Internat. J. Math. & Math. Sci.
VOL. 11 NO. 3 (1988) 517-522

ON TRANSLATIVITY OF ABSOLUTE ROGOSINSKI-BERNSTEIN SUMMABILITY METHODS

AZMI K. AL-MADI

Department of Mathematics
U.A.E. University, Al-Ain
United Arab Emirates

(Received November 26, 1986 and in revised form February 25, 1987)

ABSTRACT. The paper is concerned with the problem of translativity of an absolute Rogosinski-Bernstein summability methods (B_h); $0 \leqq h \leqq 1$. The paper contains two results that when h; $\frac{1}{2} \leqq h \leqq 1$, $|B_h|$ is translative and when h; $(o < h < \frac{1}{2})$, then $|B_h|$ is not translative neither to the left nor to the right.

KEYS WORDS AND PHRASES. Absolute summability, Rogosinski-Bernstein methods, sequence-to sequence transformation, absolutely regular, translative.
1980 AMS SUBJECT CLASSIFICATION CODE. Primary 40F05, Secondary 40G99.

1. INTRODUCTION.

A series $a_0+a_1+\ldots$ of real or complex terms a_k with its partial sums $S_n = a_0 + a_1 + \ldots + a_n$, is evaluable to t by the Rogosinski-Bernstein method (B_h) of order h; $(0 \leqq h \leqq 1)$, if $t_n \to t$ as $n \to \infty$, where

$$t_n = \sum_{k=0}^{n} \left[\cos \frac{\pi k}{2(n+h)}\right] a_k . \tag{1.1}$$

The series is evaluable to M by the Ceasa'ro mean of (C,1), if $M_n \to M$ as $n \to \infty$, where

$$M_n = \frac{1}{n+1} \sum_{k=0}^{n} S_k. \tag{1.2}$$

The series is absolutely summable (B_h) or summable $|B_h|$ if the sequence $\{t_n\}$ is of bounded variation; that is to say, if

$$\sum_{n=0}^{\infty} |t_n - t_{n-1}| = 0(1), \qquad t_{-1} = 0. \tag{1.3}$$

Similar definition for the series being absolutely summable (C,1).

A sequence-to-sequence method A is called translative to the left, if the limitability of $S_0,S_1,\ldots,S_n,\ldots$ implies the limitability of $0,S_0,\ldots, S_{n-1}$, to the same limit, it is translative to the right, if the converse holds. A is translative if and only if, A is translative to the left and right.

The method (B_h) has been the subject of many papers (see [2], [3] and [4]). Agnew [3] obtained many significant results concerning inclusion and equivalence relations between the methods (B_h), (C,1) and modified arithmetic means. The author [1; Theorems (3.1) and (3.2)] obtained results involving absolute summability methods analogous to some of those by Agnew. In [2], the author investigated the values of h; $0 \leqq h \leqq 1$, for which (B_h) is or is not translative, and left an open problem that (B_h); $\frac{1}{2}(\sqrt{3}-1) \leqq h < \frac{1}{2}$ is non-translative to the left nor do to the right.

2. OBJECT OF THE PAPER.

The object of this paper is to obtain results on translativity involving absolute summability of the method (B_h). Some of these results are analogous to some of those by the author in [2].

3. SUBSIDIARY RESULTS.

The following results are necessary for our purposes:

THEOREM 1. (The author [1; Lemma 2.2]) A necessary condition that the sequence-to-sequence transformation

$$g_n = \sum_{k=0}^{\infty} G_{n,k} S_k ,$$

be absolutely regular is that $G_{n,k}$ be bounded for all n,k.

THEOREM 2. (The author [1; Theorem 3.2]) If $\frac{1}{2} < h \leqq 1$, then $|B_h|$ and $|C,1|$ are equivalent.

4. MAIN RESULTS.

In this section we prove the two main results of this paper.

THEOREM 3. If $\frac{1}{2} < h \leqq 1$, then $|B_h|$ is translative.

PROOF. Since $|C,1|$ is translative, Theorem 2 yields the result.

THEOREM 4. If h; $h \in (0,\frac{1}{2})$, then $|B_h|$ is not translative to the left nor to the right.

