ON SOME WEAK CONDITIONS OF COMMUTATIVITY
IN COMMON FIXED POINT THEOREMS

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ABSTRACT. We generalize common fixed point theorems of Fisher and Sessa in complete metric spaces, using some conditions of weak commutativity between a set-valued mapping and a single-valued mapping. Suitable examples prove that these conditions do not imply each of the others.

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1. INTRODUCTION.

In this paper (X,d) denotes a complete metric space and $B(X)$ stands for the set of all nonempty bounded subsets of X. The function δ of $B(X) \times B(X)$ into $[0, +\infty)$ is defined as

$$\delta(A,B) = \sup \{d(a,b): a \in A, b \in B\}$$

for all A,B in $B(X)$. If $A = \{a\}$ is singleton, we write $\delta(A,B) = \delta(a,B)$ and if $B = \{b\}$, then we put $\delta(A,B) = d(a,b)$. It is easily seen that

$$\delta(A,B) = \delta(B,A) \geq 0,$$
$$\delta(A,B) \leq \delta(A,C) + \delta(C,B),$$
$$\delta(A,A) = \text{diam } A,$$
$$\delta(A,B) = 0 \text{ implies } A=B= \{a\}$$

for all A,B,C in B(X). We recall some definitions and a basic lemma of Fisher [1].
Let $\{A_n: n=1,2,\ldots\}$ be a sequence of nonempty subsets of X. We say that the
sequence $\{A_n\}$ converges to a subset A of X if each point a in A is the limit
of a convergent sequence $\{a_n\}$ with a_n in A_n for n=1,2,... and if for any
$\varepsilon > 0$, there exists an integer N such that $A_n \subseteq A_\varepsilon$ for n>N, A_ε being the union of
all open spheres with centers in A and radius ε. The following lemma holds.

    LEMMA 1. If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of (X,d) which
converge to the bounded subsets A and B respectively, then the sequence
$\{\delta(A_n,B_n)\}$ converges to $\delta(A,B)$.

    A set-valued mapping F of X into B(X) is continuous at the point x in X
if whenever $\{x_n\}$ is a sequence of points of X converging to x, the sequence $\{Fx_n\}$
in B(X) converges to Fx. F is said to be continuous in X if it is continuous
at each point x in X. We say that z is a fixed point of F if z is in Fz.

    Following the notations of our foregoing paper [2], we denote by ψ the set of
all real functions ψ of $[0,+\infty)$ into $[0,+\infty)$ such that ψ is nondecreasing,
right continuous and $\psi(t) < t$ for all t > 0.

2. SOME COMMENTS.

    Let I be a mapping of X into itself such that

$$F(X) \subseteq I(X). \tag{2.1}$$

    Let x_o be an arbitrary point of X and let z_o be an arbitrary point chosen
in Fx_o. Since (2.1) holds, let x_1 be a point in X such that $Ix_1 = z_o$. Having
defined the point x_n and chosen an arbitrary point z_n in Fx_n, then we can define
inductively the sequence $\{x_n\}$ such that $Ix_{n+1} = z_n \in Fx_n$ for n=0,1,2...

    Using this iterative process, Fisher [3] proved the following result.

    THEOREM 1. Let F be a mapping of X into B(X) and let I be a continuous
mapping of X into itself satisfying the inequality

$$\delta(Fx,Fy) \leqq c \cdot \max \{d(Ix,Iy), \delta(Ix,Fx), \delta(Iy,Fy), \delta(Ix,Fy), \delta(Iy,Fx)\} \tag{2.2}$$

for all x,y in X, where $0 \leqq c < 1$. If FIx = IFx for all x in X and if
(2.1) holds, then F and I have a unique common fixed point z and further
Fz = {z}.

    Fisher [1] also proved that

    THEOREM 2. Theorem 1 holds if one assumes the continuity of F in X instead
of the continuity of I.

