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ON BEHAVIOR OF SOLUTIONS OF NON-LINEAR DIFFERENTIAL EQUATIONS IN HILBERT SPACE II

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ABSTRACT. This paper deals with the behavior of solutions of non-linear ordinary differntial equations in a Hilbert space with applications to non-linear partial differential equations.

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I. INTRODUCTION.

In this paper, we continue our study of behavior of solutions of non-linear ordinary differential equations in a Hilbert space H with applications to non-linear partial differential equations.

We introduce here non-linear operator of self-adjoint type and we study the quasiuniqueness of Cauchy problem and the classical uniqueness of Cauchy problem for this equation.

In the special case under condition A we obtain complete results about quasi-uniqueness. Recall that we do not have this situation in a linear case because this part of our theorems have no analog in the linear case.

As usual, we study two cases of non-linear differential equations: the case of degenerate equation in bounded interval of time and the case of non-degenerate equation in unbounded interval of time.

In the first part of this paper, we study the following non-linear degenerate equation:

$$t\frac{du}{dt} = B(t,u(t)) \tag{*}$$

where $t \in I = (0,1]$. $u(t)$ for each $t \in I$ is an element of H and has derivative with respect to t, if $t > 0$. $B(t,u(t))$ is a non-linear map from H to H with domain D_B, D_B is the dense subset of H and for each $t \in I$ and $u(t) \in D_B$, $B(t,u(t))$ is an element of H also. $B(t,u(t))$ is not necessarily bounded. Special case of equation (*) is the case (1.20), that B is a product of the form

$$B(t,u(t)) = A(t,u(t))\cdot u(t)$$

where $A(t,u(t))$ is a non-linear map from H to H. In this situation, we obtain complete results about the quasiuniqueness at the point $t = 0$. Our Theorem 2.3 is the main theorem of the first part.

The non-degenerate equation which we also study is the equation of the form

$$\frac{du}{dt} = B(t,u(t)), \qquad (**)$$

where $t \in \bar{I} = [1,+\infty)$. $u(t)$ for each $t \in \bar{I}$ is an element of H and has derivative with respect to t. $B(t,u(t))$ is a non-linear map from H to H with domain D_B, D_B is the dense subset of H and for each $t \in \bar{I}$ and $u(t) \in D_B$, $B(t,u(t))$ is an element of H also. $B(t,u(t))$ is not necessarily bounded.

We obtain complete results for the same special case as above. The main theorem of this part of our study is Theorem 4.7 about the quasiuniqueness of equation (**) in a special case at the point $t = +\infty$.

In §1 we study the quasiuniqueness for equation (*). In §2 we study the quasiuniqueness in two special cases and we obtain complete Theorem 2.3 in the special case (1.20) of equation (*). In §3 we study the uniqueness of Cauchy problem for special case of equation (*). We obtain partial results only for this problem. In §4 we study non-degenerate equation (**) and we obtain for this equation the same results as for equation (*). The Theorem 4.7 is a parallel to Theorem 2.3. In §5 we study several examples of non-linear partial differential equations with condition (1.20) and we obtain for these equations, the quasiuniqueness at the point $t = 0$ in the degenerate case and at the point $t = +\infty$ in the non-degenerate case. Recall that we have no analog of these theorems in a linear case.

The method of this study was used first by Agmon-Nirenberg [1,2] for studying the classical uniqueness of Cauchy problem in the non-degenerate linear case. This method in the degenerate case was used by the author in [3].

This method was used by the author for study of the quasiuniqueness in the non-linear case for the following special equation:

$$t\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}K(u)\frac{\partial u}{\partial x}$$

and for the non-degenerate equation of the similar type [4]. Several theorems of this paper are like theorems of paper [4], but here we have the case of non-linear equation, and in [4] we studied special case of quasilinear equations. Several theorems for example, Theorem 2.1 or Theorem 4.5, were obtained in paper [5] also.

§1. On the quasiuniqueness in degenerate case, let us consider the following non-linear equation in the Hilbert space H.

$$t\frac{\partial u}{\partial t} = B(t,u(t)) \qquad (1.1)$$

where $t \in I = (0,1]$. $u(t)$ for each $t \in I$ is an element of H and has derivative with respect to t, if $t > 0$.

$B(t,u(t))$ is a non-linear map from H to H with domain D_B, D_B is the dense subset of H, and for each $t \in I$ and for each $u(t) \in D_B$, $B(t,u(t))$ is an element of H also. $B(t,u(t))$ is not necessarily bounded. H is a Hilbert space with scalar product $(.,.)$ and with norm $||.||$ correspondingly.

<u>Definition 1.1</u>. The non-linear operator $B(t,u(t))$ is called smooth operator of self-adjoint type if the following condition is satisfied:

<u>Condition S</u>. For each $t \in I$ and for each $u(t) \in D_B$ the following scalar product

$$(B(t,u(t)),u(t)) \qquad (1.2)$$

is real and differentiable with respect to t if $t > 0$. We study in this paper the behavior of the norm of the solutions of equation (1.1) under Condition S only.

Definition 1.2. Let $f(t)$ be a scalar function in interval I. $f(t)$ is called a flat at the point $t = 0$, if for each $K > 0$

$$t^{-K}f(t) \to 0 \quad \text{as} \quad t \to 0.$$

Definition 1.3. The solution $u(t)$ of equation (1.1) is called a flat solution, if $||u(t)||$ is a flat function at the point $t = 0$. The question which we have now is when equation (1.1) has no flat solutions. In the linear case with self-adjoint operator $B(t)$, the following statement is true.

Theorem 1.1. Let $B(t)$ be a linear symmetric operator with domain $D_B(t)$ and

$$\frac{d}{dt}B(t)x = \dot{B}(t)x \quad \text{for each} \quad x \in D_{B(t)}. \tag{1.3}$$

Let $u(t)$ be a solution of the following equation

$$t\frac{du}{dt} = B(t)u \tag{1.4}$$

such that one of the following conditions is satisfied:

$$||\dot{B}(t)u(t)|| \le \gamma(t)||B(t)u(t)|| + \beta(t)||u(t)|| \tag{1.5}$$

or

$$(\dot{B}),u(t)) \ge -\gamma(t)|(B(t)u(t),u(t))| - \beta(t)||u(t)||^2 \tag{1.6}$$

where $\gamma(t)$, $\beta(t)$ are non-negative continuous functions in the interval $I' = [0,1]$. Then for this solution $u(t)$, the following is true:

i) $||u(t)|| \ge M||u(t\ )||t^{\nu+\mu}$ (1.7)

where constant $\nu \ge 0$ depending on $\nu(t)$, $\nu(t)$ and $u(t)$ itself and constant $\mu \ge 0$ depending on $\nu(t)$, $\beta(t)$ only.

ii) if $u(t)$ is a flat solution of equation (1.4), then $u(t) \equiv 0$ in the interval I.

