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MULTIPLIERS ON WEIGHTED HARDY SPACES OVER CERTAIN TOTALLY DISCONNECTED GROUPS

TOSHIYUKI KITADA
Department of Mathematics
Faculty of General Education
Hirosaki University
Hirosaki 036, JAPAN

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ABSTRACT. In this note, we consider the multipliers on weighted H^1 spaces over totally disconnected locally compact abelian groups with a suitable sequence of open compact subgroups (Vilenkin groups). We first show an (H^1,L^1) multiplier result from which Onneweer's theorem follows. We also give an (H^1,H^1) multiplier result under a condition of Baernstein-Sawyer type.

KEY WORDS AND PHRASES. Totally disconnected groups, Weighted H^1 spaces, Weighted L^p spaces, Multipliers.
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1. INTROUDCTION.

Recently, Onneweer obtained a weighted L^p multiplier theorem [1, Theorem 1] over a Vilenkin group which is a generalization of Taibleson's theorem over a local field.

In this note, we show a weighted (H^1,L^1) multipler theorem under a weaker hypothesis than [1, Proposition 2], and show the Onneweer's theorem, by using an extended interpolation theorem for weighted H^1 and L^p spaces. We do not know whether this multiplier is also a weighted (H^1,H^1) multiplier. But we are able to show that a Baernstein-Sawyer type condition [2] which is stronger than Onneweer's, implies a weighted (H^1,H^1) result. This is also a generalization of Theorem 2 [2].

2. DEFINITIONS AND NOTATIONS.

Throughout this note, G will denote a locally compact abelian group with a sequence $\{G_n\}_{-\infty}^{\infty}$ such that

(i) each G_n is an open compact subgroup of G,

(ii) $G_{n+1} \subsetneqq G_n$ and order $(G_n/G_{n+1}) < \infty$,

(iii) $\bigcup_{\infty}^{\infty} G_n = G$ and $\bigcap_{\infty}^{\infty} G_n = \{0\}$.

Moreover we shall assume that G is order-bounded, i.e.;

$$B: = \sup \{\text{order } (G_n/G_{n+1});\ n \in Z\} < \infty.$$

Let Γ denote that dual group of G and for each $n \in Z$, let Γ_n denote the annihilator of G_n. Then we have

(i)' each Γ_n is an open compact subgroup of Γ,

(ii)' $\Gamma_n \subsetneqq \Gamma_{n+1}$ and order (Γ_{n+1}/Γ_n) = order (G_n/G_{n+1}),

(iii)' $\bigcup_{-\infty}^{\infty} \Gamma_n = \Gamma$ and $\bigcap_{\infty}^{\infty} \Gamma_n = \{1\}$.

We choose Haar measures μ on G and λ on Γ so that $\mu(G_0) = \lambda(\Gamma_0) = 1$, then $\mu(G_n) = (\lambda(\Gamma_n))^{-1}: = (m_n)^{-1}$ for each $n \in Z$. For an arbitrary set A we denote its indicator function by ξ_A. The symbols $\wedge$ and $\vee$ will be used to denote the Fourier and inverse Fourier transform respectively. It is easy to see that for each $n \in Z$ we have $(\xi_{G_n})^{\wedge} = (\lambda(\Gamma_n))^{-1}\xi_{\Gamma_n}$. We set $D_n := (\mu(G_n))^{-1}\xi_{G_n}$ for each $n \in Z$.

We now define the weighted L^p spaces. For $\alpha \in R$, we define the function v_α on G by $v_\alpha(x) = (m_n)^{-\alpha}$ if $x \in G_n \setminus G_{n+1}$ $(n \in Z)$; $= 0$ if $x = 0$. We denote the L^p spaces with respect to the measure $d\mu_\alpha := v_\alpha d\mu$ on G by $L^p_\alpha(G)$, simply L^p_α. Also for $1 \leq p < \infty$, we set

$$||f||_{p,\alpha} := (\int_G |f(x)|^p d\mu_\alpha)^{1/p}.$$

Let S(G) be the set of all functions ϕ on G such that ϕ has compact support and is constant on the cosets of some subgroup G_n (n depends on ϕ) of G. The functions in S(G) are called test functions on G. It is well known that if $\alpha > -1$, then S(G) is dense in L^p_α for $1 \leq p < \infty$.