    The proof of both these theorems are based on the fact that the sequence
$\{\delta(Fx_n,Fx_1): n=0,1,2,\ldots\}$ is bounded for any x_o in X. As in [2], from now on we
assume that

$$\sup \{\delta(Fx_n,Fx_1) : n=0,1,2\ldots\} < + \infty \tag{2.3}$$

for some point x_o in X. Further in [2], we defined two mappings F and I to be
weakly commuting if $IFx \in B(X)$ and

$$\delta(FIx,IFx) \leqq \max \{\delta(Ix,Fx), \text{diam } IFx\} \tag{2.4}$$

for all x in X. Clearly two commuting mappings F and I are weakly commuting,
but in general two weakly commuting mappings do not commute as it is shown in the
Example 1 of [2].

Using this concept, the first part of Theorem 3.1 of [2], that generalizes Theorem 1, runs as follows.

THEOREM 3. Let F be a mapping of X into B(X) and let I be a continuous mapping of X into itself satisfying the inequality

$$\delta(Fx,Fy) \leq \psi(\max\{d(Ix,Iy), \; \delta(Ix,Fx), \; \delta(Iy,Fy), \; \delta(Ix,Fy), \; \delta(Iy,Fx)\}) \qquad (2.5)$$

for all x,y in X, where $\psi \in \Psi$. If there exists a point x_o in X satisfying condition (2.3), if F and I weakly commute and if (2.1) holds, then F and I have a unique common fixed point z and further Fz = {z}.

Note that if we assume $\psi(t) = c \cdot t$ for all $t \geq 0$ and $0 \leq c < 1$, (2.5) becomes (2.2). Following Itoh and Takahashi [4], we also consider two mappings F and I such that IFx $\subseteq$ FIx for all x in X. In this case, we say that F and I quasi commute and it is evident that if F and I commute, they also quasi commute.

When we wrote the paper [2], we were unaware about the result of [1] and, under the same assumptions of Theorem 3, in the second part of Theorem 3.1 of [2], we assumed the continuity of F instead of the continuity of I. In [2], we established the following inequality (see inequality (3.6) of [2]),

$$d(z_m,z_n) \leq \delta(z_m,Fx_n) \leq \delta(Fx_m,Fx_n) < \epsilon \qquad (2.6)$$

for any m,n > p, p being a suitable nonnegative integer. This inequality implies that $\{z_n\}$ is a Cauchy sequence, which converges to a point z in X since X is complete . Unfortunately, the second part of the proof of Theorem 3.1 of [2] was not correctly established. Strictly speaking, the gap consists in the fact that (2.6) does not imply the following inequality of [2],

$$\delta(z,Fz_n) \leq d(z,z_n) + \delta(z_n,Fz_n) \leq d(z,z_n) + \epsilon$$

for n > p, from which, as $n \to +\infty$, one should deduce that Fz = {z}.

Here we point out that the second part of Theorem 3.1 of [2] can be substituted by the following result, which is a direct generalization of Theorem 2.

THEOREM 4. Let F be a continuous mapping of X into B(X) and let I be a mapping of X into itself satisfying the inequality (2.5) for all x,y in X, where $\psi \in \Psi$. If there exists a point x_o in x satisfying condition (2.3), if F and I quasi commute and if (2.1) holds, then F and I have a unique common fixed point z and further Fz = {z}.

PROOF. It is a minor variant of the proof of Theorem 1 of [1] and we omit it for brevity.

Note that if F and I commute and $\psi(t) = c \cdot t$ for all $t \geq 0$, $0 \leq c < 1$, we deduce Theorem 2 from Theorem 4.

3. EXAMPLES.

The following example proves the greater generality of Theorem 4 over Theorem 2.

EXAMPLE 1. Let X = [0,1] with the function δ induced by the Euclidean metric d. Define the mappings F and I as Fx = [0,x/(x+4)] for all x in X and Ix = x/2 if $0 \leq x \leq 2/3$ and Ix = 1 if $2/3 < x \leq 1$. Then we have that

$$F(X) = [0, \tfrac{1}{5}] \subseteq [0, \tfrac{1}{3}] \cup \{1\} = I(X)$$

and

$$IFx = [0, x/(2x+8)] \subseteq \begin{cases} [0, x/(x+8)] = FIx & \text{if } 0 \leq x \leq 2/3, \\ [0, 1/5] = FIx & \text{if } 2/3 < x \leq 1. \end{cases}$$

