

Using the relations (1.1) and the boundary conditions (2.1.b), we get:

$$LI = -2\int_{\Gamma_3} w(\underline{x})\nabla u(\underline{x}).\underline{n}\, ds - 2\int_{\Gamma_4} w(\underline{x})\nabla u(\underline{x}).\underline{n}\, ds \tag{3.5}$$

These line integrals over Γ_3 and Γ_4 cancel the corresponding ones in (3.4). Hence the functional J is stationary at the solution u.

4. MATRIX SET-UP.

To descibe the matrix set-up stage, we consider for convenience and simplicity the solution of the following one dimensional problem:

$$\left[\frac{d^2}{dx^2} + B(zx)\right] f(zx) = G(zx), \qquad -1 \le x \le 1 \tag{4.1.a}$$

together with the boundary conditions:

$$f(-z) = \alpha\ , \quad f(z) = \beta \tag{4.1.b}$$

where z is regarded as a parameter that takes any complex value.

We seek an approximate solution $f_N(zx)$ to $f(zx)$ of the form:

$$f_N(zx) = \sum_{n=1}^{N} a_n(z)\, h_n(x), \qquad -1 \le x \le 1 \tag{4.2}$$

Then the problem represents a one-dimensional form of (2.1); and the functional J given in (3.1) reduces to:

$$J(V) = \int_{-1}^{1} [(V')^2 + BV^2 - 2GV]\, dx - 2[\alpha - V(-1)]V'(-1) + 2[\beta - V(1)]V'(1) \tag{4.3}$$

The coefficients $a_n(z)$ are defined by the stationary point of J (at the solution where V = f); that is, by the equations:

$$L\underline{a} = [\ A + B + S\ ]\underline{a} = \underline{G} + \underline{H} \tag{4.4.a}$$

where A, B, and S are N×N matrices; G and H are N-vectors, with components:

$$A_{ij} = \int_{-1}^{1} h'_i h'_j \, dx, \quad B_{ij} = \int_{-1}^{1} h_i B(zx) h_j \, dx, \quad G_i = \int_{-1}^{1} h_i G(zx) \, dx,$$

$$S_{ij} = h_i(-1)h'_j(-1) + h_j(-1)h'_i(-1) - h_i(1)h'_j(1) - h_j(1)h'_i(1), \tag{4.4.b}$$

$$H_i = \alpha h'_i(-1) - \beta h'_i(1), \qquad i,j = 1,2,\ldots,N.$$

When using global expansion functions, it is desirable for stability reasons to use orthogonal polynomials (see Mikhlin [2]). Accordingly, in (4.2) we take

$$h_{-2} = 1 \; ; \; h_{-1} = x \; ; \; h_n = (1 - x^2)\, T_n(x) \;\; , \; n = 0,1,2,\ldots,r \tag{4.5}$$

where $r = N-3$ and $T_n(x)$ is the nth Chebyshev polynomial of the first kind. The reason for this choice of basis is the need to handle the derivative terms in the matrix A without introducing artificial singularities. To calculate the elements in (4.4.b), we expand the functions B(zx) and G(zx) by Chebyshev series and use Fast Fourier Techniques to approximate the expansion coefficients. Thence we relate the elements A_{ij}, B_{ij} and G_i of (4.4.b) to the coefficients of these expansions. This together with a numerical example will be considered in a subsequent paper.

While we do not attempt an error analysis here, the rapidity of convergence in calculating the matrix equation (4.4) has been considered formally by Delves and Mead [3], Freeman et al [4] and Delves and Bian [5]. In these papers it is shown that a complete characterisation of the convergence of the calculation can be given in terms of an assumed structure of the matix L in (4.4) and the convergence of the Fourier coefficients of the right hand function G(zx) in (4.1.a). Both a priori and a posteriori truncation error estimates are provided by Delves [6] where a very similar treatment to the one given in this section is used for Frdholm integral equations and from which we take (ignoring the a priori estimate since it contains an unknown constant):

$$\text{A posteriori estimate: } \sim \frac{C}{s-1} N^{-(s-1)} \sim N a_N \tag{4.6}$$

which is a standard bound; $s = \min(p,q)$ where p and q depend on the differentiability of B(zx) and G(zx). The procedure given in this section can easily be extended to two dimesions in a straightforward manner and details are omitted.

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