In order to define the weighted Hardy spaces on G, we first define weighted atoms on G. Let $1 < q \leq \infty$. A function a(x) on G is a $(1,q)_\alpha$ atom if there exists an interval $I = I_n(x): = x + G_n$, $x \in G$, $n \in Z$ such that

(i) supp a is contained in I,

(ii) $\left(\frac{1}{\mu_\alpha(I)} \int_I |a(x)|^q d\mu_\alpha\right)^{1/q} \leq \mu_\alpha(I)^{-1}$, if $1 < q < \infty$

and $|a(x)| \leq \mu_\alpha(I)^{-1}$, if $q = \infty$,

(iii) $\int a(x)d\mu = 0$.

The weighted Hardy space $H^{1,q}_\alpha(G)$, simply $H^{1,q}_\alpha$, is the space of all functions f on on G such that $f(x) = \sum_0^\infty \lambda_k a_k(x)$,

where the a_k's are $(1,q)_\alpha$ atoms and $\Sigma_0^\infty|\lambda_k| < \infty$. We set $||f||_{H^{1,q}_\alpha}: = \inf \Sigma_0^\infty|\lambda_k|$, where the infimum is taken over all such decompositions. Then $H^{1,q}_\alpha$ is a subspace of L^1_α and a Banach space with the norm $||\cdot||_{H^{1,q}_\alpha}$. It also follows easily from the definition that

$$H^{1,\infty}_\alpha \subset H^{1,q_2}_\alpha \subset H^{1,q_1}_\alpha$$

whenever $1 < q_1 < q_2 < \infty$. We denote $H^{1,\infty}_\alpha$ by H^1_α. In the following section, we show that $H^{1,q}_\alpha = H^1_\alpha$ if $-1 < \alpha \leq 0$ and $1 < q < \infty$.

We say that $m \in L^\infty(\Gamma)$ is an (X,Y) multiplier (or a multiplier on X, when $X = Y$) if there exists a constant $C > 0$ so that

$$||(m\hat{\phi})^\vee||_Y \leq C||\phi||_X \qquad \text{for all } \phi \in X \cap S(G)$$

where X and Y are equal to H^1_α or L^p_α.

According to [1], we say that $\phi \in L^\infty(\Gamma)$ satisfies condition $C(k,r)$ for some $k \in Z$ and $r \in [1,\infty)$ if there exist C, $\varepsilon > 0$ so that for all ℓ, $n \in Z$ with $n < \ell$ have

$$\sup \{ (\int_{G_n \setminus G_{n+1}} |(\phi^k)^\vee(x-y) - (\phi^k)^\vee(x)|^r \, d\mu)^{1/r}; \ y \in G_\ell$$

$$\leq C \, (m_n)^{1/r' + \varepsilon}(m_\ell)^{-\varepsilon}, \text{ if } 1 < r < \infty,$$

and there exists $C > 0$ so that for all $\ell \in Z$ we have

$$\sup \{ \int_{G \setminus G_\ell} |(\phi^k)^\vee(x-y) - (\phi^k)^\vee(x)| d\mu; \ y \quad G_\ell \} \leq C, \text{ if } r = 1,$$

where $\phi^k = \phi \, \xi_{\Gamma_k}$ for each $k \in Z$ and r' denotes the conjugate exponent of r.

Let $-\infty < \alpha < \infty$, $1 \leq p < \infty$ and $0 < q \leq \infty$. A function f on G belongs to the Herz space $K^{\alpha,q}_p(G)$, simply $K^{\alpha,q}_p$, if

$$||f||_{K^{\alpha,q}_p} := (\sum_{-\infty}^{\infty} ||(m_n)^{-\alpha} f \, \xi_{G_n \setminus G_{n+1}} ||_p{}^q)^{1/q} < \infty,$$

with the usual modification if $q = \infty$[3].

We now state the main theorems:

THEOREM 1. Let $\phi \in L^\infty(\Gamma)$ and suppose that ϕ satisfies condition $C(k,r)$ for some $k \in Z$ and $r \in [1,\infty)$. Then ϕ^k is an (H^1_α, L^1_α) multiplier for $-1/r' < \alpha \leq 0$.