Proof. (See [3]).

In the linear case, we do not have classical uniqueness at the point $t = 0$, only the type of uniqueness as in ii) above--the quasiuniqueness (see Definition 1.4 below).

Definition 1.4. We say that the quasiuniqueness takes place for equation (1.1) or (1.4) at the point $t = 0$ if conclusion ii) of Theorem 1.1 is true for neighborhood of the point $t = 0$ or, in other words, if we have uniqueness in the class of flat-functions at the point $t = 0$. Let $u(t)$ be a solution of (1.1). Let

$$q(t) = (u(t),u(t)). \tag{1.8}$$

If $t\frac{d}{dt} = D$, we have from (1.1) after scalar product with $u(t)$

$$Dq(t) = 2\text{Re}(b(t,u(t)),u(t)) = 2(B(t,u(t)),u(t)) \tag{1.9}$$

and if we assume that $u(t) \in C^1(I,H)$, and $B(t,u)$ has first derivative with respect to all variables (see Condition S), then

$$\begin{aligned} D^2q &= 2D(B(t,u(t)),u(t)) = 2[([DB(t,u(t))],u(t)) + (B(t,u(t)),Du(t))] \\ &= 2[([DB(t,u(t))],u(t)) + (B(t,u(t)),B(t,u(t)))] \\ &= 2([D_B(t,u(t))],u(t)) + 2||B(t,u(t))||^2 \end{aligned} \tag{1.10}$$

Let now (t_1,t_0) be the interval with $q(t) > 0$ for $t \in (t_1,t_0)$, then if

$$\ell(t) = \ln q(t), \tag{1.11}$$

then from (1.9) we have

$$D\ell(t) = \frac{Dq}{q} = \frac{2(B(t,u(t)),u(t))}{q(t)} \tag{1.12}$$

and

$$D^2\ell(t) = \frac{D^2q}{q} - \left(\frac{Dq}{q}\right)^2 \tag{1.13}$$

or from (1.9) and (1.10) we obtain

$$D^2\ell(t) = \frac{2([DB(t,u(t))],u(t))}{q(t)} + \frac{2||B(t,u(t))||^2}{q(t)} - \frac{4(Bt,u(t)),u(t))^2}{q^2(t)} \tag{1.14}$$

The following statement is true:

Lemma 1.1. Let $\ell(t)$ be a twice differentiable function in the interval I satisfying the following second-order differential inequality

$$D^2\ell(t) + ta(t)|DC(t)| + tb(t) \geq 0. \quad t \in I \tag{1.15}$$

where $a(t)$, $b(t)$ are non-negative bounded functions in I.

Then

$$\ell(t)\ \ell \geq (t_0) + 2\nu \ln\frac{t}{t_0} + 2\mu \ln\frac{t}{t_0} \tag{1.16}$$

where constant $\nu > 0$ depending on $a(t)$, $b(t)$ and $l(t)$ itself, and constant $\mu > 0$ depending on $a(t)$, $b(t)$ only.

Proof. (see [3,4]).

From (1.16) we obtain

$$\exp\ell(t) \geq [\exp\ell(t_0)]\cdot t^{2\nu+2\mu} \tag{1.17}$$

and from definition $\ell(t)$ we have

$$q(t) = \exp\ell(t),\ q(t_0) = \exp\ell(t_0) \tag{1.18}$$

and from this and from (1.17) we have the following estimate for $q(t)$

$$q(t) \geq q(t_0)t^{2\nu+2\mu}$$

where constant $\nu > 0$ depending on $a(t)$, $b(t)$, and $q(t)$ itself, and constant $\mu > 0$ depending on $a(t)$, $b(t)$ only.

From (1.18) we obtain estimate (1.7). Our problem now is to obtain the inequality of type (1.15) for non-linear equation (1.1). From this discussion we obtain that the following statement is true.

Theorem 1.2. Let $u(t)$ be a solution of equation (1.1) such that

$$\left([DB(t,u(t)],u(t)\right) + ||B(t,u(t))||^2 - \frac{2(b(t,u(t)),u(t))^2}{(u(t),u(t)))} \geq$$

$$- ta(t)|(B(t,u(t)),u(t))| - tb(t)||u(t)||^2 \tag{1.19}$$

for some non-negative bounded functions $a(t)$, $b(t)$ in the interval I. Then

i) $||u(t)|| \geq M||u(t_0)||t^{\nu+\mu}$ where constant $\nu \geq 0$ depending on $a(t)$, $b(t)$ and $u(t)$ itself, and constant $\mu \geq 0$ depending on $a(t)$, $b(t)$ only.

ii) if $u(t)$ is a flat solution, then $u(t) \equiv 0$ in the interval I.

Proof. i) follows from Lemma 1.1 and ii) follows immediately from i). Let us consider the special case of operator $B(t,u)$. Let Hilbert space H satisfy the following condition: for each $u, v \in H$, $u \cdot v \in H$ also. Let us now consider that $B(t,u(t))$ is the product of the following form:

$$B(t,u(t)) = A(t,u(t)) \cdot u(t) \tag{1.20}$$

where $A(t,u(t))$ is a non-linear map from Hilbert space H to H with domain D_A, D_A being the dense subset of H, and $A(t,u(t))$ satisfies the following condition:

Condition A. For each $u(t) \in D_A$ and for each $v \in H$ the following function

$(A(t,u(t))v,v)$ is differentiable with respect to t and for each $w \in H$, $(A(t,u(t))v,w) = (v,A(t,u(t))w)$. In a standard case H is L_2 on compact set of R^n or compact smooth n-manifold. In applications we have $A(t,u)$ in the following form:

$$A(t,u) = F(t,x,\frac{\partial u}{\partial x_1},\ldots,\frac{\partial^m u}{\partial x^m})$$

where $x \in \Omega$, Ω is compact set of R^n or smooth compact n-manifold (for example, sphere S_r).