PROOF. In (1.1) write for a_k its value $S_k - S_{k-1}$, with $S_{-1} = 0$, and consider the sequence $\{U_n\}$ given by

$$U_n = S_0 + S_1 + \dots + S_n ,$$

to obtain t_n in terms of U_n. The result is

$$t_n = \sum_{k=0}^{\infty} A_{n,k} U_k , \tag{4.1}$$

where

$$A_{n,n} = \sinh\theta \tag{4.2}$$

$$A_{n,n-1} = \sin(1+h)\theta - 2\sin\theta, \tag{4.3}$$

$$A_{n,k} = -4\sin^2\tfrac{1}{2}\theta\cos(k+1)\theta, \qquad 0 \leqq k \leqq n-2 \tag{4.4}$$

and

$$A_{n,k} = 0 \quad \text{otherwise}, \tag{4.5}$$

and where

$$\theta = \frac{\pi}{2(n+h)} \quad . \tag{4.6}$$

Let $\{t_n\}$, $\{\bar{t}_n\}$ be respectively the (B_h) transforms of $\{S_n\}$, $\{S_{n-1}\}$. Obtain $\bar{t}_{n+1}$ in terms of t_n. The result is

$$\bar{t}_{n+1} = \sum_{k=0}^{n} V_{n,k} t_k, \tag{4.7}$$

where

$$V_{n,k} = \sum_{v=k}^{n} A_{n+1,v+1} B_{v,k} , \qquad 0 \leq k \leq n \tag{4.8}$$

and

$$V_{n,k} = 0 \quad \text{otherwise} , \tag{4.9}$$

and where $A_{n,k}$ is given by (4.2)-(4.5), and $B_{n,k}$ is the reciprocal matrix of $A_{n,k}$. Using the inversion formula of (4.10, we have for each r=3,4,5,... ,

$$B_{n,n-r} = (-1)^r \frac{H_{n,n-r}}{A_{n,n} A_{n-1,n-1} \cdots A_{n-r,n-r}} \tag{4.10}$$

where

$$H_{n,n-r} = \begin{vmatrix} A_{n-r+1,n-r} & A_{n-r+1,n-r+1} & 0 & 0 & . & 0 \\ A_{n-r+2,n-r} & A_{n-r+2,n-r+1} & A_{n-r+2,n-r+2} & 0 & . & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n,n-r} & A_{n,n-r+1} & & & . & A_{n,n-1} \end{vmatrix} \tag{4.11}$$

Using (4.8),(4.9) and the fact that

$$B_{n,k} = -\left(\sum_{v=k}^{n-1} A_{n,v} B_{v,k} \right) (A_{n,n})^{-1} , \tag{4.12}$$

we have

$$V_{n,k} = \sum_{v=k}^{n-1} E_{n,v} B_{v,k} , \qquad 0 \leq k \leq n-1 \tag{4.13}$$

where

$$E_{n,v} = - \frac{A_{n+1,n+1}}{A_{n,n}} A_{n,v} + A_{n+1,v+1} . \tag{4.14}$$

Hence

$$V_{n,n-r} = \sum_{u=1}^{r} E_{n,n-u} B_{n-u,n-r} . \qquad 1 \leq r \leq n \tag{4.15}$$

Let w be any constant with $0 < w < \frac{1}{2}$. We will show that if h; $0 < h < \frac{1}{2}$, then for sufficiently large n, and $2 \leq r \leq n^w$,

$$\left| V_{n,n-r} \right| \geq C \left| E_{n,n-1} B_{n-1,n-r} \right|, \tag{4.16}$$

where C is some positive constant less that 1, and this will be satisfies if, we prove that (for sufficiently large n and $2 \leq r \leq n^{w}$) the terms of the sum of the right hand side of (4.15) alternate in sign and

$$\left| \frac{E_{n,n-u-1} \; B_{n-u-1,n-r}}{E_{n,n-u} \; B_{n-u,n-r}} \right| < \text{constant} < 1. \tag{4.17}$$

Using (4.2)-(4.4), we have that

$$nA_{n,n} \to \frac{\pi h}{2}, \quad nA_{n,n-1} \to \frac{\pi(1-h)}{2}, \quad A_{n,k} < 0, \text{ with } A_{n,k} \to 0, \ (0 \leq k \leq n-2). \tag{4.18}$$

Using (4.2)-(4.6) and (4.18), we have from (4.10) and (4.11) that

$$\frac{B_{n,n-r}}{n} \to (-1)^{r} \frac{2}{\pi h} \left(\frac{1-h}{h}\right)^{r}, \quad \text{as} \quad n \to \infty. \tag{4.19}$$

