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A NOTE ON THE VERTEX-SWITCHING RECONSTRUCTION

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ABSTRACT. Bounds on the maximum and minimum degree of a graph establishing its reconstructibility from the vertex switching are given. It is also shown that any disconnected graph with at least five vertices is reconstructible.

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1. INTRODUCTION.

A switching G_ν of a graph G at vertex ν is a graph obtained from G by deleting all edges incident to ν and inserting all possible edges to ν which are not in G. Since switching is a commutative operation, i.e., $(G_\nu)_u = (G_u)_\nu$, the definition can be naturally extended to arbitrary subsets of the vertex set V(G). Thus, G_A is defined for all $A \subseteq V(G)$.

The Vertex-Switching Reconstruction Problem, proposed by Stanley [1], asks: Is G uniquely determined up to isomorphism by the set (deck), $\{G_\nu\}_{\nu \in V(G)}$? If the answer is "yes" then G is called reconstructible.

It was shown in [1] that any graph G with $n = |V(G)| \neq 0 \pmod 4$ is reconstructible. It seems that a little is known about the case $n = 0 \pmod 4$. However, Stanley pointed out [1], that the degree sequence of a graph, and consequently, the number of edges easily reconstructible, provided $n \neq 4$. Bounds on the number of edges in a graph, e(G), establishing its reconstructibility was given [2]. Namely:

$$e(G) \notin \left[\frac{n(n-2)}{4}, \frac{n^2}{4}\right], \quad n \neq 4.$$

As might be expected, in virtue of the last result, G is reconstructible if it has a vertex of degree not close to n/2 or if G is disconnected. Here we will prove the last claim (Theorem 2) and show that for sufficiently large n a graph is reconstructible if $\max(\Delta, n - \delta) > 0.9n$, where Δ and δ are the maximum and the minimum degree of G respectively. Actually, we prove a little more, namely:

2. MAIN RESULTS.

THEOREM 1. If $\min\left(n\binom{n-1}{\Delta}, n\binom{n-1}{\delta}\right) < 2^{n/2-3}$,

then G is reconstructible.

PROOF. In virtue of the quoted result of Stanley, we may assume $n = 0 \pmod 4$. We will consider a graph G as a spanning subgraph of a fixed copy of the complete graph K_n. The switching equivalence class G^* of G is the set of all $H \subset K_n$ isomorphic to G such that $H = G_A$ for some switching $A \subseteq V(G)$.

For each subgraph $g \subset G$, let $\mu(G^* \supset g)$ be the number of those elements of G^* which contain a fixed copy of g.

First we show that G is reconstructible if

$$\frac{\mu(G^* \supset g)s(g \to K_n)}{s(g \to G)} < 2^{n/2-2}, \tag{2.1}$$

where $s(H \to F)$ is the number of the subgraphs of F isomorphic to H.

Observe that

$$|G^*|s(g \to G) \leq \mu(G^* \supset g)s(g \to K_n). \tag{2.2}$$

On the other hand, consider the set $S_i = \{A : G_A \in G^*, |A| = i\}$. Observe that $\Sigma|S_i| = 2|G^*|$ since G_A and $G_{\bar{A}}$, $\bar{A} = V(G)\backslash A$, are identical. It is known that for a nonreconstructible graph $|S_{4i}| \geq \binom{n/2}{2i}$ ([2], Corollary 2.4). Thus, if G is not reconstructible then

$$2|G^*| \geq \Sigma\binom{n/2}{2i} = 2^{n/2-1}. \tag{2.3}$$

Comparing (2.2) and (2.3), we get that (2.1) is enough for the reconstructibility of G.

Let now g be a star $K_{1,\Delta}$. Observe that $\mu(G^* \supset K_{1,\Delta}) \leq 2$ since the only proper switching, possibly preserving a fixed copy of $K_{1,\Delta}$, is $A = V(K_{1,\Delta})$. Furthermore, $s(g \to K_n) = n\binom{n-1}{\Delta}$. Hence, by (2.1), G is reconstructible if $n\binom{n-1}{\Delta} < 2^{n/2-3}$.