In this special case, it is possible to obtain more complete results and more simple form of condition (1.19). Namely, it is possible to rewrite the first term of (1.19) in the following form:

$$([DB(t,u(t)],u(t)) = ([DA(t,u)u],u) = ([DA(t,u)]u,u) + (A(t,u)Du,u) = ([DA(t,u)]u,u) + (Du,A(t,u)u) \tag{1.21}$$

And from (1.1) and (1.20) we have that the last term is

$$||A(t,u(t))u(t)||^2$$

or

$$||B(t,u(t))||^2$$

From this and (1.19) we obtain

$$([DA(t,u)]u,u) + 2||A(t,u(t))u||^2 - \frac{2(A(t,u(t)u,u)^2}{(u(t),u(t))} \geq - ta(t)|(A(t,u(t)u(t),u(t)| - tb(t)||u(t)||^2 \tag{1.22}$$

and since

$$|(A(t,u(t))u(t),u(t))| \leq ||A(t,u(t))u(t)|| \cdot ||u(t)||,$$

we obtain that (1.22) is satisfied, for example, if the following condition is satisfied:

$$([DA(t,u(t))]u(t),u(t)) \geq - ta(t)|(A(t,u(t))u(t),u(t))| - tb(t)||u(t)||^2 \tag{1.23}$$

or, since $D = t\frac{d}{dt}$, after dividing on t:

$$([\tfrac{d}{dt}A(t,u(t))]u(t),u(t)) \geq - a(t)|(A(t,u(t))u(t),u(t))| - b(t)||u(t)||^2. \tag{1.24}$$

This condition is very similar to condition (1.6) in the linear case. From the previous discussion we obtain that the following statement is true.

Theorem 1.3. Let $u(t)$ be a solution of equation (1.1) under Condition A such that condition (1.24) is satisfied for some non-negative bounded functions $a(t)$, $b(t)$ in the interval I.

Then:

i) $$||u(t)|| \geq M||u(t_0)||^{\nu+\mu} \tag{1.25}$$

where constant $\nu \geq 0$ depending on $a(t),b(t)$ and $u(t)$ itself, and constant $\mu \geq 0$ depending on $a(t)$, $b(t)$ only.

ii) if $u(t)$ is a flat solution, then $u(t) \equiv 0$ in the interval I.

Proof. i) follows from previous discussion and Lemma 1.1 and ii) follows immediately from estimate (1.25).

Remark 1.1. Our condition (1.24) is not simple enough, and in concrete situations, it is difficult to check it. But in the following special case, when

$$(A(t,u(t))v,v) \geq 0 \quad \text{for each } v \in H, \tag{1.26}$$

it is possible to write other conditions in the following form.

If $(A(t,u)v,v)$ as a function of t satisfies the following condition for solution $u(t)$ of equation (1.1):

$$\frac{d}{dt}[(A(t,u)v,v)] \geq -C(A(t,u(t))v,v) \tag{1.27}$$

for some constant $C > 0$, and for each v, $||v|| = 1$, then for this solution $u(t)$, the conclusion of Theorem 1.3 is true.

It is possible to obtain the following statement about the quasiuniqueness.

Theorem 1.4. Let conditions A and (1.26) be satisfied. Let condition (1.27) be satisfied for each function $u(t) \in D_A$ with flat norm $||u(t)||$. Let $u(t)$ be a flat solution of equation (1.1). Then $u(t) \equiv 0$ in the interval I. Or, in other words, under these conditions, the quasiuniqueness takes place at the point $t = 0$ for solutions of equation (1.1).

Proof. Let $u(t)$ be a flat solution. Then for $u(t)$, condition (1.27) is satisfied, and therefore for $\ell(t) = \ln||u(t)||^2$ we have inequality (1.15). From Lemma 1.1, we have for $||u(t)||^2$ an estimate of the type (1.18). This is a contradiction with flatness $u(t)$. Therefore, $u(t)$ is a trivial solution in the interval I, or the quasiuniqueness takes place at the point $t = 0$ for solutions of equation (1.1) under conditions A, (1.26), (1.27).

Remark 1.2. Let $f(t)$ be a scalar function in the interval I. Let $f(t)$ satisfy the following condition

$$f'(t) \geq -Cf(t)$$

or

$$f'(t) + Cf(t) \geq 0.$$

It is possible to rewrite this condition in the following form

$$[e^{Ct}f(t)]' \geq 0.$$

In other words, the following function

$$e^{Ct}f(t)$$

is monotonic and not decreasing in the interval I. It is easy to see that each function $f(t)$ will satisfy this condition in the interval $(0,\varepsilon)$, if

i) $f(0) > 0$

ii) $f'(0)$ is bounded.

Interval $(0,\varepsilon)$ depends on function $f(t)$.

From this discussion, we obtain that the following statement is true.

Theorem 1.5. Let conditions A and (1.26) be satisfied. Let $A(t,u)$ for each $u(t) \in D_A$ with flat norm $||u(t)||$ satisfy the following conditions:

i) $(A(t,u(t))v,v) > 0$ in some interval $(0,\varepsilon)$ with ε depending on $u(t)$ for each $||v \in H$, $||v|| = 1$.

ii) $\frac{d}{dt}[(A(t,u(t))v,v)]$ is bounded in interval $(0,\varepsilon)$ for each $v \in H$, $||v|| = 1$ (this condition follows from Condition A,).

If $u(t)$ is a flat solution of equation (1.1), then $u(t) \equiv 0$ in the interval $(0,\varepsilon)$ with ε depending on $u(t)$, or the quasiuniqueness takes place at the point $t = 0$ for solutions of equation (1.1) under those conditions.

§2. The special cases

In this § we study two special cases for equation of type (1.1) for completing Theorem 1.5. The first special case is the following:

$$(B(t,u),u(t)) \geq 0 \tag{2.1}$$

for each $u(t)$ from the domain D_B. In this case, if

$$q(t) = ||u(t)||^2$$

then

$$t\dot{q}(t) = 2(B(t,u),u) \leq 0$$

or

$$q(t) \geq q(t_0) \quad \text{for} \quad t \leq t_0 \tag{2.2}$$

and we obtain that the following statement is true. Theorem 2.1. Let $u(t)$ be a solution of equation (1.1) under condition (2.1). If $u(0) = 0$, then $u(t) = 0$ in the interval I. In this situation, we have classical uniqueness at the point $t = 0$. Remark 2.1. Let operator $B(t,u)$ satisfy condition A and let $A(t,u)$ satisfy the following condition:

$$[(A(t,u(t))v,v)] \leq 0$$

for each $u(t) \in D_A$ with the flat $||u(t)||$ in the interval $[0,\varepsilon]$ with $\varepsilon > 0$ depending on $u(t)$ may be and for each $v \in H$, $||v|| = 1$. Let $u(t)$ be a flat solution of equation (1.1) under those conditions. Then $u(t) \equiv 0$ in the interval $[0,\varepsilon]$ or the quasiuniqueness take place at the point $t = 0$ for solutions of equation (1.1) under those conditions.

The second special case is the following:

Let $B(t,u)$ satisfy condition A and let $A(t,u)$ as a function of t satisfy the following condition:

$$(A(t,u)v,v) \to 0 \text{ as } t \to 0 \tag{2.3}$$

for each $u(t)$ with flat norm and for each $v \in H$ with $||v|| = 1$.