As a Corollary we obtain Theorem 1 of [1]:

COROLLARY. Let $\phi \in L^\infty(\Gamma)$. (i) Suppose that condition $C(k,r)$ holds for all $k \in Z$, for some $r \in (1,\infty)$, and with constants C and ε independent of $k \in Z$. If ϕ is a multiplier on $L^2_{\alpha_0}$ for some α_0 with $-1/r' < \alpha_0 < 1/r'$, then ϕ is a multiplier on L^p_α for all p, α such that $1 < p < \infty$ and $-|\alpha_0| \leq \alpha \leq (p-1)|\alpha_0|$.
(ii) If $C(k,1)$ holds for all $k \in Z$, and with C independent of $k \in Z$, then ϕ is a multiplier on L^p for $1 < p < \infty$.

THEOREM 2. Let $\phi \in L^\infty(\Gamma)$ and suppose that there exist $r \in [1,\infty)$ and $\varepsilon > 0$ such that

$$\|(\phi_j)^\vee\|_{K_r^{\varepsilon+1/r',\infty}} \le C\,(m_j)^{-\varepsilon} \text{ for all } j \in Z,$$

where $\phi_j := \phi\, \xi_{\Gamma_{j+1}\setminus\Gamma_j}$ for each $j \in Z$, then ϕ is a multiplier on H^1_α for $-1 < r'\alpha \le 0$.

3. PRELIMINARY RESULTS.

To prove Theorem 2, we need the "maximal function" characterization of H^1_α. For f locally in $L^1_\alpha(G)$ we define the maximal function $M_\alpha f$ of f by

$$M_\alpha f(x) := \sup_I \{\frac{1}{\mu_\alpha(I)} \int_I |f(y)|\, d\mu_\alpha(y)\},$$

where the I's are intervals containing x. When $\alpha = 0$, we denote M_α by M, simply.

LEMMA. Let $\alpha > -1$.

(a) $\mu_\alpha(x+G_n) \le C\,\mu_\alpha(x+G_{n+1})$ for all $x \in G$ and $n \in Z$,

(b) M_α is of weak-type (1.1) on L^1_α and is of type (p,p) on L^p_α for $1 < p < \infty$,

(c) If $\alpha \le 0$, then for all interval I

$$\mu_\alpha(I) \le C\,\mu(I) \inf\{v_\alpha(y);\ y \in I,\ y \neq 0\}.$$

(d) If $\alpha \le 0$, then M is of weak-type (1,1) on L^1_α.

PROOF. (a) and (c) are Lemmas 1(b) and (c) in [1]. (b) follows from (a). By (c), we have that $Mf(x) \le C\,M_\alpha f(x)$ for each $x \in G$. Then (d) follows from (b).

THEOREM A. Let $\alpha > -1$. An f L^1_α belongs to H^1_α if and only if $f^* := Mf$ L^1_α. Moreover $\|f\|_{H^1_\alpha}$ is equivalent to $\|f^*\|_{1,\alpha}$.

A slight modification of the argument in [2] establishes the result, so we omit the proof.

THEOREM B. Let $-1 < \alpha \le 0$. Then $H^{1,q}_\alpha \simeq H^1_\alpha$, for $1 < q < \infty$.

PROOF. We have already seen that H^1_α is continuously included in $H^{1,q}_\alpha$, for each $1 < q < \infty$. In order to establish the opposite inclusion, it suffices to show that a $(1,q)_\alpha$ atom a has the representation

$$a(x) = \sum_0^\infty \lambda_j a_j(x) \tag{3.1}$$

where each a_j is a $(1,\infty)_\alpha$ atom and $\sum_0^\infty |\lambda_j| \le C$, C independent of a. Like the non-weighted case, this can be done by using the Calderon-Zygmund decomposition [4], [5].

Let a be a $(1,q)_\alpha$ atom that is suppported on $I := x_0 + G_{n_0}$ $(x_0 \in G,\ n_0 \in Z)$. We let $b(x) := \mu_\alpha(I)a(x)$, then $\operatorname{supp} b \subset I$, $\int b(x)\,d\mu(x) = 0$, and $\|b\|^q_{q,\alpha} \le \mu_\alpha(I)$.

