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MAXIMAL SUBALGEBRA OF DOUGLAS ALGEBRA

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ABSTRACT. When q is an interpolating Blaschke product, we find necessary and sufficient conditions for a subalgebra B of $H^{\infty}[\bar{q}]$ to be a maximal subalgebra in terms of the nonanalytic points of the noninvertible interpolating Blaschke products in B. If the set $M(B) \cap Z(q)$ is not open in $Z(q)$, we also find a condition that guarantees the existence of a factor q_0 of q in H^{∞} such that B is maximal in $H^{\infty}[\bar{q}]$. We also give conditions that show when two arbitrary Douglas algebras A and B, with $A \subseteq B$ have property that A is maximal in B.

KEY WORDS AND PHRASES. Maximal subalgebra, Douglas algebra, interpolating sequence, sparse sequence, Blaschke product, inner functions, open and closed subset, nonanalytic points, support set, Q-C level sets.
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1. INTRODUCTION.

Let D be the open unit disk in the complex plane and T be its boundary. Let L^{∞} be the space of essentially measurable functions on T with respect to the Lebesgue measure. By H^{∞} we mean the family of all bounded analytic functions in D. Via identification with boundary functions, H^{∞} can be considered as a uniformly closed subalgebra of L^{∞}. A uniformly closed subalgebra B between H^{∞} and L^{∞} is called a Douglas algebra. If we let C be the family of continuous functions on T, then it is well known that $H^{\infty}+C$ is the smallest Douglas algebra containing H^{∞} properly. For any Douglas algebra B, we denote by $M(B)$ the space of nonzero multiplicative linear functionals on B, that is, the set of all maximal ideals in B. An algebra B_0 is said to be a maximal subalgebra of B, if B_1 is another algebra with the property that $B_0 \subseteq B_1 \subseteq B$, then either $B_1 = B_0$ or $B_1 = B$.

An interpolating sequence $\{z_n\}_{n=1}^{\infty}$ is a sequence in D with the property that for any bounded sequence of complex numbers $\{\lambda_n\}_{n=1}^{\infty}$, there exists f in H^{∞} such that $f(z_n) = \lambda_n$ for all n. A well-known condition states that a sequence $\{z_n\}_{n=1}^{\infty}$ is interpolating if and only if

$$\inf_{n} \prod_{n \neq m}^{\infty} \left| \frac{z_m - z_n}{1 - \bar{z}_n z_m} \right| = \delta > 0. \tag{1.1}$$

A Blaschke product

$$q(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \left(\frac{z - z_n}{1 - \bar{z}_n z} \right) \tag{1.2}$$

is called an interpolating Blaschke product if its zero set $\{z_n\}_{n=1}^{\infty}$ is an interpolating sequence ($|z_n|/z_n \equiv 1$ is understood whenever $z_n = 0$). A sequence $\{z_n\}_{n=1}^{\infty}$ is said to be sparse if it is an interpolating sequence and

$$\lim_{n\to\infty} \prod_{n\neq m} \left| \frac{z_m - z_n}{1 - z_n z_m} \right| = 1. \tag{1.3}$$

For a function q in $H^{\infty}+C$, we let $Z(q) = \{m \in M(H^{\infty}+C) : q(m) = 0\}$ be the zero set of ϕ in $M(H^{\infty}+C)$. An inner function is a function in H^{∞} of modulus 1 almost everywhere on T. We denote by $H^{\infty}[\bar{b}]$ the Douglas algebra generated by H^{∞} and the complex conjugate of the inner function b.

We put $X = M(L^{\infty})$. Then X is the Shilow boundary for every Douglas algebra. For a point in $M(H^{\infty})$, we denote by μ_x the representing measure on X for x and by supp μ_x the support set for μ_x. For a function q in L^{∞} (in particular if q is an interpolating Blaschke product), we put $N(\bar{q})$ the closure of the union set of supp μ_x such that $x \in M(H^{\infty}+C)$ and $\bar{q}|_{\text{supp}\,\mu_x} \notin H^{\infty}|_{\text{supp}\,\mu_x}$. Roughly speaking, $N(\bar{q})$ is the set of nonanalytic points of q. Set $QC = H^{\infty} + C \cap \overline{H^{\infty} + C}$ and for x_0 in X, let $Q_{x_0} = \{x \in X: f(x)=f(x_0)$ for $f \in QC\}$. Q_{x_0} is called the QC-level set for x_0 [9]. For an inner function q, K. Izuchi has shown the following [5, Theorem 1(i)].