Let $q(t) = (u(t),u(t))$. If $q(t) > 0$ in the interval (t_1,t_0) we introduce a new function by formula

$$v(t) = \frac{u(t)}{q^{\frac{1}{2}}(t)} \tag{2.4}$$

or

$$u(t) = q^{\frac{1}{2}}(t)v(t)$$

After scalar produce (1.1) with $u(t)$ we obtain

$$t\dot{q} = 2((A(t,q^{\frac{1}{2}}v)u,u) = 2q(A(t,q^{\frac{1}{2}}v)v(t),v(t)).$$

Let now $u(t)$ be a solution of (1.1) under condition A and $A(t,u)$ satisfies condition (2.3). The following scalar product

$$|(A(t,q^{\frac{1}{2}}v)v(t),v(t))| < \delta$$

in some neighborhood of the point $t = 0$. This neighborhood depends on a solution $u(t)$ itself. From this we have that for each $u(t) \in D_A$ with flat norm $q(t)$ there exist numbers $\varepsilon,\delta > 0$, such that

$$|(A(t,q^{\frac{1}{2}}v)v(t),v(t))| < \delta \quad \text{for} \quad 0 < t < \varepsilon$$

where δ depends on $u(t)$.

From this we have for $q(t)$ the following inequality:

$$\frac{t\dot{q}}{q} < 2\delta \quad \text{for } 0 < t < \varepsilon \tag{2.5}$$

where δ depends on $u(t)$. From (2.5) we have the following estimate for $q(t)$

$$q(t) > \left(\frac{t}{t_0}\right)^{2\delta} q(t_0) \quad \text{for} \quad t < t_0 < \varepsilon \tag{2.6}$$

and from this estimate, we have the quasiuniqueness. From the previous discussion we obtain that the following statement is true.

Theorem 2.2. Let $A(t,u)$ satisfy the following condition:

$$(A(t,u)v,v) \to 0 \quad \text{as} \quad t \to 0$$

for each $u(t) \in D_{A(t)}$ with flat norm and for each $v \in H$ with $||v|| = 1$. Then if $u(t)$ is a flat solution of equation (1.1) under conditions A and (2.6), then $u(t)$ is a trivial in the interval $(0,\varepsilon)$ with $\varepsilon > 0$ depending on $u(t)$ itself.

From Theorems 1.5, 2.1, and 2.2, we have that the following statement is true.

Theorem 2.3. Let operator $B(t,u(t))$ satisfy condition A. Then the quasiuniqueness takes place at the point $t = 0$ for solutions of equation (1.1) under those conditions.

Proof. Let $u(t)$ be a flat solution of (1.1). Let us consider the following function:

$$(A(t,u(t))u(t),u(t)) = q(t)(A(t,u(t))v(t),v(t))$$

where

$$q(t) = ||u(t)||^2$$

and

$$u(t) = v(t)q^{\frac{1}{2}}(t) \quad ||u(t)|| = 1.$$

It follows from condition A that the following function

$$f(t) = (A(t,u(t))v(t),v(t))$$

is continuous and smooth in the same interval $(0,\varepsilon)$ with ε depending on $u(t)$. Then we have three cases:

i) $f(t) \leq 0$ in some interval $[0,\varepsilon']$.

ii) $f(t) > 0$ in some interval $[0,\varepsilon']$,

iii) $f(t) \to 0$ as $t \to 0$.

The quasiuniqueness follows for the case i) from Remark 2.1, for the case ii) from Theorem 1.5, for the case iii) from Theorem 2.2.

Remark 2.2. Smoothness in the Theorem 2.3 is necessary for the quasiuniqueness. In the finite-dimension Hilbert space, the smoothness also is necessary. Let us consider the following equation:

$$t\frac{dx}{dt} = -x\ln x \tag{2.7}$$

in the interval $I = (0,1]$. It is easy to see that the following function

$$x(t) = \begin{cases} \exp(-\frac{1}{t}) & t > 0 \\ 0 & t \leq 0 \end{cases} \tag{2.8}$$

is a solution of equation (2.7) for all $t \in R^1$. This function is flat and $x(t) \in C^\infty(R^1)$. More than this, let us consider the following first order non-linear inequality

$$t\dot{q}(t) \geq -Cq(t)\ln q(t) \quad q(t) \geq 0 \tag{2.9}$$

and we will look for the solutions with $q(t) < 1$ for $0 < t < \delta$ only. It is possible to rewrite inequality (2.9) in the following form

$$t\dot{q}(t) + Cq(t)\ln q(t) = \phi(t) \geq 0. \tag{2.10}$$

Let

$$l(t) = -\ln q(t) \tag{2.11}$$

and

$$l(t) \geq 0.$$

After dividing (2.10) over $-q$ we obtain

$$-\frac{t\dot{q}}{q} - C\ln q(t) = -\frac{\phi(t)}{q(t)} \leq 0 \qquad (2.12)$$

or after substitution (2.11) we obtain from (2.12) the following inequality for $l(t)$

$$t\dot{l}(t) + Cl(t) = -\frac{\phi(t)}{q(t)} \leq 0 \qquad (2.13)$$

Now we introduce a new function $m(t)$ by formula

$$m(t) = \ln l(t). \qquad (2.14)$$

This function is defined, since $l(t) \geq 0$. From (2.13) we obtain for $m(t)$

$$t\dot{m}(t) + C = \phi_1(t) \leq 0 \qquad (2.15)$$

and

$$\phi_1(t) = \frac{-\phi(t)}{C(t)q(t)}. \qquad (2.16)$$

After integrating (2.15) we get

$$m(t) - m(t_0) = -C\ln\frac{t}{t_0} + \int_{t_0}^{t} \phi_1(\tau)d\tau \qquad (2.17)$$

or

$$\frac{l(t)}{l(t_0)} = -C\ln\frac{t}{t_0} + \int_{t_0}^{t} \phi_1(\tau)d\tau \qquad (2.18)$$

From (2.18) we obtain for $l(t)$

$$l(t) = l(t_0)\left(\frac{t}{t_0}\right)^{-C}\exp\int_{t_0}^{t}\phi_1(\tau)d\tau \qquad (2.19)$$

and from (2.11) and (2.19) we obtain for $q(t)$

$$\ln q(t) = -l(t_0)\left(\frac{t}{t_0}\right)^{-C}\exp\int_{t_0}^{t}\phi_1(\tau)d\tau \qquad (2.20)$$

or for $q(t)$ we have

$$q(t) = \exp\left[-l(t_0)\left(\frac{t}{t_0}\right)^{-C}\exp\int_{t_0}^{t}\phi_1(\tau)d\tau\right] \qquad (2.21)$$

In (2.20) $l(t_0) > 0$ and $-l(t_0) < 0$. The following integral

$$\int_{t_0}^{t}\phi_1(\tau)d\tau = -\int_{t}^{t_0}\phi_t(\tau)d = \int_{t}^{t_0}-\phi\ (\tau)d = \int_{t}^{t_0}\frac{\phi(t)}{l(t)q(\tau)}d\tau \geq 0 \quad \text{for} \quad t < t_0,$$

since $\phi(t) \geq 0$, $l(t) \geq 0$, $q(t) \geq 0$. From this we have that

$$\exp\int_{t_0}^{t}\phi_1(\tau)d\tau \geq 1 \qquad (2.22)$$

and since $-l(t) < 0$, then we have from (2.21) and (2.22) that the following estimate holds

$$q(t) \leq \exp\left(-l(t_0)\left(\frac{t}{t_0}\right)^{-C}\right) = \chi(t) \qquad (2.23)$$

Function $\chi(t)$ in (2.23) is a flat and from this discussion we obtain that the following statement is true.