For $t > 0$ (we shall be explicit later), we denote the open set $\{x \in G: M_\alpha^q(b) > t\}: = \{x \in G; M_\alpha(|b|^q)(x) > t^q\}$ by U_t. We note that $U_t \subset I$ for $t > 1$. (This is easily seen from the fact that for any two intervals in G, they are disjoint or one contains the other). Lemma (b) implies that

$$\mu_\alpha(U_t) < C||b||_{q,\alpha}^q /t^q < C\mu_\alpha(I)/t^q \tag{3.2}$$

and $\mu_\alpha(G_k) \to \infty$ as $k \to -\infty$ [1, Lemma (a)]. Thus we have the decomposition

$U_t: = \bigcup_j I_j$; where the I_j's are maximal disjoint sub-intervals of U_t. The Calderon-Zygmund decomposition is now that $b(x) = g_0(x) + \sum_j h_j$, where $g_0(x) = b(x)$ if $x \notin U_t$; $= m(b,I_j)$ if $x \in I_j$ and $h_j(x) = (b(x) - g_0(x))\, \xi_{I_j}(x)$, and where $m(b,I_j)$ denotes the average of b over I_j with respect to μ. Then the maximality of the I_j's and Lemma (a), (b) imply that $|g_0(x)| < C_0 t$, μ_α-a.e. and

$$\frac{1}{\mu_\alpha(I_j)} \int_{I_j} |h_j| d\mu_\alpha < \left(\frac{1}{\mu_\alpha(I_j)} \int_{I_j} |h_j|^q d\mu_\alpha\right)^{1/q} < 2CC_0 t := C_1 t$$

by Lemma (c). If we set $(C_1t)^{-1}h_j = b_j$, then b_j is supported in I_j, $\int b_j \, d\mu = 0$ and $||b_j||_{q,\alpha}^q < \mu_\alpha(I_j)$ for each j.

The idea will be now to do for each b_j the same kind of decomposition that we performed for b (with the same t) and to build an induction process which will eventually lead to the decomposition (3.1). We shall use multi-indices for the successive decomposition, in the following way:

$$b(x) = g_0(x) + \sum_{j_0} h_{j_0}(x) = g_0(x) + C_1 t \sum_{j_0} b_{j_0}(x)$$

$$= g_0(x) + C_1 t \sum_{j_0} (g_{j_o}(x) + \sum_{j_1} h_{j_0,j_1}(x))$$

$$= g_0(x) + C_1 t \sum_{j_0} g_{j_0}(x) + C_1 t \sum_{j_0,j_1} h_{j_0,j_1}(x)$$

$$= g_0(x) + C_1 t \sum_{j_0} g_{j_0}(x) + \dots + (C_1t)^n \sum_{j_0,\dots,j_{n-1}} g_{j_0\dots j_{n-1}}(x)$$

$$+ (C_1t)^n \sum_{j_0,\dots,j_n} h_{j_0,\dots,j_n}(x) \tag{3.3}$$

for each $n \in N$, where, $b_{j_0,\dots,j_{n-1}} := (C_1t)^{-1} h_{j_0,\dots,j_{n-1}}$ and

(i) $\mu_\alpha(\{M_\alpha^q(b_{j_0,\dots,j_{n-1}}) > t\} < C\,\mu_\alpha(I_{j_0,\dots,j_{n-1}})/t^q$

(ii) $\{M^q_\alpha (b_{j_0,\dots,j_{n-1}}) > t\} = \bigcup_{j_n} I_{j_0,\dots,j_n}$

(iii) $\operatorname{supp} h_{j_0,\dots,j_n} \subset I_{j_0,\dots,j_n}$, $\int h_{j_0,\dots,j_n}\, d\mu = 0$

(iv) $\left(\frac{1}{\mu_\alpha(I_{j_0,\dots,j_n})} \int_{I_{j_0,\dots,j_n}} |h_{j_0,\dots,j_n}|^q\, d\mu_\alpha\right)^{1/q} \le C_1 t,$

