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NONPARAMETRIC MINIMAL SURFACES IN R^3 WHOSE BOUNDARIES HAVE A JUMP DISCONTINUITY

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ABSTRACT. Let Ω be a domain in R^2 which is locally convex at each point of its boundary except possibly one, say (0,0), ϕ be continuous on $\partial\Omega/\{(0,0)\}$ with a jump discontinuity at (0,0) and f be the unique variational solution of the minimal surface equation with boundary values ϕ. Then the radial limits of f at (0,0) from all directions in Ω exist. If the radial limits all lie between the lower and upper limits of ϕ at (0,0), then the radial limits of f are weakly monotonic; if not, they are weakly increasing and then decreasing (or the reverse). Additionally, their behavior near the extreme directions is examined and a conjecture of the author's is proven.

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1. INTRODUCTION.

How does the generalized solution of the Dirichlet problem for the minimal surface equation with boundary values ϕ behave when ϕ has a jump discontinuity (say at the origin)? Under certain mild conditions on the domain $\Omega \subseteq R^2$, we shall show that the radial limits at (0,0) of the solution, denoted $Rf(\theta)$, exist for all $\theta \in (\alpha,\beta)$, where $\Omega = \{(r,\theta)|\alpha < \theta < \beta,\ 0 < r < r(\theta)\}$. Further, on at most three intervals (i.e. $[\alpha,\alpha']$, $[\theta_L, \theta_R]$, $[\beta',\beta]$) $Rf(\theta)$ is constant and elsewhere it is strictly monotonic.

If $Rf(\theta)$ lies between the lower and upper limits of ϕ at (0,0), then Rf is weakly monotonic on $[\alpha,\beta]$. If not, then Rf is not monotonic on $[\alpha,\beta]$ but it is weakly monotonic on $[\alpha, \alpha+\pi]$ and on $[\beta-\pi,\beta]$. Under some smoothness and nontangency assumptions, we shall show that $\alpha' = \alpha$ or $\alpha' = \alpha+\pi$ and $\beta' = \beta$ or $\beta' = \beta-\pi$. We shall also show that $\theta_R = \theta_L + \pi$ when θ_L and θ_R occur. Thus there is at most one interval on which $Rf(\theta)$ is constant.

2. PRELIMINARIES.

By Ω we will mean a bounded open subset of R^2 with the following properties: (a) Ω is connected and simply connected. (b) $\partial\beta$ is Lipschitz and $N = (0,0) \in \partial\Omega$. (c) Ω is locally convex at each point of its boundary except possibly N. (d) In polar coordinates (r,θ) about $N, \Omega = \{(r,\theta) \mid \alpha < \theta < \beta,\ 0 < r < r(\theta)\}$ with $-\pi < \alpha < 0 < \beta < \pi$. From (d) we see that near N, the x-axis divides $\partial\Omega$ into two components.

DEFINITION. Let Ω be as above. We will denote by $C^*(\partial\Omega)$ those functions $\phi \in C^0(\partial\Omega/\{N\})$ such that

$$\phi(N+) = \lim\ \phi(P) \text{ as } P \in \partial\Omega \cap \{(x,y) \mid y > 0\} \text{ approaches N and}$$

$$\phi(N-) = \lim\ \phi(P) \text{ as } P \in \partial\Omega \cap \{(x,y) \mid y < 0\} \text{ approaches N}$$

each exist.

Notice $\phi \in C^*(\partial\Omega)$ implies ϕ has a jump discontinuity at N (possibly with jump 0).

DEFINITION. Let $\phi \in C^*(\partial\Omega)$. Define $f = f(\cdot, \phi)$ to be the function in $BV(\Omega)$ which minimizes

$$J(v) = J(v,\phi) = \iint_\Omega \sqrt{1 + |Dv|^2} + \int_{\partial\Omega} |v - \phi|$$

for $v \in BV(\Omega)$.

Notice $f \in C^2(\Omega) \cap C^0(\bar{\Omega}/\{N\})$ and $f = \phi$ on $\partial\Omega/\{N\}$.

We set

$$S_0 = S_0(\phi) = \{(x,y,f(x,y)) \mid (x,y) \in \Omega\}$$

and

$$\Gamma_0 = \Gamma_0(\phi) = \{(x,y,\phi(x,y)) \mid N \neq (x,y) \in \partial\Omega\}.$$

Let S be the closure of S_0, Γ be the closure of Γ_0, Γ^+ be the closure of $\Gamma \cap \{(x,y) \mid y > 0\}$, and Γ^- be the closure of $\Gamma \cap \{(x,y) \mid y < 0\}$.