Theorem 2.4. Let $q(t)$ be a solution of inequality (2.9) with the condition

$$q(t) \leq 1 \quad \text{for } 0 < t < \delta.$$

Then in this interval, $q(t)$ will be flat-function which satisfies the estimate (2.23).

Remark 2.3. From Theorem 2.4, we obtain that if we have the following equation

$$t\frac{du}{dt} = B(t,u) \tag{2.24}$$

and $B(t,u)$ satisfies the following condition

$$\mathrm{Re}(B(t,u),u) \geq -C||u(t)||^2 \ln||u(t)||^2, \tag{2.25}$$

then each solution of (2.24) such that

$$q(t) = ||u(t)||^2 < 1 \quad \text{for} \quad 0 < t < \delta$$

must be flat.

This is not a theorem about existence of flat solutions, but if there exists solution (2.24) such that $||u(t)|| < 1$, then $||u(t)||$ must be flat.

§3. On the uniqueness

In §§1-2 we obtained results about the quasiuniqueness for solutions of equation (1.1) at the point $t = 0$. It is possible to obtain some results about uniqueness for solutions of this equation.

The first and the simplest result about uniqueness was obtained in Theorem 2.1. In the special case of operator $B(t,u)$ with condition A we obtain the following statement.

Theorem 3.1. Let condition A be satisfied and let for each $u(t) \in D_A$, $u(t) \neq 0$ and for each $t \in I$ the following condition be satisfied:

$$([\tfrac{d}{dt}A(t,u)]u(t),u(t)) > -C\{|(A(t,u)u(t),u(t))| + ||u(t)||\} \tag{3.1}$$

for every $t \in I$ and for some constant $C \geq 0$.

Then for each non-trivial solution $u(t)$ under those conditions, the following estimate holds

$$||u(t)|| > ||u(t\ )||(\frac{t}{t_0})^\mu \quad \text{for} \quad t < t \tag{3.2}$$

where constant $\mu \geq 0$ depends on C from (3.1) and $u(t)$ itself.

Proof. Proof of this theorem is similar to proof of Theorem 1.3 and is based on the following statement (like Lemma 1.1).

Lemma 3.1. Let $l(t)$ be a twice differentiable non-trivial function in the interval I, satisfying the following second-order differential inequality

$$D^2 l(t) + ta(t)[Dl(t)| + tb(t) > 0, \ t \quad I \tag{3.3}$$

where $D = t\frac{d}{dt}$, $a(t)$, $b(t)$ are non-negative functions, bounded in I. Then

$$l(t) > l(t_0) + 2\nu \ln\frac{t}{t_0} + 2\mu \ln\frac{t}{t_0} \tag{3.4}$$

where constant $\nu \geq 0$ depending on $a(t)$, $b(t)$ only and constant $\mu \geq 0$ depending on $a(t)$, $b(t)$ and $u(t)$ itself. From this lemma, we obtain the estimate (3.2). From estimate (3.2) we obtain that the following function

$$t^{-\mu}||u(t)|| \tag{3.5}$$

is a strongly monotonic in the interval $I = (0,1]$ and this function is decreasing in this interval. From this we have the classical uniqueness under these conditions for every $t = t_0 > 0$.

Theorem 3.2. Under conditions of Theorem 3.1, we have the classical uniqueness for solutions of equation (1.1) in the following sense:

i) if $u(t_0) \neq 0$ for $t_0 \in I = (0,1]$, then $u(t) \neq 0$ in the interval $I = (0,1]$.

ii) if $u(t_0) = 0$ for $t_0 \in I = (0,1]$, then $u(t) = 0$ in the interval $I = (0,1]$.

Proof. ii) follows from i) and i) follows immediately from the fact that function (3.5) is strongly monotonic. If $A(t,u)$ satisfies the following condition

i) $(A(t,u)v,v)$ is strongly positive for each $u(t) \in D_A$, and for each $v \in H$ $||v|| = 1$ and for each $t \in I$, or $(A(t,u)v,v) \geq \varepsilon > 0$, ε, may be dependent on $u(t)$.

ii) $(A(t,u)v,v)$ has bounded first derivatives for each $u(t) \in D_A$ and for each v with $||v|| = 1$ and for each $t \in I$ with respect to t, maximum of those derivatives depends on $u(t)$, then the classical uniqueness takes place in the interval I in the sense of Theorem 3.2 or the following statement is true.

Theorem 3.3. Let $B(t,u)$ satisfy condition A and $A(t,u)$ satisfy the conditions i) - ii) of the previous discussion. Then, the conclusion of Theorem 3.2 is true.

Proof. Let $u(t)$ be a non-trivial solution of equation (1.1) under conditions of this theorem. Then there exists constant $C \geq 0$ which depends on $u(t)$ maybe, such that the following is true for each v with $||v|| = 1$

$$(\frac{d}{dt}[A(t,u)]v,v) + C(A(t,u)v,v) > 0 \quad \text{in the interval } I.$$

This condition is enough for obtaining second-order differential inequality (3.3) and by using Lemma 3.1, we have that the function (3.5) is strongly decreasing in the interval I. From this, statement of our theorem follows immediately.

Remark 3.1. Recall that in this section, we require for the classical uniqueness of solutions of equation (1.1) in the interval I, more tough conditions than for the quasiuniqueness at the point $t = 0$ in §§1-2. Recall also, that we do not have here complete results about the classical uniqueness in the interval I, as Theorem 2.3 about the quasiuniqueness at the point $t = 0$.