(v) $|g_{j_0,\dots,j_{n-1}}(x)| \le C_0 t,$

for every $j_0,\dots,j_n$ and $n \in N$.
By using (i), (ii) and (iv), we see that the L^1_α-norm of the last term in the right hand side of (3.3) is bounded by $(Ct^{1-q})^{n+1}\, \mu_\alpha(I)$. Hence for large $t > 0$ so that $Ct^{1-q} < 1$, we have that

$$b(x) = g_0(x) + C_1 t \sum_{j_0} g_{j_0}(x) + \dots$$
$$+ (C_1 t)^n \sum_{j_0,\dots,j_{n-1}} g_{j_0,\dots,j_{n-1}}(x) + \dots, \text{ in } L^1_\alpha .$$

Let $a_0 := (C_0 t \mu_\alpha(I))^{-1} g_0$ and $a_{j_0,\dots,j_{n-1}} := (C_0 t)\mu_\alpha(I_{j_0,\dots,j_{n-1}}))^{-1} g_{j_0,\dots,j_{n-1}}$ for each $j_0,\dots,j_{n-1}$, $n \in N$, then these are $(1,\infty)_\alpha$ atoms by (iii) and (v). Thus we obtain that

$$a(x) = \mu_\alpha(I)^{-1} b(x)$$
$$= C_0 t \mu_\alpha(I)^{-1}(\mu_\alpha(I) a_0(x) + C_1 t \sum_{j_0} \mu_\alpha\, I_{j_0})\, a_{j_0}(x) + \dots.$$
$$+ (C_1 t)^n \sum_{j_0,\dots,j_{n-1}} \mu_\alpha(I_{j_0,\dots,j_{n-1}}) a_{j_0,\dots,j_{n-1}}(x) + \dots),$$

which is the desired representation (3.1). For, the sum of the absolute value of the coefficients of the right hand side is bounded by $C_0 t \sum_0^\infty (Ct^{1-q})^k := C$, C independent of a. This completes the proof.

THEOREM C. Let $-1 < \alpha \le 0$ and $1 < p_1 \le \infty$. Suppose that T is a sublinear operator of weak-type (1,1) on H^1_α, by which we mean that there exists B_0 such that for every $f \in H^1_\alpha$ and $t > 0$;

$$\mu_\alpha\{x \quad G; |Tf(x)| > t\}) \le B_0 \|f\|_{H^1_\alpha} /t,$$

and T is of weak-type on $L^{p_1}_\alpha$ with constant B_1. Then for $1 < p < p_1$, T is of type (p,p) on L^p_α with constant depending only on B_0, B_1, p_1 and p.

PROOF. The proof is similar to the non-weighted case [4], [5].

Let f L^p_α and choose a q so that $1 < q < p < p_1 < \infty$. As in the proof of Theorem B, we consider the open set $E_t := \{M^q_\alpha f > t\} = \{M_\alpha(|f|^q) > t^q\}$, for $t > 0$.

Then we have the same kind of decomposition; $E_t = \bigcup_j I_j$. From this we obtain a Calderon-Zygmund decomposition $f = g_t + h_t$, where $g_t = f$ if $x \notin E_t$; $= m(f, I_j)$ if $x \in I_j$ for each j, and $h := h_t = \sum_j h_j$, where $h_j := (f - g_t)\,\xi_{I_j}$. We then have $|g_t(x)| \leq C_0 t$ and

$$\left(\frac{1}{\mu_\alpha(I_j)} \int_{I_j} |h_j|^q \, d\mu_\alpha\right)^{1/q} \leq C_1 t$$

for each $j \in N$. Hence $a_j := (C_1 t \mu_\alpha(I_j))^{-1} h_j$ is a $(1,q)_\alpha$ atom and

$h = C_1 t \sum_j \mu_\alpha(I_j) a_j \in H_\alpha^{1,q}$. And Theorem B implies that $h \in H_\alpha^1$ with norm bounded by $Ct\mu_\alpha(E_t)$. The rest of proof proceed as in [4], [5] with a few modifications, so we omit the details.