THEOREM 1. If q is an inner function that is not a finite Blaschke product, then,

$$N(\bar{q}) = \cup \{Q_x;\ x \quad Z(q)\}. \tag{1.4}$$

In particular, the right side of 1.4 is a closed set. Now assume that q is an interpolating Blaschke product, and let B be a Douglas algebra contained in $H^{\infty}[\bar{q}]$. We will always assume that $M(B) \cap Z(q)$ is not an open set in Z(q), for Izuchi has shown [6] that if B is a maximal subalgebra of $H^{\infty}[\bar{q}]$, then $M(B) \cap Z(q)$ is not open in Z(q). We will give answers to the following two questions. When is B a maximal subalgebra of $H^{\infty}[\bar{q}]$ or when is there a factor q_0 of q in H^{∞} such that B is maximal in $H^{\infty}[\bar{q}_0]$? These answers will be in terms of the nonanalytic points of q and the invertible inner functions of $H^{\infty}[\bar{q}]$ that are not invertible in B.

For a Douglas algebra B, we denote by N(B) the closure of $\cup\{\text{supp}\,\mu_x;\ x \in M(H^{\infty}+C)/M(B)\}$. In particular $N(H^{\infty}[\bar{q}]) = N(\bar{q})$. In general if A and B are Douglas algebras such that

$A \subseteq B$, we put $N_A(B)$ = the closure of $\cup\{\text{supp } \mu_x : x \in M(A)/M(B)\}$ and for any inner function b, $N_A(b)$ = the closure of $\cup\{\text{supp } \mu_x : x \in M(A), |b(x)| < 1\}$.

It is shown in [7, Corollary 2.5] that if $B \subseteq H^\infty[\bar{q}]$, then $N(B) \subseteq N(\bar{q})$, and it is not hard to show that $N(\bar{q})/N(B) \supseteq N_B(\bar{q})$ (in a sense the set $N_B(\bar{q})$ is generated by the nonanalytic points $M(B)/M(H^\infty[\bar{q}]) \subseteq M(H^\infty + C)/M(H^\infty[\bar{q}])$).

2. OUR MAIN RESULT.

We'll need the following lemma. It shows how small $M(B)/M(H^\infty[\bar{q}])$ must be if B is to be a maximal subalgebra of $H^\infty[\bar{q}]$. Let $\Omega = \{b : b$ is an interpolating Blaschke product with $\bar{b} \in H^\infty[\bar{q}]\}$, and $\Omega(B) = \{b_0 \in \Omega : \bar{b}_0 \notin B\}$.

LEMMA 1. *Let q be an interpolating Blaschke product and B be a Douglas algebra contained in $H^\infty[\bar{q}]$. Suppose for all $b_0 \in \Omega(B)$, we have that $N_B(\bar{q}) \subseteq N_B(\bar{b}_0)$. Then B is a maximal subalgebra of $H^\infty[\bar{q}]$.*

PROOF. It suffices to show that if $b \in \Omega(B)$, then $B[\bar{b}] = H^\infty[\bar{q}]$. Hence the only Douglas algebra between B and $H^\infty[\bar{q}]$ that contains B properly is $H^\infty[\bar{q}]$. It is clear that $M(H^\infty[\bar{q}]) \subseteq M(B[\bar{b}])$. We show that $M(B[\bar{b}]) \subseteq M(H^\infty[\bar{q}])$. Now $M(B[\bar{b}]) = \{m \in M(B) : |b(m)|=1\}$. It suffices to show that if $m \notin M(H^\infty[\bar{q}])$, then $m \notin M(B[\bar{b}])$. Let $m \in M(B)$ such that $m \notin M(H^\infty[\bar{q}])$. Then $\bar{q}|_{\text{supp } \mu_m} \notin H^\infty|_{\text{supp } \mu_m}$ and since $N_B(\bar{q}) \subseteq N_B(\bar{b})$, we have that $\bar{b}|_{\text{supp } \mu_m} \notin H^\infty|_{\text{supp } \mu_m}$. Thus $|b(m)| < 1$ and we get $m \notin M(B[\bar{b}])$. This shows that $M(B[\bar{b}]) \subseteq M(H^\infty[\bar{q}])$, and B is maximal in $H^\infty[\bar{q}]$.