Throughout this paper, we will make the following

ASSUMPTION. $f \notin C^0(\bar{\Omega})$.

We will need to represent S parametrically. Let us set $E = \{(u,v) \mid u^2 + v^2 < 1\}$, $B = \{(u,v) \in E \mid v > 0\}$, $\partial' B = \{(u,v) \in \partial E \mid v > 0\}$, $\partial'' B = \{(u,0) \mid -1 < u < 1\}$, $B' = B \cup \partial' B$, and $B'' = B \cup \partial'' B$. Using the methods of [1] or [2], we can prove the following propositions.

PROPOSITION 1. There exists $X = (z,y,z) \in C^0(\bar{B}: R^3) \cap C^2(B: R^3)$ such that X maps B homeomorphically onto S_0, X maps $\partial' B$ strictly monotonically onto Γ_0, X maps $\partial'' B$ into the z axis, $X(-1,0) = (0,0,\phi(N-))$, $X(1,0) = (0,0,\phi(N+))$, and

$$X_u \cdot X_v = 0$$

$$X_u^2 = X_v^2$$

$$X_{uu} + X_{vv} = 0$$

on B. Also, X extends across $\partial'' B$ by reflection to a function in $C^2(E: R^3)$ and

$$X_u(u,0) = (0,0,z_u(u,0))$$

$$X_v(u,0) = (x_v(u,0), y_v(u,0), 0)$$

for $-1 < u < 1$.

For each $\alpha < \theta < \beta$ and $t > 0$, define

$$\lambda(t,\theta) = (t\cos(\theta),\ t\sin(\theta)),$$

$$\omega(t,\theta) = \omega(t,\theta,\phi) = X^{-1}(\lambda(t,\theta),\ f(\lambda(t,\theta))),$$

$$Rf(\theta) = \lim_{t \to 0+} f(\lambda(t,\theta)) \text{ if this exists.}$$

Set $Rf(\alpha) = \phi(N-)$, $Rf(\beta) = \phi(N+)$, $u(\alpha) = -1$, and $u(\beta) = 1$.

PROPOSITION 2. For all $\alpha < \theta < \beta$, there is a unique $u(\theta) \in [-1,1]$ such that

$$\omega(t,\theta) \to (u(\theta),0) \quad \text{as} \quad t \to 0+$$

and

$$Rf(\theta) = z(u(\theta),0).$$

Further, $u(\cdot) \in C^0([\alpha,\beta])$, $Rf \in C^0([\alpha,\beta])$, and

$$X_v(u(\theta),0) = |z_u(u(\theta),0)| \quad (\cos(\theta),\ \sin(\theta),0)$$

for all $\theta \in (\alpha,\beta)$ with $|u(\theta)| < 1$.

REMARK. If X has no branch points on $\{(u(\theta),0) \mid \theta_1 < \theta < \theta_2\}$, then $u(\cdot)$ is strictly increasing on $[\theta_1,\theta_2]$. Also, $u(\cdot)$ is weakly increasing on $[\alpha,\beta]$.

From the proof of Theorem 3.2 of [1], we have the following

LEMMA 1. Suppose $\alpha \leqq \theta_1 < \theta_2 \leqq \beta$ and $\theta_2 - \theta_1 \leqq \pi$. Then Rf is weakly monotonic on $[\theta_1,\theta_2]$. Further, X maps $\{(u,0) \mid u(\theta_1) \leqq u \leqq u(\theta_2)\}$ strictly monotonically into the z-axis.

3. BOUNDARY BEHAVIOR.

DEFINITION. We will say condition * holds (for $\phi \in C^*(\partial\Omega)$) if $Rf(\theta) \equiv Rf(\theta,\phi)$ lies between $\phi(N-)$ and $\phi(N+)$ whenever $\alpha < \theta < \beta$.

REMARK. If $\beta - \alpha \leqq \pi$, it follows from Lemma 1 or from standard barrier arguments that * holds for all $\phi \in C^*(\partial\Omega)$.

THEOREM 1. Suppose * holds. Then

(i) X is strictly monotonic on $\partial''B$, Rf is weakly monotonic on $[\alpha,\beta]$, S has no branch points in E, Rf is constant on $[\alpha,\alpha']$ and $[\beta',\beta]$, and Rf is strictly monotonic on $[\alpha',\beta']$, for some $\alpha',\beta' \in [\alpha,\beta]$ with $\alpha' < \beta'$.