4. PROOFS OF THE MAIN RESULTS.

PROOF OF THEOREM 1. Let $-1/r' < \alpha \leq 0$. To prove the conclusion, it suffices to show that $\|K*a\|_{1,\alpha} \leq C$ for every $(1,\infty)_\alpha$ atom a, where $K := (\phi^k)^\vee$. Let a be such an atom, supported on an interval $I = x_0 + G_n$ $(x_0 \in G,\ n \in Z)$. We write

$$\int_G |K*a| \, d\mu_\alpha = \int_I + \int_{G\setminus I} = A + B, \text{ say.}$$

Let first $r = 1$ (hence $\alpha = 0$). Then

$$A \leq \left(\int_I |K*a(x)|^2 \, d\mu\right)^{1/2} \left(\int_I d\mu\right)^{1/2} \leq C\|a\|_2 \, \mu(I)^{1/2} \leq C\, \mu(I)^{-1} \mu(I) = C.$$

On the other hand,

$$B = \int_{G\setminus I} \left| \int_G K(x-y)a(y) \, d\mu\,(y)\right| d\mu(x) = \int_{G\ I} \left| \int_G K(x-y)-K(x-x_0))a/y)d\mu(y)\right| d\mu(x)$$

$$\leq \int_I |a(y)| \, d\,\mu(y) \int_{G\ I} |K(x-y)-K(x-x_0)| \, d\mu(x)$$

$$= \int_{G_n} |a(x_0+y)| \, d\mu(y) \int_{G\setminus G_n} |K(x-y)-K(x)| d\mu(x) \leq C \int_G |a(x_0+y)| \, d\mu(y) \leq C.$$

hence the conclusion follows, when $r = 1$. This together with Theorem C implies the conclusion of Corollary (ii).

Next, let $r > 1$. To estimate A, we use Corollary (ii) and Lemma (c). Then

$$A \leq \left(\int_I |K*a(x)|^r \, d\mu\right)^{1/r} \left(\int_I (v_\alpha(x))^{r'} \, d\mu\right)^{1/r'} \leq C\|a\|_r \left(\int_I v_{\alpha r'}(x)d\mu\right)^{1/r'}$$

$$\leq C\, \mu_\alpha(I)^{-1} \mu(I)^{1/r} \mu(I)^{1/r'} \inf\{v_\alpha(x);\ x \in I,\ x \neq 0\} \leq C\mu_\alpha(I)^{-1} \mu_\alpha(I) = C.$$

On the other hand, using Lemma (c) again,

$$B \leq \int_I |a(y)| d\mu(y) \int_{G\ I} |K(x-y)-K(x-x_0)| \, v_\alpha(x) \, d\mu(x)$$

$$= \int_{G_n} |a(x_0+y)|d\mu(y) \int_{G\setminus G_n} |K(x-y)-K(x)|\ v_\alpha(x+x_0)\ d\mu(x)$$

$$= \sum_{\ell=-\infty}^{n-1} \int_{G_n} |a(x_0+y)|\ d\mu(y) \int_{G_\ell \setminus G_{\ell+1}} |K(x-y)-K(x)|\ v_\alpha(x+x_0)\ d\mu(x)$$

$$\leq \sum_{-\infty}^{n-1} \int_{G_n} |a(x_0+y)|\ d\mu(y)\ \left(\int_{G_\ell \setminus G_{\ell+1}} |K(x-y)-K(x)|^r\ d\mu(x)\right)^{1/r}$$

$$\times \left(\int_{G_\ell \setminus G_{\ell+1}} v_{\alpha r'}\ (x+x_0)\ d\mu(x)\right)^{1/r'}$$

$$\leq \sum_{-\infty}^{n-1} \int_{G_n} |a(x_0+y)|\ d\mu(x)\ (m_\ell)^{\varepsilon+1/r'}(m_n)^{-\varepsilon}\ m_\ell)^{-1/r'}$$

$$\times \inf\ \{v_\alpha(x);\ x \in I,\ x \neq 0\}$$

$$\leq C(m_n)^{-\varepsilon} \sum_{-\infty}^{n-1} (m_\ell)^{\varepsilon} \int_{G_n} |a(x_0+y)|\ v_\alpha(x_0+y)\ d\mu(y) \leq C(m_n)^{-\varepsilon}\ (m_{n-1})^{\varepsilon}\ ||a||_{1,\alpha} \leq C.$$

This completes the proof.