Thus   F   and   I satisfy condition (2.1) and quasi commute. Further, we have that

$$\delta(Fx,Fy) = \max\{\tfrac{x}{x+4}, \tfrac{y}{y+4}\} \leq \tfrac{1}{2} \cdot \max\{\tfrac{x}{2}, \tfrac{y}{2}\}$$

$$= \tfrac{1}{2} \cdot \max\{\delta(Ix, Fx), \delta(Iy, Fy)\}$$

if x and y are in [0,2/3] and

$$\delta(Fx,Fy) = \max\{\tfrac{x}{x+4}, \tfrac{y}{y+4}\} < \tfrac{1}{2} \cdot 1 = \tfrac{1}{2} \cdot \max\{\delta(Ix,Fx), \delta(Iy,Fy)\}$$

if at least x or y is in (2/3,1]. Then the inequality (2.5) is satisfied if one assumes $\psi(t) = t/2$ for all $t \geq 0$ and clearly (2.3) holds since X is a bounded metric space. So all the assumptions of Theorem 4 hold but Theorem 2 is not applicable since F and I do not commute. Note that F and I also weakly commute since

$$\delta(FIx,IFx) = \begin{cases} x/(x+8) \leq x/2 = \delta(Ix,Fx) & \text{if } 0 \leq x \leq 2/3, \\ 1/5 < 1 = \delta(Ix,Fx) & \text{if } 2/3 < x \leq 1, \end{cases}$$

but Theorem 3 is not applicable since I is not continuous in X.

The Example 2 of [2] proves that the weak commutativity is a necessary condition in Theorem 3. Now we prove that the quasi commutativity is a necessary condition in Theorem 4.

EXAMPLE 2. Let X = {x,y,z} be a finite set with metric d defined as d(x,y)= d(x,z)=1,d(y,z)=2. Define F and I as Fx=Fz={x}, Fy={x,z} and Ix=y, Iy=x, Iz=z.

Of course F is continuous in X and the conditions (2.1) and (2.3) are trivially satisfied. Further, we have that

$$\delta(Fy,F\alpha) = \max\{d(x,y),d(x,z)\} = 1 = \tfrac{1}{2} \cdot 2 = \begin{cases} \tfrac{1}{2} \cdot \delta(Ix,Fy) & \text{if } \alpha=x, \\ \tfrac{1}{2} \cdot \delta(Iz,Fy) & \text{if } \alpha=z, \end{cases}$$

and $\delta(Fx,Fz) = 0$. Hence the inequality (2.5) is satisfied if $\psi(t)=t/2$ for all $t \geq 0$ and then all the assumptions of Theorem 4 hold, except the quasi commutativity since IFy= I{y,z} = {x,z} $\not\subseteq$ {x} = Fx = FIy but F and I do not have common fixed points.

The Example 3 of [2] proves that the condition (2.1) is necessary in Theorem 3. The next example proves that the same condition is necessary in Theorem 4, even if F and I are single-valued mappings.

EXAMPLE 3. Let X = [0,+ ∞) with the Euclidean metric d,Fx=x for all x in X and Ix=1 if x=0, Ix=2x if x > 0. Then we have that F and I (quasi) commute and F is continuous in X. The sequence $\{d(F_{x_n}, F_{x_1}) : n=0,1,2,\dots\}$ is bounded for any x_0 in X-{0} since it is easily seen that

$$Ix_{n+1} = Fx_n = 2^{-n} x_0$$

for any $x_0 \in X - \{0\}$. Further we have that

$$d(Fx,Fy) = \begin{cases} |x-y| = \dfrac{1}{2}\,|2x-2y| = \dfrac{1}{2}\,d(Ix,Iy) & \text{if } x > 0,\ y > 0, \\[3mm] y = \dfrac{1}{2} \cdot 2y = \dfrac{1}{2} \cdot d(Iy,Fx) & \text{if } x=0,\ y > 0. \end{cases}$$

Thus the inequality (2.5) is satisfied if $\psi(t)=t/2$ for all $t\geq 0$ and hence all the assumptions of Theorem 4 hold except the condition (2.1) since $F(X)=X \nsubseteq X-\{0\} = I(X)$, but I does not have fixed points.