Now, to complete the proof, one has to consider the complementary graph $\bar{G}$, which is reconstructible iff G is. □

Now we will prove that disconnected graphs are reconstructible. First we need the following simple lemma:

LEMMA 1. Suppose that nonisomorphic graphs G and H have the same deck. Then for any $\nu \in V(G)$ there is $u \in V(G)$, $\nu \neq u$, such that $G_{\nu u} \cong H$.

PROOF. Since the decks of G and H are equal then there is a bijection $\phi: V(G) \to V(H)$ such that $G_\nu \cong H_{\phi(\nu)}$. Let $h_\nu : H_{\phi(\nu)} \to G_\nu$ be an isomorphism. Choosing $u = h(\phi(\nu))$ we obtain $G_{\nu u} \cong H$. Moreover, since $G_{\nu\nu} = G$, then $\nu \neq \phi(\nu)$. □

COROLLARY 1. Let $n \neq 4$. If $G_{\nu u}$ and G, $\nu \neq u$, have the same deck then $\deg(\nu) + \deg(u) = n$ or $n - 2$, depending on whether ν and u are adjacent in G or are not.

PROOF. Let $e(\nu,u)$ be the number of edges between ν and u. Since $e(G) = e(H)$ then

$$\deg(\nu) + \deg(u) - 2e(\nu,u) = \tfrac{1}{2} \cdot 2(n-2) = n - 2. \ \square$$

COROLLARY 2. If G is not reconstructible and $n \neq 4$ then $n - 2 \leq \delta + \Delta \leq n$.

PROOF. This easily follows from Lemma 1 and Corollary 1. We omit the details. □

THEOREM 2. Any disconnected graph is reconstructible, provided $n \neq 4$.

PROOF. Assume the contrary. Then there are two nonisomorphic graphs G and H with the same deck, $n \neq 4$, and, say, G is disconnected. Denote by C a minimal connected component of G. First we show that G has exactly two connected components and $C \cong K_{\delta+1}$.

Let ν be a vertex of the minimal degree in C, and let u be such a vertex that $G_{\nu u} \cong H$. We claim that either $u = \phi(\nu) \in \bar{C}$ or G is regular of degree $\frac{n-2}{2}$. Indeed, otherwise,

$$|C| \geq \max(\deg(\nu) + 1, \deg(u) + 1) > n/2,$$

which contradicts the minimality of C. Furthermore, if G is regular then again ν and u are in different components since, otherwise, the degree sequences of G and $G_{\nu u}$ are different. Now it follows by Corollary 1, $\deg(\nu) + \deg(u) = n - 2$. Therefore, G has exactly two components, C is regular, and $\Delta \geq n/2$.

Let us show that C is just $K_{\delta+1}$. Since all vertices of degree Δ are in $\bar{C}$, we have

$$\deg(\nu) + 1 \leq |C| \leq n - \Delta - 1.$$

Hence, applying Corollary 2, we get

$$n - 2 \leq \delta + \Delta \leq \deg(\nu) + \Delta \leq n - 2.$$

Thus, $\deg(\nu) = \delta$, $\deg(u) = \Delta$, and $C \cong K_{\delta+1}$.

Finally, $G_{\nu u} \cong G$ since $\deg(\nu) = |C| - 1$, $u \in \bar{C}$ and $\deg(u) = \Delta = |\bar{C}| - 1$, which is a contradiction. This completes the proof. □

REFERENCES

1. STANLEY, R.P. Reconstruction from vertex switching. J. Combin. Theory (B) 38, (1985), 132-138.
2. KRASIKOV, I. and RODITTY, Y. Balance equations for reconstruction problems. Archiv der Mathematik, Vol. 48 (1987), 458-464.