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BIEBERBACK FUNCTIONS AND PERIODIC DISTRIBUTIONS

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ABSTRACT. This paper deals with a correspondence between the periodic distributions and holomorphic functions. Periodic distributions whose "negative" Fourier coefficients are zero are characterised as the boundary values of certain holomorphic functions.

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1. INTRODUCTION

Let f(z) be a holomorphic function in $U^n \subset C^n$ where U is the unit disc of the complex plane C. Let the power series expansion of f be given by

$$f(z) = \Sigma a_k z^k \tag{1.1}$$

Where $k = (k_1, k_2, \dots, k_n)$ is a multi-index of non-negative integers z^{*n} and $z^k = z_1^{k_1} z_2^{k_2} \dots z_n^{k_n}$ and $z = (z_1, z_2, \dots, z_n) \in U^n$. We will denote by B the set of all holomorphic functions given by (1) satisfying the following inequality.

$$|a_k| \leq M\ |k|^p \tag{1.2}$$

for some non-negative integers M and p. This set will also be called the set of Bieberbach functions. It may be noted that in case n = 1, the class B includes a variety of functions that are geometrically interesting. For example certain classes of finitely valent functions [2], the class of star-like or convex functions [3], or more generally univalent functions [4]. The name is also suggestive in this context. In this paper we investigate the geometric significance of the inequality (1.2) and thus establish a correspondence between elements of B and certain type of periodic distributions. i.e. distributions on T^n where T is the unit circle in C. This investigation enables us to obtain the coefficients a_k in (1.2) as the Fourier coefficients of the corresponding distribution just as the coefficients of a H^2 function on U in one variable are identified with the Fourier coefficients of its radial limit function [Theorem 17.10, p. 366 of [6]].

Our main result is the complete description of the distribution and the holomorphic function involved in this correspondence.

2. MAIN RESULT

Here we quickly recall the basic facts about the space of test functions on T^n and its dual namely the space of periodic distributions and their Fourier coefficients as outlined in Exercise 22 on page 190 of [5].

$$T^n = \{(e^{ix_1}, e^{ix_2}, \ldots, e^{ix_n}): x_i \text{ real}\} \tag{2.1}$$

Functions ϕ on T^n can be identified with functions $\tilde{\phi}$ on R^n that are 2π- periodic in each variable by setting

$$\tilde{\phi}(x_1, x_2, \ldots, x_n) = \phi(e^{ix_1}, e^{ix_2}, \ldots, e^{ix_n}) \tag{2.2}$$

Let Z^n be the set of n-tuples of integers, z^{*n} be the set of n-tuples of non-negative integers. For $k \varepsilon Z^n$ the function e_k is defined on T^n by

$$e_k(e^{ix_1}, e^{ix_2}, \ldots, e^{ix_n}) = e^{ik.x} = \exp[i(k_1x_1 + k_2x_2 + \ldots + k_nx_n)] \tag{2.3}$$

If σ_n is the Haar measure on T^n then the Fourier coefficients of ϕ are given by

$$\phi(k) = \int_{T^n} e_{-k}\, \phi\, d\sigma_n \quad (k \varepsilon Z^n, \phi \varepsilon L^1(\sigma_n)) \tag{2.4}$$

$D(T^n)$ is the space of all functions ϕ on T^n such that $\tilde{\phi} \varepsilon C^\infty(R^n)$. If $\phi \varepsilon D(T^n)$ then

$$\Big(\sum_{k \varepsilon Z^n} (1 + |k|^2)^N |\tilde{\phi}(k)|^2\Big)^{1/2} < \infty \quad (N = 0,1,2,\ldots) \tag{2.5}$$

This family of seminorms defined a Frechet topology on $D(T^n)$ which coincides with the space given by the seminorms

$$\operatorname*{Max}_{|\alpha| \leq N} \; \sup_{x \varepsilon R^n} |(D^\alpha \tilde{\phi})(x)| \quad (N = 0,1,2\ldots) \tag{2.6}$$

$D'(T^n)$ is the space of all continuous linear functionals on $D(T^n)$ also called the space of periodic distributions. The Fourier coefficients of any $u \varepsilon D'(T^n)$ are given by

$$\hat{u}(k) = u(e_{-k}) \quad (k \varepsilon Z^n) \tag{2.7}$$

To each $u \varepsilon D'(T^n)$ there exists N and Q such that

$$|\hat{u}(k)| \leq Q(1+|k|)^N \quad (k \varepsilon Z^n) \tag{2.8}$$

Conversely if g is a complex function on Z^n such that

$$|g(k)| \leq Q(1+ |k|)^N \tag{2.9}$$

for some Q and N, then $g = \hat{u}$ for some $u \in D'(T^n)$. There is thus a linear one-to-one correspondence between the periodic distributions on the one hand and functions of polynomial growth on Z^n on the other.