Thus

$$\frac{B_{n-u,n-r}}{n} \to (-1)^{r-u} \frac{2}{\pi h} \left(\frac{1-h}{h}\right)^{r-u}, \quad \text{as} \quad n \to \infty. \tag{4.20}$$

Next, we will show that $E_{n,n-u} > 0$ for $1 \leq u \leq r \leq n^{w}$. Using (4.14), we have from (4.2) and (4.4) that if $2 \leq u \leq n$, then

$$E_{n,n-u} = A_{n,n-u} \left[\frac{\sin^{2} \frac{\pi}{4(n+h+1)} \cos \frac{\pi(n-u+2)}{2(n+1+h)}}{\sin^{2} \frac{\pi}{4(n+h)} \cos \frac{(n-u+1)\pi}{2(n+h)}} - \frac{\sin \frac{\pi}{2(n+1+h)}}{\sin \frac{\pi}{2(n+h)}} \right]. \tag{4.21}$$

It may be easily seen that the quantity inside the square brackets is negative. This together with (4.4) imply that when $2 \leq u \leq n^{w}$,

$$E_{n,n-u} > 0 \quad \text{for large n uniformly in} \quad 2 \leq u \leq n^{w}. \tag{4.22}$$

Using (4.2),(4.3) and note that

$$\sin \frac{\pi h}{2(n+h)} = \frac{\pi h}{2(n+h)} - \frac{\pi^{3} h^{3}}{48(n+h)^{3}} + 0\left(\frac{1}{n^{5}}\right), \tag{4.23}$$

we have from (4.14) that

$$E_{n,n-1} \sim \frac{\pi^{3}(1+h)(1+2h)}{12n^{4}}. \tag{4.24}$$

Thus, by (4.22) and (4.24), we have that

$$E_{n,n-u} > 0 \qquad 1 \leq u \leq n^{w}. \tag{4.25}$$

Thus, if $h \in (0,\frac{1}{2})$, then (4.20) and (4.25) imply that the terms on the right hand side of (4.15) are alternate in sign for sufficiently large n uniformly in $2 \leq r \leq n^{w}$. To show that (4.17) is satisfied, we use (4.4) and (4.23) to have from (4.21), (after straightforward calculations) the result that

$$E_{n,n-u} = \frac{\pi^3(u+h-1)}{4n^4} + O(\frac{u^3}{u^5}) \quad , \text{ for large n uniformly in } 2 \leq u \leq n^w . \quad (4.26)$$

This implies that

$$\frac{E_{n,n-u-1}}{E_{n,n-u}} \to \frac{u+h}{u+h-1} . \quad (4.27)$$

Using (4.20), we have

$$\frac{B_{n-u-1,n-r}}{B_{n-u,n-r}} \to -\frac{h}{1-h} . \quad (4.28)$$

Using (4.27) and (4.28), we see that when $h \in (0,\frac{1}{2})$, (4.17) is satisfied, and so (4.16) is satisfied, which by (4.20) and (4.24) shows that the numbers $V_{n,n-r}$ are not bounded for large n uniformly in $2 \leq r \leq n^w$. Hence the necessary condition of Theorem (3.1), fails to hold. Thus in case $h \in (0,\frac{1}{2})$, the transformation given by (4.7) is not absolutely regular, consequently, when $h \in (0,\frac{1}{2})$, $|B_h|$ is not translative to the left.

As for right translativity, obtain t_n in terms of $\bar{t}_{n+1}$ and using similar technique to that of left translativity, we see that when $h \in (0,\frac{1}{2})$, $|B_h|$ is not translative to the right, and this completes the proof.

REFERENCES

1. AL-MADI, A.K. On Absolute Rogosinski-Bernstein Trigonometric Summability Methods, Indian Journal of Mathematics 25 (1983), 53-66.

2. AL-MADI, A.K. On Translativity of Rogosinski-Bernstein Trigonometric Summability Methods, International Journal of Maths. and Mathematical Sciences (To Appear).

3. AGNEW, R.P. Rogosinski-Bernstein Trigonometric Summability Methods and Modified Arithmetic Means, Annals of Maths. 56(3) (1952), 537-559.

4. PETERSEN, G.M. A Note on Divergent Series, Canadian Journal of Maths. 4 (1952), 445-454.