In the case of condition (2.6) it is possible to obtain also the classical uniqueness. Namely, let condition (2.6) be satisfied and let $u(t)$ be a non-trivial solution of equation (1.1). Then, if

$$q(t) = ||u(t)||^2,$$

we have for $q(t)$ the following equation

$$t\dot{q}(t) = 2q(A(t,q^{\frac{1}{2}}v)v,v) \quad \text{where} \quad u(t) = q^{\frac{1}{2}}(t)v(t). \tag{3.6}$$

From (2.6) we obtain that the following form

$$(A(t,q^{\frac{1}{2}}v)v,v) \to 0 \quad \text{as} \quad t \to 0$$

and for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|(A(t,q^{\frac{1}{2}}v)v,v)| \leq \varepsilon \quad \text{whenever} \quad t < \delta \tag{3.7}$$

with ε, δ depending on $u(t)$, then from this we have for $q(t)$ the following inequality

$$\frac{t\dot{q}}{q} \leq \varepsilon$$

or

$$\frac{\dot{q}}{q} \leq \frac{\varepsilon}{t}$$

or

$$\ln\frac{q(t)}{q(t_0)} \leq \varepsilon \ln\frac{t}{t_0}$$

or

$$q(t) \le q(t_0)\exp^{\varepsilon}\ln\frac{t}{t_C} = q(t_0)\left(\frac{t}{t_C}\right)^{\varepsilon} \tag{3.8}$$

and since this is true for each $\varepsilon > 0$, from this discussion we obtain that the following statement is true.

Theorem 3.4. Let $B(t,u)$ satisfy the condition A and $A(t,u)$ satisfy the condition (2.6). If for each $u(t) \in D_A$ the following function

$$(A(t.u)v,v) \to 0 \quad \text{as} \quad t \to 0 \quad \text{for each} \quad v \quad \text{with} \quad ||v|| = 1,$$

then the conclusion of Theorem 3.2 is true.

Proof. The proof follows immediately from our estimate (3.8).

§4. The non-degenerate case

Let us consider the following non-degenerate non-linear equation in the Hilbert space H:

$$\frac{du}{dt} = B(t,u(t)) \quad \text{where} \quad t \in \bar{I} = [1,+\infty). \tag{4.1}$$

$u(t)$ for each $t \in \bar{I}$ is an element of H and has derivative with respect to t. $B(t,u(t))$ is non-linear map from H to H with domain D_B. D_B is the dense subset of H, and for each $t \in \bar{I}$ and for each $u(t) \in D_B$, $B(t,u(t))$ is an element of H also. $B(t,u(t))$ is not necessarily bounded. H is a Hilbert space with scalar produce $(.,.)$ and with norm $||.||$ correspondingly.

As in the degenerate situation, operator $B(t,u)$ is called a smooth operator of self-adjoint type if the scalar product

$$(B(t,u),u)$$

is real and has derivative with respect to t for each $t \in \bar{I}$ and for each $u(t) \in D_B$. After the change of variable t by formula

$$s = e^{-t}$$

we obtain from equation (4.1) the following equation

$$d\frac{du}{ds} = B(s,u) \quad \text{for} \quad s \in I = (0,1]. \tag{4.2}$$

The equation (4.2) is the equation of the type (1.1) and because of this, it is possible to rewrite our results for equation (4.1). It is easy to see that in this situation, the following class of functions plays the role of the flat functions:

Class A:

$$\{f(t),\ t \in \bar{I}: \ \text{for each} \ C > 0 \quad e^{Ct}f(t) \to 0 \quad \text{as} \quad t \to +\infty\}$$

In this situation, we have the following type of quasiuniqueness.

Definition 4.1. We say that the quasiuniqueness takes place for solutions of equation (4.1) at the point $t = +\infty$, if the following statement is true:

If $u(t)$ is a solution of equation (4.1) and $||u(t)||$ belongs to Class A or, in other words: for each $C > 0$, $e\ ||u(t)|| \to 0$ as $t \to +\infty$, then $u(t) \equiv 0$ for $t > N$ for some $N < \infty$. Now it is possible to rewrite all theorems of §§1-3 for the case of equation (4.1).

Theorem 4.1. Let $u(t)$ be a solution of equation (4.1) such that

$$\left(\frac{d}{dt}[B(t,u(t)],u(t)\right) + ||B(t,u(t))||^2 - \frac{2B(t,u(t)),u(t))^2}{||u(t)||^2} \ge$$

$$-Ce^{-t}[|(B(t,u(t)),u(t))| + ||u(t)||^2] \quad \text{for some constant} \quad C \ge 0. \tag{4.3}$$

Then the following is true:

i) $||u(t)|| \geq M||u(t_0)||e^{-\mu t}$ (4.4)

where constant $\mu \geq 0$ depends on C from (4.3) and $u(t)$ itself.

ii) If $u(t)$ belongs to class A, then $u(t) \equiv 0$ in the interval $\bar{I} = [1,+\infty)$, or the quasiuniqueness takes place for solution of equation (4.1) under those conditions at the point $t = +\infty$.

Proof. The proof follows from Theorem 1.2.

Remark 4.1. In the linear case, these type of theorems and these type of estimates were obtained first by Agmon-Nirenberg [1,2]. It is possible also to study the special case of equation (4.1) under condition A. From the Theorem 1.3, we have that the following statement is true.

Theorem 4.2. Let $u(t)$ be a solution of equation (4.1) under condition A such that

$$([\tfrac{d}{dt}A(t,u)]u(t),u(t)) \geq -Ce^{-t}[|(A(t,u)u,u)| + ||u(t)||^2] \qquad (4.5)$$

for some constant $C \geq 0$. Then the conclusion of Theorem 4.1 is true.

It is possible also to study equation (4.1) under condition (1.26) and if $A(t,u)$ as a function of t satisfies the following condition

$$(\tfrac{d}{dt}[A(t,u)]v,v \geq -Ce^{-t}(A(t,u)v,v) \qquad (4.6)$$

for each v with $||v|| = 1$ with some constant $C \geq 0$, then the conclusion of Theorem 4.1 is true also.

From (4.6) we have, if $f(t)$ denotes $(A(t,u)v,v)$ as function of t, that

$$f'(t) \geq -Ce^{-t}f(t) \qquad (4.7)$$

or

$$\frac{f'(t)}{f(t)} \geq -Ce^{-t}$$

or

$$\ln\frac{f(t)}{f(t_0)} \geq -C\int_{t_0}^{t} e^{-\tau}d\tau = Ce^{-\tau}\Big|_{t_0}^{t} = C(e^{-t}-e^{-t_0})$$

or

$$f(t) \geq f(t_0)e^{Ce^{-t}}e^{-Ce^{-t}} = f(t_0)e^{C(e^{-t}-e^{-t_0})} \qquad (4.8)$$

or

$$f(t) \geq Mf(t_0)e^{Ce^{-t}}.$$

From (4.7) we obtain that the following function

$$f(t)e^{-Ce^{-t}}$$

is a monotonic and it is not decreasing in the interval $\bar{I} = [1,+\infty)$. For example, if $f(t)$ is not decreasing in the interval $\bar{I} = [1,+\infty)$, this condition is enough to satisfy the condition (4.7) or the condition (4.8) with $C = 0$.