Using Theorem 1 above, it is not hard to show directly that $N(B[\bar{b}]) = N(\bar{q})$. However, by Proposition 4.1 of [7], this condition is not sufficient.

We let $E = N_B(\bar{q})$. This can be a very complicated set. For example, it can contain $\text{supp } \mu_x$ where x belongs to a trivial Gleason part or a Gleason part where $|q| < 1$, but yet $q \neq 0$ on this part [see 3]. So for B to be maximal in $H^\infty[\bar{q}]$, E must be as simple as possible. To see how simple, we set $\Lambda(B) = \{b \in \Omega(B) : B \subseteq H^\infty[\bar{b}]\}$ and $\Lambda^*(B) = \{a \in \Omega(B) : a \notin \Lambda(B)\}$. Now let $E^* = \bigcap_{b \in \Lambda(B)} N(\bar{b})$, $E^{**} = \bigcap_{b_0 \in \Omega(B)} N(\bar{b}_0)$, $E_0^* = E^* \cap E$ and $E_0^{**} = E^{**} \cap E$. Note that if $E_0^{**} = \phi$, then there are interpolating Blaschke products a_0 and a_1 in $\Lambda^*(B)$ such that $N_B(\bar{q}) \cap N(\bar{a}_0) \cap N(\bar{a}_1) = \phi$. Thus we get $B \subset B[\bar{a}_0] \subset H^\infty[\bar{q}]$. To see this, just note that both $N_B(\bar{q}) \cap N(\bar{a}_0) \neq \phi$ and $N_B(\bar{q}) \cap N(\bar{a}_1) \neq \phi$ since a_0 and a_1 belong to $\Lambda^*(B)$. Since their intersection is empty, there is an $x_1 \in M(B)$ such that $\bar{a}_0|_{\text{supp } \mu_{x_1}} \in H^\infty|_{\text{supp } \mu_{x_1}}$. Thus $N_{B[\bar{a}_0]}(\bar{q}) \subset N(\bar{q})$, which implies that $B[\bar{a}_0] \subset H^\infty[\bar{q}]$. Obviously, $B \subset B[\bar{a}_0]$, so B cannot be maximal in $H^\infty[\bar{q}]$ unless $E_0^{**} \neq \phi$. We now state.

PROPOSITION 1. *Let B be a Douglas algebra properly contained in $H^\infty[\bar{q}]$, and suppose $E_0^{**} \neq \phi$. Then the following statements are equivalent:*

(i) $N(B) = N(\bar{q})$;

(ii) *B is a maximal subalgebra of $H^\infty[\bar{q}]$;*

(iii) $E_0^{**} = E_0^* = E$;

(iv) $E_0^{**} = N_B(\bar{q})$.

PROOF. We prove the following: (i) → (ii) → (iii) → (iv) → (ii) → (i). Suppose (i) holds. We will show that $N_B(\bar{q}) \subseteq N_B(\bar{b})$ for all $b \in \Omega(B)$. Using Lemma 1, this will prove that B is a maximal subalgebra of $H^\infty[\bar{q}]$. Let $b \in \Omega(B)$ and consider the Douglas algebra $B[\bar{b}]$. We have $B \subseteq B[\bar{b}] \subseteq H^\infty[\bar{q}]$, hence $N(B) = N(B[\bar{b}]) = N(\bar{q})$. Now $N(\bar{q}) = N(B) \cup N_B(\bar{q})$, so by the above equality we have that $N_B(\bar{q}) \subseteq N(B[\bar{b}])$. Thus, if $x \in M(B)$ such that $\bar{q}|_{\text{supp}\,\mu_x} \notin H^\infty|_{\text{supp}\,\mu_x}$ implies that $\text{supp}\,\mu_x \subseteq N(B[\bar{b}])$. Thus $\bar{b}|_{\text{supp}\,\mu} \notin H^\infty|_{\text{supp}\,\mu_x}$ and $N_B(\bar{q}) \subseteq N_B(\bar{b})$. We have (i) → (ii).