The Example 4 of [2] proves that the continuity of I is a necessary condition in Theorem 3. Now we show that the continuity of F is necessary in Theorem 4.

EXAMPLE 4. Let $X= [0,1]$ with the function δ induced by the Euclidean metric d and define F and I as $Fx= 1/2$ if $x=0$ and $Fx=(0,x/2]$ if $x>0$, $Ix=1$ if $x=0$ and $Ix=x$ if $x>0$. Of course, condition (2.3) holds since X is a bounded metric space and (2.1) is satisfied since

$$F(X) = (0, \frac{1}{2}] \subseteq (0,1] = I(X).$$

Note that F quasi commutes with I since

$$IF0 = I(\frac{1}{2}) = \frac{1}{2} \in (0, \frac{1}{2}] = F1 = FI0$$

and

$$IFx = I(0,\frac{x}{2}] = (0, \frac{x}{2}] = Fx = FIx$$

if $x>0$. Further we have that

$$\delta(Fx,Fy) = \begin{cases} \dfrac{1}{2}\,.\max\{x,y\} = \dfrac{1}{2} \cdot \max\{\delta(Ix,Fx),\ \delta(Iy,Fy)\} & \text{if } x > 0, y > 0 \\[3mm] \dfrac{1}{2} = \dfrac{1}{2} \cdot 1 = \dfrac{1}{2}\,.\delta(Ix,Fy) & \text{if } x =0, y > 0 \end{cases}$$

Then the inequality (2.5) is satisfied if $\psi(t)=t/2$ for all $t\geq 0$ and all the assumptions of Theorem 4 hold except the continuity of F, but F and I do not have common fixed points.

4.   ANOTHER FIXED POINT THEOREM.

In this Section we establish another result by using a weaker condition than the commutativity between two single-valued mappings of X into itself, but, following the ideas of this work, we give this condition between a set-valued mapping F of X into $B(X)$ and a single-valued mapping I of X into itself. Precisely, we say that F and I slightly commute if $IFx \in B(X)$ and

$$\delta(FIx,IFx) \leq \max\{\delta(Ix,Fx),\ \text{diam } Fx\} \tag{4.1}$$

for all x in X. Note that if F is a single-valued mapping, then diam $Fx=$ diam $IFx = 0$ for all x in X and hence (2.4) and (4.1) become $d(FIx,IFx) \leq d(Ix,Fx)$ for all x in X, that is the condition given in [5].

In the sequel we use the following lemma of [6].

LEMMA 2.   Let $\{A_n\}$ be a sequence of nonempty bounded subsets of (X,d) and y be a point of X such that

$$\lim_{n\to\infty} \delta(A_n,y) = 0.$$

Then the sequence $\{A_n\}$ converges to the set $\{y\}$.

Now we give the following result.

THEOREM 5. Let F be a mapping of X into B(X) and I be a mapping of X into itself satisfying the inequality (2.5) for all x,y in X, where $\psi \epsilon \Psi$. If there exists a point x_0 in X satisfying condition (2.3), if F and I slightly commute, if (2.1) holds and if F or I is continuous in X, then F and I have a unique common fixed point z and further $Fz = \{z\}$.

PROOF. We omit the first part of this proof since it is identical to the first part of the proof of Theorem 3.1 of [2].