From this discussion and Theorem 1.4, we obtain that the following statement is true.

Theorem 4.3. Let conditions A and (1.26) be satisfied and let $(A(t,u)v,v)$ be a monotonic function with unique minimum for $t = +\infty$ for each $u(t)$ from class A and for each v with $||v||= 1$. Let $u(t)$ be a solution of equation (4.1) from class A. Then $u(t) \equiv 0$ in the interval $\bar{I}$. In other words, the quasiuniqueness takes place under those conditions for solutions of equation (4.1) at the point $t = +\infty$. From the previous discussion and Theorem 1.5, we obtain that the following theorem is true.

Theorem 4.4. Let conditions A and (1.26) be satisfied. Let $(A(t,u)v,v)$ as function of t for each $u(t) \in D_A$, $u(t)$ from class A satisfies the following condition:

$(A(t,u)v,v)$ is monotonic with local minimum at the point $t = +\infty$ in some neighborhood of the $t = +\infty$ with respect to t in the interval $\bar{I} = [1,+\infty)$ for each $u(t)$ from class A and for each v with $||v|| = 1$. If $u(t)$ is a solution of equation (4.1) from class A, then $u(t) \equiv 0$ in the interval $(N,+\infty)$ with N depending on $u(t)$ itself. From Theorem 2.1, we obtain that the following statement is true.

Theorem 4.5. Let $u(t)$ be a solution of equation (4.1) under condition (2.1). If $||u(t)|| \to 0$ as $t \to +\infty$, then $u(t) = 0$ in the interval $\bar{I} = [1,+\infty)$. In this situation, we have classical uniqueness at the point $t = +\infty$. Let now $A(t,u)$ satisfy the following condition

$$(A(t,u)v,v) \to 0 \quad \text{as} \quad t \to +\infty \tag{4.9}$$

for each $u(t)$ from class A and for each $v \in H$ with $||v|| = 1$. From Theorem 2.2, we obtain that the following statement is true.

Theorem 4.6. Let $A(t,u)$ satisfy the condition (4.9). Then if $u(t)$ is a solution of equation (4.1) from class A, $u(t)$ is a trivial in the interval $(N,+\infty)$ with $N < \infty$ depending on $u(t)$ itself. From the previous theorems and Theorem 2.3, we obtain that the following statement is true.

Theorem 4.7. Let $B(u)$, not dependent on t, satisfy condition A for u from class A or $(A(t,u)v,v) = f(t)$ a function from C^1 $H_1(\bar{I})$ with respect to t, $t > N$ for each $u(t)$ from class A and for each $v(t)$ with $||v|| = 1$. Let $u(t)$ be a solution of equation (4.1) from class A. Then $u(t) \equiv 0$ in the interval $(N,+\infty)$ with $N < +\infty$ depending on $u(t)$ itself.

Proof.

i) If $(A(t,u)v,v) \to 0$ as $t \to 0$ this statement follows from Theorem 4.6;

ii) If $(A(t,u)v,v) \le 0$ in the interval $(N,+\infty)$, this statement follows from Theorem 4.5.

iii) If $(A(t,u)v,v)|_{u=0} > 0$ in the interval $(N,+\infty)$ and $(\frac{d}{dt}[A(t,u)]v,v)|_{u=0} \ge 0$ in this interval, this statement follows from Theorem 4.4 since condition (4.7) is satisfied with $C = 0$.

iv) If $(A(u)v,v)|_{u=0} > 0$ in the interval $(N,+\infty)$ and $(\frac{d}{dt}[A(u)]v,v)|_{u=0} < 0$ in this interval, this statement follows from condition (4.7) since in this situation in the derivative $\frac{d}{dt}[A(u)]$ we will have terms with $u_t^1(t)$ and from having $u(t)$ belong to class A, it follows that in the interval $(N,+\infty)$ we can choose constant C such that (4.7) will be satisfied in some neighborhood of the point $t = +\infty$. In this situation, C depends on $u(t)$ itself. Recall that interval $(N,+\infty)$ depends also on $u(t)$.

Remark 4.2. It is possible to rewrite our theorems from §3 about uniqueness for the case of equation (4.1) and to prove these using Lemma 3.1 and monotonicity of functions of type

$$e^{\mu t}||u(t)|| \tag{4.10}$$

in the interval $\bar{I} = [1,+\infty)$ with some $\mu > 0$.

§5. Examples

Let us consider the following equation

$$t\frac{\partial u}{\partial t} = F(t,u,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2},\ldots,\frac{\partial^m u}{\partial x^m})u \tag{5.1}$$

where $t \in I = (0,1]$ and $x \in \Omega \subset R^1$, Ω is compact or compact 1 manifold, F is a C^1-function with respect to all variables $(t,u,z_i,\ldots,z_m)$ for all values of these variables. F is real-values function.

From Theorem 2.3 we obtain that for equation (5.1) the quasiuniqueness takes place at the point $t = 0$ for classical solutions. If function F belongs to C^1 in some neighborhood of the origin only, we obtain from Theorem 2.3 that the quasi-uniqueness takes place at the point $t = 0$ too.

From flatness of solution $u(t,x)$ of equation (5.1) we obtain that there exists a neighborhood $[0,\varepsilon]$ with $\varepsilon > 0$ depending on $u(t)$ itself and in this neighborhood $u(t) = 0$ for classical solutions.

Remark 5.1. Recall that this statement is true for classical solutions of equation (5.1) only, because in this situation we have that

$$\frac{\partial^i u}{\partial x^i} \to 0 \quad i = 1,\ldots,m \quad \text{as } t \to 0$$

for flat function $u(t,x)$. In this case also from flatness of $u(t,x)$ follows the flatness of $\partial^i u/\partial x^i$ for $i = 1,\ldots m$. And if Ω is compact, it is possible to choose for each $\delta > 0$ the neighborhood of the point $t = 0$ the interval $[0,\varepsilon]$ such that

$$\left|\frac{\partial^i u}{\partial x^i}\right| < \delta \quad \text{for } i = 0,1,\ldots,m \quad \text{if } t < \varepsilon .$$