PROOF OF COROLLARY. (i) Since $\phi \in L^\infty(\Gamma)$ is amultiplier on L^2, it follows from a classical interpolation theorem for weighted spaces [6] and [1, Proposition 1] that ϕ is a multiplier on L^2_α for all $-|\alpha_0| \leq \alpha \leq |\alpha_0|$. As in the proof of [1, Theorem 1], the case where $1 < p < 2$ and $-|\alpha_0| \leq \alpha \leq 0$, has to be proved.

Let $1 < p < 2$ and $-|\alpha_0| \leq \alpha \leq 0$. Since each ϕ^k, $k \in Z$ is a multiplier on L^2_α and also a (H^1_α, L^1_α) multiplier by Theorem 1, it follows from Theorem C that ϕ^k is a multiplier on L^p_α. The assumption that the constants C and ε are independent of k, implies that ϕ is a multiplier on L^p_α.

(ii) This is already seen in the proof of Theorem 1.

PROOF OF THEOREM 2. According to Theorem A, it suffices to show that $||(\check{\phi}*a)^*||_{1,\alpha} \leq C$ for all $(1,\infty)_\alpha$ atom a. Let a be a $(1,\infty)_\alpha$ atom, supported on an interval $I:=x_0 + G_n(x_0 \in G,\ n \in Z)$. We set $\check{\phi}*a = f$. The case where $r = 1$ (hence $\alpha = 0$) is known [2, Corollary]. So we let $1 < r < \infty$ and $-1/r' < \alpha \leq 0$. Now we write

$$\int_G f^* d\mu_\alpha = \int_I + \int_{G\setminus I} = A + B, \quad \text{say.}$$

We first estimate A. Since $K_r^{1/r' + \varepsilon,\infty}$ $K_1^{\varepsilon,\infty}$ Lemma (b) and [2, Corollary] imply that $||f^*||_r \leq C||f||_r \leq C||a||_r$. Thus as in the proof of Theorem 1, we have that

$$A \leq \left(\int_I (f^*)^r d\mu\right)^{1/r} \left(\int_I v_{\alpha r'} d\mu\right)^{1/r'} \leq C\ ||a||\ \mu(I)^{1/r'}\ \inf\{v_\alpha(x);\ x \in I,\ x \neq 0\}$$

$$\leq C\mu_\alpha(I)^{-1}\ \mu_\alpha(I) = C \tag{4.1}$$

Let $\psi(\gamma) := \overline{(\gamma, x_0)}\phi(\gamma)$ and $b(x) := a(x+x_0)$. Then it is easily seen that $f = \phi * a = \psi * b$, supp $b \subset G_n$, and $\int b d\mu = 0$. Thus we have that

$b*D_k = 0$ if $k \le n$, and supp $(b*D_k) \subset G_n$ if $k > n$. Also $(b*D_k)_j :$

$= (b*D_k)*(D_{j+1}-D_j) = 0$ if $j \ge k$ and $(b*D_k)_j = b_j$ if $j < k$. Moreover $b_j = 0$, if $j < n$. Hence

$$f^*(x) = (\check{\psi}*b)^* = \sup_k |(\check{\psi}*b)*D_k(x)| = \sup_{k>n} |\check{\psi}*(b*D_k)(x)|$$

$$= \sup_{k>n} \left| \sum_{-\infty}^{\infty} (\psi_j)^\vee *(b*D_k)_j(x) \right| = \sup_{k>n} \left| \sum_{n}^{k-1} (\psi_j)^\vee * b_j(x) \right| \le \sum_{n}^{\infty} |(\psi_j)^\vee * b(x)|.$$