Next suppose that (ii) holds. It is clear that $E_0^{**} \subseteq E_0^* \subseteq E$. We must show that $E_0^*|E_0^{**}$ and $E|E_0^*$ are empty sets. First we show that $E_0^*|E_0^{**}$ is empty. Suppose not. Then there is an $x \in M(B)$ and a $b_0 \in \Lambda^*(B)$ such that $\bar{b}_0|_{\text{supp}\,\mu_x} \in H^\infty|_{\text{supp}\,\mu_x}$ and $\text{supp}\,\mu_x \subseteq E_0^*$. It is clear by Theorem 1 that $\text{supp}\,\mu_x \cap N(\bar{b}_0) = \phi$. Consider the algebra $B[\bar{b}_0]$. Since $b_0 \in \Lambda^*(B)$, $E \subseteq B[\bar{b}_0]$. Since $\text{supp}\,\mu_x \subseteq N(\bar{q})$ and $\text{supp}\,\mu_x \not\subseteq N(\bar{b}_0)$, we have that $|b_0(x)| = 1$, so we have $\text{supp}\,\mu_x \subseteq N(\bar{q})/N_{B[\bar{b}_0]}(\bar{q})$. This implies that $B[\bar{b}_0] \subseteq H^\infty[\bar{q}]$, which is a contradiction. So $E_0^{**} = E_0^*$.

Now we show that E/E_0^* is empty. Again suppose not. Hence there is a $y \in M(B)$ such that $\text{supp}\,\mu_y \subseteq E$, but $\text{supp}\,\mu_y \not\subseteq E_0^*$. There is a $b \in \Lambda(b)$ such that $\text{supp}\,\mu_y \subseteq N(\bar{b})$. Again this implies that $\bar{b}|_{\text{supp}\,\mu_y} \in H^\infty|_{\text{supp}\,\mu_y}$. Thus we have that $B \subsetneq B[\bar{b}]$ (since $b \in \Lambda(B)$) and $B[\bar{b}] \subsetneq H^\infty[\bar{q}]$ (since $\text{supp}\,\mu_y \subseteq N(\bar{q})/N_{B[\bar{b}]}(\bar{q})$), which is a contradiction. So we get $E_0^* = E$. This shows that (ii) → (iii).

It is trivial that if (iii) holds, $E_0^{**} = N_B(\bar{q})$.

If (iv) holds and b is any interpolating Blaschke product in $\Omega(B)$, then by (iv) $N_B(\bar{q}) \subseteq N_B(\bar{b})$ so by Lemma 1, B is a maximal subalgebra of $H^\infty[\bar{q}]$.

Finally, suppose (ii) holds. We are going to show that $N(B) = N(\bar{q})$. Suppose not. Then $N(B) \subseteq N(\bar{q})$. By Theorem 1 there is a Q-C level set Q with $N(B) \cap Q = \phi$. Put $B_0 = [H^\infty, \bar{I}$; I is an interpolating Blaschke product with $\bar{I} \in H^\infty[\bar{q}]$ and $\bar{I}|_Q \in H^\infty|_Q]$. By Proposition 4.1 of [7], we have $B_0 \subsetneq H^\infty[\bar{q}]$ and $N(B_0) = N(\bar{q})$. Since $N(B) \cap Q = \phi$, we also have $B \subseteq B_0$ (because $N(B) \subseteq N(B_0)$). This implies that B is not a maximal subalgebra of $H^\infty[\bar{q}]$, which is a contradiction. Thus $N(B) = N(\bar{q})$.

Now suppose we have that $E_0^{**} \subseteq E_0^* \subseteq E$ ($E_0^{**} = \phi$ is possible). When is there a factor q_0 of q in H^∞ such that B is a maximal subalgebra of $H^\infty[\bar{q}_0]$ ($B = H^\infty[\bar{q}_0]$ is not possible)? To answer this question, let $\Omega_0 = \{q_0 : q\bar{q}_0 \in H^\infty\}$, and $\Omega_0(B) = \{q_0 \in \Omega_0 : B \subseteq H^\infty[\bar{q}_0]\}$.