From the above theory it is clear that any $f \in B$ given by (1.1) and satisfying (1.2) gives raise to a periodic distribution $\mu_f = v$ whose Fourier coefficients satisfy

$$\hat{v}(k) = \begin{cases} a_k & k \in Z^{*n} \\ 0 & \text{otherwise} \end{cases} \tag{2.10}$$

On the other hand any periodic distribution v satisfies

$$|\hat{v}(k)| < Q|k|^N \tag{2.11}$$

by virtue of (2.8) and so if only $\hat{v}(k) = 0$ for $k \in Z^n - Z^{*n}$ the power series.

$$g(z) = \sum \hat{v}(k)\, z^k \tag{2.12}$$

will represent a holomorphic function in U^n as we shall see later and hence $\mu_g = v$. Let G denote the class of all periodic distributions v such that

$$\hat{v}(k) = 0 \quad \text{if } k \in Z^n - Z^{*n}.$$

Consider the equality $\mu_f = v$ $(f \in B,\ v \in G)$. The following theorem gives a complete description of either v or f if the other is given.

THEOREM. Let $f \in B$ be given. Then $\mu_f = v$ is completely given by

$$v(\phi) = \operatorname*{Limt}_{r \to 1-} \int_{T^n} f(rx)\, \phi(x)\, d\sigma_n(x) \tag{2.13}$$

for any $\phi \in D(T^n)$, and in particular for a fixed $z = (z_1, z_2, \ldots z_n) \in U^n$

$$v\left(\prod_{i=1}^{n} (x_i - z_i)^{-1}\right) = f(z),\ x = (x_1, x_2, \ldots x_n) \in T^n. \tag{2.14}$$

Conversely if $v \in G$ then the function f given by (2.14) is holomorphic in U^n belongs to B and satisfies (2.13) so that $\mu_f = v$ holds.

PROOF. Let f be given. By definition $v = \mu_f$ is the distribution given by (2.10). Since any distribution is uniquely determined by its Fourier coefficients (See the last parts of Exercise 22 on page 190 of [5]) all we have to prove is that the following functional u on $D(T^n)$ is linear, continuous and has the Fourier coefficients given by (2.10), with u replacing v.

$$u(\phi) = \operatorname*{Limt}_{r \to 1-} \int_{T^n} f(rx)\, \phi(x)\, d\sigma_n(x). \tag{2.15}$$

First we will show that for a fixed $\phi \in D(T^n)$

$$\int_{T^n} f\,(rx)\, \phi(x)\, d\,\sigma_n(x) \tag{2.16}$$

has a limit as $r \to 1-$. To see this we use the power series expansion (1.1) and rewrite (2.16) as

$$\sum_{k \in Z^{*n}} a_k r^{|k|} \int_{I^n} \exp[i(\theta_1 k_1+\theta_2 k_2+\ldots+\theta_n k_n)] \ \phi(e^{i\theta},\ldots,e^{i\theta}) \frac{d\theta_1\ldots d\theta_n}{(2\pi)^n} \quad (2.17)$$

where $I = [-\pi, \pi]$. The term by term integration is justified by the holomorphic nature of f in U^n which implies the local uniform convergence. The series in (2.17) is the same as

$$\sum_{k\in Z^{*n}} a_k \ \phi(-k) \ r^{|k|} \quad (2.18)$$

Now consider the power series

$$\sum_{m=o}^{\infty} b_m \lambda^m \quad (2.19)$$

in one complex variable λ where

$$b_m = \sum_{\substack{k\in Z^{*n} \\ |k|=m}} a_k \ \hat{\phi}(-k) \quad (2.20)$$

If the series (2.19) were absolutely convergent for $|\lambda| < 1$, then this series at $\lambda = r$ is the same as (2.18). Further if (2.19) were convergent at $\lambda = 1$ then by Abel's limit theorem of one complex variable [1, p.42] this series will converge to $\sum_o^\infty b_m$ as $\lambda \to 1$ the series (2.19) will have a radius of convergence greater than or equal to one and hence will be absolutely convergent for $|\lambda| < 1$. Thus for our purposes it is sufficient to prove that (2.19) converges at $\lambda = 1$.

We know that

$$\sum_{k\in Z^{*n}} (1 + |k|^2)^N \ |\phi(k)|^2 \leq M \quad (N = 0,1,2,\ldots).$$

Thus if $|k| = m$ and $N > 0$ integer, then

$$|\phi\ (-k)\ | \leq M_1\ (1 + |k|^2)^{-N/2} \quad (2.21)$$

Using (2.21) and (1.2) in (2.20)

$$|b_m| \leq C_1 m^p (m + 1)^n (1 + m^2)^{-N/2} \quad (2.22)$$

since the number of $k \in Z^{*n}$ with $|k| = m$ is atmost $(m + 1)^n$. But an estimate of the form (2.22) forces $\Sigma|b_m|$ to be convergent if N is sufficiently large. Hence our claim is established.