Suppose * does not hold. Then

(ii) X has one branch point, $(u(0),0)$, in E, $z(\cdot,0)$ is strictly increasing (decreasing) on $[-1,u(0)]$ and strictly decreasing (increasing) on $[u(0),1]$, Rf is constant on $[\alpha,\alpha']$, $[\theta_L,\theta_R]$, and $[\beta',\beta]$, Rf is strictly increasing (decreasing) on $[\alpha',\theta_L]$ and Rf is strictly decreasing (increasing) on $[\theta_R,\beta']$, for some α',β',θ_L, $\theta_R \in [\alpha,\beta]$ with $\alpha' < \theta_L$ and $\theta_L + \pi \leqq \theta_R < \beta'$.

PROOF. From Lemma 1, we see that * holds iff Rf is weakly monotonic on $[\alpha,\beta]$ and if * fails to hold, then Rf is weakly monotonic on $[\alpha,\alpha+\pi]$ and on $[\beta-\pi,\beta]$. From [3] we know that X is strictly monotonic on a subset of ∂B iff it is weakly monotonic there. Since $X(u(\theta),0) = (0,0,Rf(\theta))$, X has at most one branch point in

E, which can only occur at (u(0),0) ([4]). Using Proposition 2 and the subsequent remark, we see either that one of the conclusions of Theorem 1 holds or that X is monotonic on $\partial''B$ and has a branch point at (u(0),0). We will eliminate this possibility.

In the case to be eliminated, Rf is weakly monotonic (say increasing) on $[\alpha,\beta]$, strictly increasing on $[\alpha',\theta_L]$, constant on $[\theta_L,\theta_R]$, and strictly increasing on $[\theta_R,\beta']$, for some $\alpha \leqq \alpha' < \theta_L < \theta_L + \pi \leqq \theta_R < \beta' \leqq \beta$. We may rotate the x-y plane so that $\theta_R = 0$ and (by a conformal map of B into B fixing (-1,0) and (1,0)) we may assume that u(0) = 0. As in [5], there exist neighborhoods U and U′ of 0 in E and a C^{-1}-diffeomorphism $F: U' \to U$ with $DF(0) = e \cdot id$ for some $0 \neq e \in \mathbb{R}$ such that

$$(z + ix)(w) = (F(w))^m$$

$$y(w) = \mathrm{Im}(A(F(w))^n) + o(|w|^n)$$

for all $w \in U'$, where $0 \neq A = a + ib$ and $n > m > 1$ are integers. Suppose we set $\omega = s + it = F(w)$ and $\mathcal{X} = x \circ F^{-1}$, $\mathcal{Y} = y \circ F^{-1}$, $\mathcal{Z} = z \circ F^{-1}$. Then $(\mathcal{Z} + i\mathcal{X})(\omega) = \omega^m$ for $\omega \in U$. Let γ be the image of the real axis under F. Then γ is tangent to the real axis at the origin and, since x(w) = 0 for w real, $\mathcal{X}(\omega) = 0$ for $\omega \in \gamma$. If $\omega = re^{i\delta}$, then $\mathcal{X}(r,\delta) = r^m \sin(m\delta)$ and the only curves on which $\mathcal{X}$ vanishes are $\delta = k\pi/m$ for all integers k. Thus γ must be the real axis in U. Since y(w) = 0 for w real, $\mathcal{Y}(\omega) = 0$ for ω real. This means that b = 0 and $\mathcal{Y}(\omega) = a\mathrm{Im}(\omega^n) + o(|\omega|^n)$. If σ is a curve in U from $(r,\delta) = (\varepsilon,0)$ to $(r,\delta) = (\varepsilon,\pi)$ (ε small) such that $(\mathcal{X}(\sigma), \mathcal{Y}(\sigma))$ is star-shaped with respect to the origin, then the sign pattern of $\mathcal{X}(\sigma)$ is +,- and $\mathcal{Y}(\sigma)$ is +,-,+. Thus m must be 2, n must be 3, $\mathcal{Z}(s,o) = s^2$, and so $Rf(\theta) = \mathcal{Z}(F(u(\theta)))$ cannot be monotonic on (α,β). Q.E.D.

In [1], the case $\phi \in C^0(\partial\Omega)$ and $\beta - \alpha > \pi$ is considered and the conjecture that $\theta_R - \theta_L = \pi$ is mentioned. The following theorem proves that this is always true.

THEOREM 2. In case (ii) of Theorem 1, $\theta_R - \theta_L = \pi$.