As in [2], we can prove that the sequence $\{Ix_n\}$ converges to a point z in X and the sequence of sets $\{Fx_n\}$ converges to the set $\{z\}$. Since F and I slightly commute, we have that

$$\delta(FIx_n, IFx_n) \leq \max \{\delta(Ix_n, Fx_n), \delta(Fx_n, Fx_n)\}$$

for any n=0,1,2,... and as n→∞, we deduce by Lemma 1 that

$$\lim_{n \to \infty} \delta(FIx_n, IFx_n) = d(z,z) = 0. \tag{4.2}$$

Now we assume that F is continuous in X. Then the sequence of subsets $\{FIx_n\}$ converges to the set $\{Fz\}$ and using inequality (2.5), we have that

$$\delta(FIx_{n+1}, Fx_n) \leq \psi(\max\{d(I^2 x_{n+1}, Ix_n), \delta(I^2 x_{n+1}, FIx_{n+1}), \delta(Ix_n, Fx_n),$$

$$\delta(I^2 x_{n+1}, Fx_n), \delta(Ix_n, FIx_{n+1})\})$$

$$\leq \psi(\max\{\delta(IFx_n, Ix_n), \delta(IFx_n, FIx_{n+1}), \delta(Ix_n, Fx_n),$$

$$\delta(IFx_n, Fx_n), \delta(Ix_n, FIx_{n+1})\})$$

$$\leq \psi(\max \{ \delta(IFx_n, FIx_n) + \delta(FIx_n, F_{x_n}) + \delta(Fx_n, Ix_n),$$

$$\delta(IFx_n, FIx_n) + \delta(FIx_n, FIx_{n+1}), \delta(Ix_n, F x_{n+1})\}),$$

since $I^2 x_{n+1}$ is in IFx_n and ψ is nondecreasing.

Since ψ is right continuous, as n→∞, using Lemma 1 and (4.2), we obtain that

$$\delta(Fz, z) \leq \psi(\max \{\delta(Fz, z), \delta(Fz, Fz))\}). \tag{4.3}$$

But again using (2.5) and the nondecreasingness of ψ, we deduce that

$$\delta(FIx_{n+1}, FIx_{n+1}) \leq \psi(\max\{d(I^2 x_{n+1}, I^2 x_{n+1}), \delta(I^2 x_{n+1}, FIx_{n+1})\})$$

$$\leq \psi(\delta (IFx_n, FIx_n) + \delta(FIx_n, FIx_{n+1})),$$

which implies, as n→∞, by (4.2) that

$$\delta(Fz, Fz) \leq \psi(\delta(Fz, Fz))$$

and hence $\delta(Fz, Fz) = 0$ since $\psi(t) < t$ for t>0. From (4.3), it follows that $Fz = \{z\}$. Since (2.1) holds, there exists a point w in X such that Iw=z and using inequality (2.5), we have that

$$\delta(Fx_n, Fw) \leq \psi(\max\{d(Ix_n, z), \delta(Ix_n, Fx_n), \delta(z, Fw), \delta(Ix_n, Fw), \delta(z, Fx_n)\}),$$

which implies, an n→∞, that

$$\delta(z, Fw) \leq \psi(\delta H(z, Fw)).$$

Thus Fw=z and since F and I slightly commute we have that

$$d(Fz,Iz)=\delta(FIw,IFw)\leqq max \{\delta(Iw,Fw), \delta(Fw,Fw)\} = d(z,z)= 0.$$

It follows that $\{z\}=Fz=\{Iz\}$ and thus z is also a fixed point of I.

Now we assume the continuity of I instead of the continuity of F. Then the sequence $\{I^2x_n\}$ converges to the point Iz and the sequence of sets $\{IFx_n\}$ converges to the set $\{Iz\}$.

We have that

$$\delta(FIx_n,Iz)\leqq\delta(FIx_n,IFx_n)+\delta(IFx_n,Iz)$$

and, as n→∞, we deduce from (4.2) and Lemma 2 that the sequence of sets $\{FIx_n\}$ also converges to the set $\{Iz\}$.

Using inequality (2.5) and since I^2x_{n+1} is in IFx_n, we get that

$$d(I^2x_{n+1},Ix_{n+1})\leqq \delta(IFx_n,Fx_n)\leqq \delta(IFx_n,FIx_n) + \delta(FIx_n,Fx_n)$$