2. Let us consider the following equation

$$t\frac{\partial u}{\partial t} = F(t,x,u,\frac{\partial u}{\partial x},\ldots,\frac{\partial^m u}{\partial x^m}) \tag{5.2}$$

where $t \in I = (0,1]$ and $x \in \Omega$ R^1 and compact or Ω is a compact manifold, F is a C^1-function with respect to all variables $(t,x,u,z_1,\ldots,z_m)$ for all values of these variables or for some neighborhood of the point $(0,x,0,\ldots0)$ for each $x \in \Omega$. If F does not change sign for each $x \in \Omega$, or in other words, if one of the following conditions is satisfied:

i) $\min_{x\in\Omega} |F(0,x,0,\ldots,0)| > 0$,

ii) for each $x \in \Omega$ $F(t,x,u,z_1,\ldots,z_m) \le 0$ for all values $t,u,z_1,\ldots,z_m$ such that $t \le \varepsilon$, $|u| \le \varepsilon,\ldots,|z_m| \le \varepsilon$,

iii) $F(t,x,u,z_1,\ldots,z_m) \to 0$ wherever $|u|\cdot|z_1|\cdot|z_2|\cdot\ldots\cdot|z_m| \to 0$ and $|u|^2 + |z_1|^2 + \ldots + |z_m|^2 \le \delta > 0$, then the quasiuniqueness takes place at the point $t = 0$ for classical solutions of this equation. From flatness of solution $u(t)$ of equation (5.2), we obtain that there exists $\varepsilon > 0$ depending on $u(t)$ itself such that $u(t) = 0$ in the interval $[0,\varepsilon]$.

3. Let us consider the following equation

$$t\frac{\partial u}{\partial t} = F(t,u,\frac{\partial u}{\partial t_i},\frac{\partial^2 u}{\partial x_{i_1}\partial x_{i_2}},\ldots,\frac{\partial^m u}{\partial x_{i_1}\ldots\partial x_{i_m}})u(t,x) \tag{5.3}$$

where $t \in I = (0,1]$, $x \in \Omega \subset R^n$ and compact or Ω is compact manifold, $i_k = 1,\ldots n$

and F is a C^1-function with respect to all variables $(t,u,z_1,\dots z_n)$ for all values of these variables. F is real-values function.
From Theorem 2.3, we obtain that for equation (5.3) the quasiuniqueness takes place at the point $t = 0$ for classical solutions. If function F belongs to C^1 in some neighborhood of the origin only, we obtain from Theorem 2.3 that the quasi-uniqueness takes place at the point $t = 0$ too.

From flatness of solution $u(t,x)$ of equation (5.3), we get that there exists $\varepsilon > 0$ dependent on $u(t,x)$ itself such that $u(t) = 0$ in the interval $[0,\varepsilon]$.
Remark 5.2. It is possible to obtain results about the classical uniqueness of equation (5.1)-(5.4) in the interval I, but for this we must require that function F satisfies one of the following conditions:

i) $F \le 0$ for all values of all variables.

ii) $F \ge \delta > 0$ for all values of all variables.

Then from results of §3 we obtain that the classical uniqueness takes place in the sense of Theorem 3.2, or

i) if $u(t_0) \ne 0$ for $t_0 \in I = (0,1]$, then $u(t) \ne 0$ in the interval $I = (0,1]$.

ii) if $u(t_0) = 0$ for $t_0 \in I = (0,1]$, then $u(t) = 0$ in the interval $I = (0,1]$ including $t = 0$.

Recall that i)-ii) are like the standard conditions for uniqueness in the linear case.

4. Let us consider the following equation

$$t\frac{\partial u}{\partial t} = F\left(t,x_i,\dots,x_n,u,\frac{\partial u}{\partial t_i},\frac{\partial^2 u}{\partial x_i\partial x_j},\dots,\frac{\partial^m u}{\partial x_{i_1}\dots\partial x_{i_m}}\right)u(t,x) \tag{5.4}$$

where $t \in I = (0,1]$, $x \in \Omega \subset R^n$ and compact $i_k = 1,\dots n$ and F is a real C^1-function with respect to all variables $(t,x,u,z_1,\dots z_n)$ for all values of these variables for some neighborhood of the point $(0,x,0,\dots 0)$ for each $x \in \Omega$. If function F satisfies one of the conditions i)-iii) from example 2, then the quasiuniqueness takes place at the point $t = 0$ for classical solutions of equation (5.4). From flatness of classical solution $u(t)$ of equation (5.4), we obtain that there exists $\varepsilon > 0$ dependent on $u(t)$ itself such that $u(t) = 0$ in the interval $[0,\varepsilon]$.

5. Let us consider the following equation

$$\frac{\partial u}{\partial t} = F\left(t,u,\frac{\partial u}{\partial x_i},\frac{\partial^2 u}{\partial x_{i_1}\partial x_{i_2}},\dots,\frac{\partial^m u}{\partial x_{i_1}\dots\partial x_{i_m}}\right)u(t,x) \tag{5.5}$$

where $t \in \bar{I} = [1,+\infty)$, $x \in \Omega$ R^n and Ω is compact or compact manifold $i_k = 1, \dots, n$ and F is a C^1-function with respect to all variables $(t,u,z_1,\dots,z_n)$ for all values of these variables. F is a real-values function.

From Theorem 4.7 we obtain that for equation (5.5), the quasiuniqueness takes place at the point $t = +\infty$ for classical solutions. If function F belongs to C^1 in some neighborhood of the point $(t,0,0,\dots,0)$ for each $t > N_0 < +\infty$, we obtain from Theorem 4.7 that the quasiuniqueness takes place at the point $t = +\infty$ also.

If solution $u(t)$ of equation (5.5) belongs to class A, we get that there exists $\varepsilon > 0$ dependent on $u(t)$ itself such that $u(t) = 0$ in the interval $(1/\varepsilon, +\infty)$.

6. Let us consider the following equation

$$\frac{\partial u}{\partial t} = F(t,x_1,\ldots,x_n,u,\frac{\partial u}{\partial x_i},\ldots,\frac{\partial^m u}{\partial x_{i_1}\ldots\partial x_{i_m}})u(t,x) \qquad (5.6)$$

where $t \in \bar{I} = [1,+\infty)$, $x \in \Omega \subset R^n$ and compact of Ω is a compact manifold, $i_k = 1,\ldots n$ and F is a C^1-function with respect to all variables for all values of variables or for some neighborhood of the point $(t,x,0,\ldots 0)$ for each $x \in 0$, and for each $t > \frac{1}{\varepsilon}$ for $\varepsilon > 0$.

If F satisfies one of the conditions of example 2, then the quasiuniqueness takes place at the point $t = +\infty$ for classical solutions of equation (5.6). If solution $u(t,x)$ of equation (5.6) belongs to class A, we get that there exists $\varepsilon > 0$ dependent on $u(t)$ itself such that $u(t) = 0$ in the interval $(1/\varepsilon,+\infty)$.

Remark 5.3. It is possible to obtain also results about the classical uniqueness of equation (5.5)-(5.6) in the interval $\bar{I}$, as we obtained in Remark 5.2.

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