PROOF. If Q is an interior branch point of X, then there is a unique unit vector n(Q) such that as $P \in E$ approaches Q, the unit normal n(P) to X(E) at P approaches n(Q) ([6]). Since

$$X_u(u(\theta),0) = (0,0,z_u(u(\theta),0) \quad \text{and}$$

$$X_v(u(\theta),0) = |z_u(u(\theta),0)|(\cos(\theta), \sin(\theta),0),$$

we see that $n(\theta) = n(u(\theta),0) = \pm(\sin(\theta), -\cos(\theta),0)$ when $\alpha' < \theta < \theta_L$ or $\theta_R < \theta < \beta'$. If we let $\theta \to \theta_L-$, we get $n(Q) = \pm(\sin(\theta_L), -\cos(\theta_L),0)$ and if we let $\theta \to \theta_R+$, we get $n(Q) = \pm(\sin(\theta_R), -\cos(\theta_R),0)$ where Q = (u(0),0). Thus $\theta_R = \theta_L + \pi$. Q.E.D.

A question of interest is to determine the asymptotic behavior of $Rf(\theta)$ for $\theta > \theta_R$ near θ_R. A discussion of the asymptotic behavior of $Rf(\theta)$ for $\theta < \theta_L$ near θ_L is similar. We may assume that Rf is increasing on $[\theta_R,\beta']$.

As in the proof of Theorem 1, let us assume that $\theta_R = 0$ and u(0) = 0; then $Rf(\theta) = \mathcal{Z}(F(u(\theta)))$ and $\mathcal{Z}(s,0) = s^2$. Since $\mathcal{Z}(\omega) + i\mathcal{X}(\omega) = \omega^2$, $\omega = (\mathcal{Z} + i\mathcal{X})^{1/2}$ and $\mathcal{Y} = a\mathrm{Im}((\mathcal{Z} + i\mathcal{X})^{3/2}) + o(|\mathcal{Z}|+ i\mathcal{X}|^{3/2})$. Thus

$$y_\chi = 3a\ \mathrm{Re}((z + i\chi)^{1/2})/2 + o(|z + i\chi|^{1/2}).$$

When $\chi = 0$, we get

$$y_\chi(z) = 3/2\ a\ z^{1/2}/2 + o(|z|^{1/2}).$$

Next, if $0 = \theta_R < \theta < \beta'$, then $Rf(\theta)$ is equal to that value of $z > o$ for which $y_\chi(z) = \tan(\theta)$. For this value of z,

$$z + o(|z|) = (2\tan(\theta)/3a^2)$$

and so asymptotically as $\theta \to 0+$,

$$Rf(\theta) = (2/3a)^2\ \theta^2.$$

We wish to examine the behavior of $Rf(\theta)$ near $\theta = \alpha$ and $\theta = \beta$.

THEOREM 3. Let $\phi \in C^*(\partial\Omega)$ and let $f \in BV(\Omega)$ minimize $J(\cdot,\phi)$ over $BV(\Omega)$. Suppose that Γ^+ (Γ^-) is a C^1 curve in a neighborhood of $(N,\phi(N+))$ $((N,\phi(N-)))$ which meets the z-axis nontangentially. Suppose further that the unit normal to the graph of f extends continuously to the corner formed by Γ^+ (Γ^-) and the z-axis. Then $\beta' = \beta$ or $\beta' = \beta - \pi$ ($\alpha' = \alpha$ or $\alpha' = \alpha + \pi$).

PROOF. The proof is essentially the same as that of Theorem 2. We will prove $\beta' = \beta$ or $\beta' = \beta - \pi$. Let $\theta < \beta'$ approach β'; then $n(\theta)$ approaches $n(\beta') = \pm(\sin(\beta'), -\cos(\beta'), 0)$. Since the normal to the corner is $\pm(\sin(\beta), -\cos(\beta), 0)$, we see that $\beta' = \beta$ or $\beta' = \beta - \pi$. Q.E.D.

REMARK. If $\Gamma^+(\Gamma^-)$ is a line segment in a neighborhood of $(N,\phi(N+))$ $(N,\phi(N-)))$ which meets the z-axis nontangentially, then [7] (also [9]) implies that the hypotheses of Theorem 3 are satisfied.

Let us say that a "fan" exists at θ_0 when $Rf(\theta)$ is constant on a nontrivial interval containing θ_0. Since $\beta - \alpha < 2\pi$, we get

COROLLARY. Suppose that the hypotheses of Theorem 3 are satisfied for Γ^+ and Γ^-. Then no more than one "fan" can occur.