Set $F = \bigcap_{q_0 \in \Omega_0(B)} N(\bar{q}_0)$. Suppose $F = N(\bar{q}_0)$ for some factor q_0 of q in H^∞. Then $B \subseteq H^\infty[\bar{q}_0]$. So q_0 is our possible candidate. Next, let Ω_{q_0} = {c:c is an interpolating Blaschke product with $\bar{c} \in H^\infty[\bar{q}_0]$},

$\Omega_{q_0}(B) = \Omega_{q_0} \cap \Omega(B)$, $\Lambda_{q_0}(B) = \Omega_{q_0}(B) \cap \Lambda(B)$, $\Lambda^*_{q_0}(B) = \Omega_{q_0}(B) \cap \Lambda^*(B)$,

$F_0 = E \cap N(\bar{q}_0)$, $F^* = E^*_0 \cap F$, $F^{**} = \bigcap_{c \in \Omega_{q_0}(B)} N(\bar{c})$, $F^*_0 = F^* \cap F_0$, and finally

$F^{**}_0 = F^{**} \cap F_0$.

We have the following.

COROLLARY 1. Let q_0 be a factor of q in H^∞ such that $F = N(\bar{q}_0)$ and assume $F^{**}_0 \neq \phi$. If any of the following conditions hold:

(i) $F_0 = F^*_0 = F^{**}_0$

(ii) $F^{**}_0 = N_B(H_0)$, where $H_0 = \bigcap_{q_0 \varepsilon \Omega(B)} H^\infty[\bar{q}_0]$.

Then B is a maximal subalgebra of $H_0 = H^\infty[\bar{q}_0]$ where $q_0 \varepsilon \Omega_0(B)$.

The fact that $F = N(q_0)$ for some $q_0 \varepsilon \Omega_0(B)$ implies that $H_0 = H^\infty[\bar{q}_0]$ and our corollary follows from Proposition 1.

We now consider this question for the genral Douglas algebras. Let A and B be Douglas algebras such that $A \subseteq B$ and there is an inner function q with $B \subseteq A[\bar{q}]$. When this occurs we say that A is near B. It is well known that if $B = L^\infty$ and A is any Douglas algebra properly contained in B, then A is not near B, that is, $B \subseteq A[\bar{q}]$ for any inner function q. In fact L^∞ is not countably generated over any Douglas algebra A [10]. So by the results of C. Sundberg [10] any Douglas algebra B which is countably generated over A is also near it.

The following result comes from [2, Lemma 5] and gives equivalent conditions for two Douglas algebras to be near each other [see 11, Theorem 1 for a similar result].

THEOREM 2. Let A and B be Douglas algebras with $H^\infty + C \subsetneq A \subseteq B$ and q be an inner function. Then the following statements are equivalent.

(i) $M(A) = Z_A(q) \cup M(B)$

(ii) $\phi B \subseteq A$.

where $Z_A(q) = Z(q) \cap M(A)$.

PROOF. Assume (i) holds; we show that $\phi B \subseteq A$. Let b be any interpolating Blaschke product for which $\bar{b}$ is in B. If x is in $Z_A(b)$, we show that x is also in $Z_A(q)$. Now x is in M(A) and $b(x) = 0$ implies that x is not in M(B), since $\bar{b}$ is in B. So by (i) we have that x must be in $Z_A(q)$. Thus $Z_A(b) \subseteq Z_A(q)$, and by Theorem 1 of [4] we have $\bar{b}$ is in A. Now let f be any function in B. By the Chang Marshall Theorem [1,8] there is a sequence of functions $\{h_n\}$ in H^∞ and a sequence of interpolating Blaschke products $\{b_n\}$ with $\bar{b}_n \in B$ for all n, such that $h_n \bar{b}_n \to f$. But $h_n(\bar{b}_n) \to f$ belongs to A since $\bar{b}_n$ is in A for all n. This proves (ii).

Assume (ii) holds. Let x be in M(A) but not in M(B). Then there is an inner function b which is invertible in B such that $|b(x)| < 1$. For any positive integer n, the function $f_n = q\bar{b}^n$ is in A, so

$$|g(x)| = |b(x)|^n |f_n(x)| \leq |b(x)|^n.$$

Letting $n \to \infty$ we get $(x) = 0$. This proves (i).