$$\leqq \delta(IFx_n,FIx_n)+ \psi(max\{d (I^2x_n,Ix_n), \delta(I^2x_n,FIx_n),$$

$$\delta(Ix_n,Fx_n), \delta(I^2x_n,Fx_n), \delta(Ix_n,FIx_n)\}).$$

As n→∞, it follows from (4.2) that

$$d(Iz,z)\leqq \psi(d(Iz,z)),$$

which implies Iz=z. Using again the inequality (2.5), we have that

$$\delta(Fz,Fx_n)\leqq \psi(max\{d(z,Ix_n), \delta(z,Fz), \delta(Ix_n,Fx_n), \delta(z,Fx_n), \delta(Ix_n,Fz)\})$$

and this implies, as n→∞, that

$$\delta(Fz,z)\leqq \psi(\delta(Fz,z)).$$

Then $Fz=\{z\}$ and hence z is also a fixed point of F. In any case, z is a common fixed point of F and I suppose that F and I have another common fixed point z'. Using inequality (2.5), we have that

$$d(z,z') = \delta(Fz,Fz') \leqq \psi(d(z,z')).$$

This means that z=z' and therefore z is the unique common fixed point of F and I. This completes the proof.

REMARK 1. We note that the mappings F and I of Example 2 satisfy all the assumptions of Theorem 5 except the slight commutativity since

$$\delta(FIx,IFx)=d(y,z)=2> max \{1,0\}=max\{\delta(Ix,Fx), diam Fx\}.$$

Therefore the slight commutativity is a necessary condition in Theorem 5.

REMARK 2. The Example 3 proves that the condition (2.1) is also necessary in Theorem 5.

REMARK 3. In Example 4, we note that F slightly commutes with I since $\delta(FIO,IFO)=1/2=d(IO,FO)$ and $\delta(FIx,IFx)=x/2=diam Fx$ if $x > 0$. Since all the assumptions of Theorem 5 are satisfied except the continuity of F and I in X, we can say that the continuity of F or I in X is a necessary condition in Theorem 5.

REMARK 4. It is not yet known if (2.3) is a necessary condition in Theorems 3,4 and 5.

If F is a single-valued mapping, we obtain the result of [5] from Theorem 5.

In Example 1, we explicitly point out that Theorem 5, assuming the continuity of F in X, holds good since the mappings F and I are also slightly commuting.

5. CONCLUDING COMMENTS.

Independently from the fixed point considerations until now established, we conclude this paper exhibiting some easy examples which show that the concepts of weak, quasi and slight commutativity between a set-valued mapping and a single-valued mapping do not imply each of the other two.

EXAMPLE 5. Let $X=\{x,y,z\}$ a finite set with function δ induced by the metric defined as $d(x,y)=d(y,z)=2,d(x,z)=1$. Define F and I as $Fx=\{x\}$, $Fy=\{x,z\},Fz=\{y,z\}$ and $Ix=x$, $Iy=z,Iz=y$. Thus it is easily seen that F and I weakly commute, but they do not quasi commute since $IFy=I\{x,z\}=\{x,y\} \subseteq \{y,z\}=Fz=FIy$ and further they do not slightly commute since

$$\delta(FIy,IFy)=2>1=\max\ \{1,1\} = \max\ \{\delta(Iy,Fy),\text{diam } Fy\}.$$

EXAMPLE 6. Let $X=\{x,y,z\}$ a finite set with function δ induced by the metric defined as $d(x,y)=2$, $d(x,z)=d(y,z)=1$. Let I be as in Example 5 and define F as $Fx=\{x\}$, $Fy=Fz=\{x,y\}$. Then it is easily seen that F and I slightly commute but they do not quasi commute since $IFy=I\{x,y\}=\{x,z\} \leq \{x,y\}=Fz=FIy$ and further they do not weakly commute since

$$\delta(FIy,IFy)=2>1=\max\{1,1\}=\max\{\delta(Iy,Fy),\ \text{diam } IFy\}.$$

EXAMPLE 7. Let (X,d) be as in Example 6 and define F and I as $Fx=\{x\}$, $Fy=\{x,y\}$, $Fz=\{y\}$ and $Ix=x$, $Iy=Iz=y$. Thus F and I quasi commute but they do not slightly nor weakly commute since

$$\delta(FIz,IFz)=2>0=\max\{\delta(Iz,Fz),\ \text{diam } Fz,\ \text{diam } IFz\}.$$

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