Then,

$$B = \int_{G\setminus I} f^* d\mu_\alpha \le \sum_{n}^{\infty} \int_{G\setminus I} |(\psi_j)^\vee * b| d\mu_\alpha = \sum_{j=n}^{\infty} \sum_{i=-\infty}^{n-1} \int_{I_i \, I_{i+1}} |(\psi_j)^\vee * b| d\mu_\alpha, \quad (4.2)$$

where $I_i := x_0 + G_i$ for each $i \in Z$.
Now for $i < n$,

$$(\psi_j)^\vee * b(x) = \int_G (\psi_j)^\vee (y) b(x-y)\, d\mu(y)$$

$$= \int_{I_{i+1}} + \int_{I_i \setminus I_{i+1}} + \int_{G\setminus I_i}$$

If $x \in I_i \setminus I_{i+1}$ and $y \in I_{i+1}$, $x-y \in G_i \setminus G_{i+1} \subset G\, G_n$. Also

if $x \in I_i\, I_{i+1}$ and $y \notin I_i$, $x-y \in G\setminus G_i \subset G\setminus G_n$. These, together with supp $b \subset G_n$ imply that the first and last terms of the right hand side of the equality are zero. Thus (4.2) is bounded by

$$\sum_{j=n}^{\infty} \sum_{i=-\infty}^{n-1} \int_{J_i} |((\psi_j)^\vee \xi_{J_i}) * b| d\mu_\alpha$$

$$\le \sum_{n}^{\infty} \sum_{-\infty}^{n-1} \int_G |b(y)| d\,\mu(y) \int_{J_i} |(\psi_j)\xi_{J_i})(x-y)| d\mu_\alpha(x) \quad (4.3)$$

where $J_i := I_i \setminus I_{i+1}$ for each $i \in Z$. Now, Lemma (c)

$$\int_{J_i} |((\psi_j)^\vee \xi_{J_i})(x-y)| d\mu_\alpha(x)$$

$$\le \left(\int_{J_i} |((\psi_j)^\vee \xi_{J_i})(x-y)|^r d\mu(x)\right)^{1/r} \left(\int_{I_i} v_{\alpha r'}(x)\, d\mu(x)\right)^{1/r'}$$

$$\le \left(\int_{J_i} |((\psi_j)^\vee (x)|^r d\mu(x)\right)^{1/r} (m_i)^{-1/r'} \inf\{v_\alpha(x);\ x \in I_i,\ x \ne 0\}$$

$$= \left(\int_{G_i \setminus G_{i+1}} |(\phi_j)^\vee (x)|^r d\mu(x)\right)^{1/r} (m_i)^{-1/r'} \inf\{v_\alpha(x);\ x \in I_i,\ x \ne 0\}$$

Hence (4.3) is bounded by

$$\leq \sum_{j=n}^{\infty} \sum_{i=-\infty}^{n-1} \int_G |a(y+x_0)| d\mu(y) \inf\{v_\alpha(x);\ x \in I_i,\ x \neq 0\}$$

$$\times (m_i)^{-1/r'} \left(\int_{G_i \setminus G_{i+1}} |(\phi_j)^{\vee}(x)|^r d\mu(x)\right)^{1/r}$$

Since $I = I_n \subset I_i$ $(i < n)$ and $||a||_{1,\alpha} \leq 1$,

$$B \leq C \sum_{j=n}^{\infty} \sum_{i=-\infty}^{n-1} (m_i)^{-1/r'} \left(\int_I |a(y)| d\mu(y) \inf\{v_\alpha(x);\ x \in I,\ x \neq 0\} \right.$$

$$\left. \times \left(\int_{G_i \setminus G_{i+1}} |(\phi_j)^{\vee}(x)|^r d\mu(x)\right)^{1/r} \right.$$

$$\leq C \sum_{n}^{\infty} \sum_{-\infty}^{n-1} m_i^{\varepsilon} (m_i)^{-(1/r' + \varepsilon)} \left(\int_{G_i \setminus G_{i+1}} |(\phi_j)^{\vee}|^r d\mu\right)^{1/r}$$

$$\leq C \sum_{n}^{\infty} \sum_{-\infty}^{n-1} m_i^{\varepsilon} ||(\phi_j)^{\vee}||_{K_r^{\varepsilon + 1/r'},\infty} \leq C \sum_{n}^{\infty} (m_j)^{-\varepsilon} \sum_{-\infty}^{n-1} m_i^{\varepsilon} \leq C. \qquad (4.4)$$

Hence, we have that $||f^*||_{1,\alpha} \leq C$ by (4.1) and (4.4). This completes the proof.

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