4. EXAMPLES.

EXAMPLE 1. (the helicoid). Consider the functions $f(x,y)$ over $\Omega = \{(r,\theta) | \alpha < \theta < \beta,\ 0 < r < 1\}$ with $-\pi < \alpha < \beta < \pi$ whose graph is given parametrically by

$$Y(s,t) = (t\cos(s),\ t\sin(s), s).$$

Then $\phi = f \in C^*(\partial\Omega)$, $Rf(\theta) = \theta$, and $\Gamma^\pm$ meet the z-axis at right angles. Here we see that Rf is strictly increasing, $\alpha' = \alpha$, and $\beta' = \beta$.

EXAMPLE 2. (Scherk's surface). Consider

$$f(x,y) = \ln(\sin(y)) - \ln(\sin(x))$$

over $\Omega = \{(r,\theta) |\ 0 < r < 1,\ \alpha < \theta < \beta\}$, where $0 < \alpha < \beta < \pi/2$. Then $Rf(\theta) = \ln(\tan(\theta))$ and $\Gamma^\pm$ meet the z-axis at right angles. Notice Rf is strictly increasing on $[\alpha,\beta]$, $\alpha' = \alpha$, and $\beta' = \beta$.

EXAMPLE 3. Here we have an example in which Ω is convex and $\alpha' \neq \alpha$. Let $\Omega' = \{(r,\theta) | -3\pi/4 < \theta < 3\pi/4,\ 0 < r < 1\}$, $\phi \in C^0(\partial\Omega')$ be zero on $r = 1$, $-3\pi/4 \leqq \theta \leqq 3\pi/4$ and $\theta = 1 - r$ on $\theta = \pm 3\pi/4$, $0 \leqq r \leqq 1$, and $f \in C^2(\Omega') \cap C^0(\bar{\Omega}')/\{N\})$ be the variational solution of the Dirichlet problem (for the minimal surface equation) in Ω' with boundary data ϕ. Next let $0 < \varepsilon < \pi/4$ and define $\Omega = \{(r,\theta) | \varepsilon - \pi/2 < \theta < \varepsilon + \pi/2,\ 0 < r < 1\}$. If we set $\phi = f$ on $\partial\Omega$, then $\phi \in C^*(\partial\Omega)$ and f minimizes J. Notice $\alpha = -\pi/2 + \varepsilon$, $\beta = \pi/2 + \varepsilon$, $\alpha' = \pi/2$, and $\beta' = \beta$. Also Γ^- meets the z-axis tangentially.

EXAMPLE 4. (See the discussion of this example in [8].) Let $\xi \in (\pi/2, \pi)$. Set $A = (0,0,1)$, $B = (\sin(\xi),0, \cos(\xi))$, $C = (\sin(2\xi), 0,\cos(2\xi))$, $D = (0,1,0)$, $E = (0,-1,0)$ and $M = (0,0,0)$. Consider the quadrilateral Q1 with successive vertices B,D,C,M and let S1 be the surface of least area spanning Q1. Since Q1 has a convex injective projection on the x-y plane, S1 is the graph of a function $g(x,y)$ over the x-y plane. Now extend S1 by reflection across the line segment BM to a surface S; the boundary of S is the polygon Γ with successive vertices A,E,B,D,C,M. Let Ω be the open subset of the x-y plane bounded by the projection of Γ on the x-y plane; notice $\alpha = -\pi$ and $\beta = \pi/2$. Using Theorem 1, we see that $S_0 = S/\Gamma$ is the graph of a function $f(x,y)$ over Ω. Notice $Rf(\theta)$ is 0 if $-\pi \leq \theta \leq 0$, $Rf(\cdot)$ is increasing on $[0,\pi/2]$ (by Theorem 1 (i) and the Corollary to Theorem 3), and Γ^- makes an angle of $2(\pi-\varepsilon)$ with the positive z-axis.

This last part shows that for any angle $\delta \in (0,\pi)$, we can set $\xi = \pi-\delta/2$ and find an example in which $\alpha' = \alpha + \pi$, $Rf(\theta)$ is (weakly) increasing on $[\alpha,\beta]$, and Γ^- intersects the positive z-axis in an angle of δ.

REMARK. In [2], the behavior of a (nonparametric) solution of an equation of prescribed mean curvature with prescribed boundary values in a domain with a reentrant corner is examined. The results of [2] can be extended to the case in which ϕ has a jump discontinuity. In fact, by combining the work in [2] with the techniques used above, Theorems 1, 2, and 3 and the Corollary can be proven in this new situation.

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