Set $Z_B(q) = M(B) \cap Z_A(q)$ and $Z_B^*(q) = Z_A(q)/Z_B(q)$; then $M(A)/M(B) = \bigcup_{x \in Z_B^*(q)} P_x$, since $M(A) = M(B) \cup Z_A(q)$.

As we have previously done, let $\Omega(B,A)$ be the set of interpolating Blaschke products b such that $b \in B$ but $b \notin A$ and set $W^* = \bigcap_{b \in \Omega(B,A)} N_A(\bar{b})$. We assume $W^* \neq \phi$.

Using Proposition 1, Theorem 2 and Lemma 1, we have the following result.

PROPOSITION 2. <u>Let</u> A <u>and</u> B <u>be arbitrary Douglas algebras such that</u> A <u>is near</u> B. <u>Then the following statements are equivalent</u>:

(i) $N_A(B) \subseteq N_A(\bar{b})$ <u>for all</u> $b \in \Omega(B,A)$;

(ii) A <u>is a maximal subalgebra of</u> B;

(iii) $W^* = N_A(B)$.

PROOF. Assume that (i) holds. Since A is near to B, there is an inner function such that $M(A) = M(B) \cup \{ \bigcup_{x \in Z_B^*(\phi)} P_x \}$. If we set $A^* = \bigcup_{x \in Z_B^*(\phi)} P_x$, then it is immediate that

$$N_A(B) = \text{closure of } \cup \{\text{supp } \mu_x : x \in A^*\}.$$

Let b be any alement in $\Omega(B,A)$. By (i) we have that $N_A(B) \subseteq N_A(\bar{b})$. As in proof of Lemma 1 we have that $A[\bar{b}] = B$. Thus is a maximal in B.

Assume that (ii) holds, and let $x \in A^*$. Since A is near B, we have that $M(A) = M(B) \cup A^*$. If $b \in \Omega(B,A)$, then by our hypothesis $A[\bar{b}] = B$, which implies that if $y \in M(A)$ and $|b(y)| = 1$, then $y \in M(B)$ (since $M(A[\bar{b}]) = \{g \in M(A) : |b(g)| = 1\} = M(B)$). So, if supp $\mu_x \subset N_A(B)$, the $\bar{b}|_{\text{supp}\mu_x} \notin H^\infty|_{\text{supp}\mu_x}$. Thus $N_A(B) \subseteq N_A(\bar{b})$ for all $b \in \Omega(B,A)$. This implies that $N_A(B) \subseteq W^*$

To show what $W^* \subseteq N_A(B)$, let $b \in \Omega(B,A)$. Hence $\bar{b} \in B$; therefore we have

$$\begin{aligned} N_A(\bar{b}) &= \text{closure of } \cup\{\text{supp } \mu_x;\ x \in M(A),\ |b(x)| < 1\} \\ &= \text{closure of } \cup\{\text{supp } \mu_x;\ x \in M(A)/M(B),\ |b(x) < 1\} \\ &\subseteq \text{closure of } \cup\{\text{supp } \mu_x;\ x \in M(A)/M(B)\} \\ &= N_A(B). \end{aligned}$$

Since this is true for any $b \in \Omega(B,A)$, we have $N_A(B) \supseteq W^*$. Thus $W^* = N_A(B)$ if A is maximal in B.

It is trivial that if (iii) holds, $N_A(B) \subseteq N_A(\bar{b})$ for all $b \in \Omega(B,A)$.
We are done.

In Proposition 4.1 of [7] Izuchi constructed a family of Douglas algebras B contained in $H^\infty[\bar{q}]$ with the property that $N(B) = N(\bar{q})$. By Proposition 1, we have that this family is a family of maximal subalgebras of $H^\infty[\bar{q}]$.

Finally we close this paper with the following question that I have been uable to answer.

QUESTION 1. Recall that if q is an interpolating Blaschke product, then $N(\bar{q}) = N(B) \cup N_B(\bar{q})$ for any Douglas algebra with $B \subseteq H^\infty[\bar{q}]$. Does there exist a Douglas algebra $B_o \subseteq H^\infty[\bar{q}]$ with $N_{B_o}(\bar{q}) = N(\bar{q})$?

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