Thus u is a well defined map from $D\ (T^n)$ to C. Clearly u is linear. We contend that u is in fact continuous and hence a periodic distribution. For this it suffices to show that if

$$\phi_m \to 0 \text{ in } D(T^n) \text{ as } m \to \infty \quad (2.23)$$

then

$$u(\phi_m) \to 0 \text{ in } C \text{ as } m \to \infty \tag{2.24}$$

Now (2.23) implies that for N positive integer

$$D^x \tilde{\phi}_m \to 0 \tag{2.25}$$

uniformly in R^n for all multi-indices α with $|\alpha| \leq N$ as $m \to \infty$. By our construction

$$u(\phi_m) = \sum_{k\varepsilon Z*^n} a_k \hat{\phi}_m(-k). \tag{2.26}$$

$$\hat{\phi}_m(-k) = \int_{T^n} e_k \phi_m \, d\sigma_n \tag{2.27}$$

If $k \in Z*^n$ and α is a multi-index then

$$k^\alpha \hat{\phi}_m(-k) = \int e_k k^\alpha \phi_m d\sigma_n = A \int e_k D^\alpha \tilde{\phi}_m \text{ (A - constant)} \tag{2.28}$$

using integrations by part and the periodicity of $\tilde{\phi}_m$. (2.25) and (2.28) now assures us that

$$|k^\alpha|^2 \; |\hat{\phi}_m(-k)|^2 \to 0 \text{ as } m \to \infty \text{ uniformly in k.} \tag{2.29}$$

Using suitable multi-indices α in (2.24) we can also get that for any N positive integer

$$(1 + |k|^2)^N |\hat{\phi}_m(-k)|^2 \to 0 \text{ as } m \to \infty \text{ uniformly in k.} \tag{2.30}$$

But if N is large enough to ensure

$$\sum_{k\varepsilon Z*^n} (1 + |k|^2)^{-N} < \infty$$

then (2.30) can also lead us to conclude that

$$\sum_{k\varepsilon Z*^n} (1 + |k|^2)^N \; |\hat{\phi}_m(-k)|^2 = \sum \frac{1}{(1+|k|^2)^N} (1 + |k|^2)^{2N} |\hat{\phi}_m(-k)|^2$$

goes to zero as $m \to \infty$. Hence

$$|u(\phi_m)|^2 = |\sum a_k \hat{\phi}_m(-k) \;|^2 \leq |\sum a_k (1 + |k|^2)^{-M/2} \hat{\phi}_m(-k) \; (1 + |k|^2)^{M/2}|$$

$$\leq \sum |a_k|^2 (1 + |k|^2)^{-M} \sum (1 + |k|^2)^M | \; \hat{\phi}_m(-k)|^2.$$

On the right side the first sum can be made finite using (1.2) and choosing a large M. Further if M is sufficiently large the second sum goes to zero as $m \to \infty$ and so $u(\phi_m) \to 0$ as $m \to \infty$, completing the proof.

Now we proceed to calculate the Fourier coefficients of this distribution u. Let $m \in Z^n$.

$$\hat{u}(m) = u(e_{-m}) = \sum_{k\epsilon Z*^n} a_k \widehat{(e_{-m})}(-k). \tag{2.31}$$

But

$$\widehat{(e_{-m})}(-k) = \int_{T^n} e_k e_{-m} \, d\sigma_n = \begin{cases} o & \text{if } k \neq m \\ 1 & \text{if } k = m \end{cases} \tag{2.32}$$

(2.31) and (2.32) show that (2.10) holds for this u. Hence $\hat{u} = \hat{\mu}_f$ and so by uniqueness

$$u = \mu_f.$$

A repeated application of Cauchy integral formula gives

$$\int_{T^n} f(rx) \prod_{i=1}^{n} (x_i - z_i)^{-1} d\sigma_n(x) = f(rz) \tag{2.33}$$

for $z = (z_1, z_2, .., z_n)$ and $r < 1$ since $f(rz)$ is analytic in the polydisc,

$$\{z \in C^n / \; |z_i| < r^{-1} \supset U^n \cup T^n.$$

Now (2.33) implies (2.14).

Next we come to the converse. Let $v \in G$ be given. Consider the series

$$g(z) = \sum_{k\epsilon Z*^n} \hat{v}(k) \, z^k \tag{2.34}$$

If $z_i \epsilon U$ for $i = 1,2....n$ we choose r such that $|z_i| < r < 1$ for all i. Then

$$|\hat{v}(k)| \, |z^k| \leq M|k|^N r^{|k|}$$

An application of Cauchy's root test ensures us that the series (2.34) is convergent in U^n and hence g(z) represents a holomorphic function there. By the fact that v is a periodic distribution v satisfies (2.11)and so $g \in B$. Now consider μ_g .

By the definition $\hat{\mu}_g(k) = \hat{v}(k)$ for all $k \in Z^n$ and so

$$\mu_g = v. \tag{2.35}$$

Now applying the first part for this $g \in B$ we get (2.13) nd (2.14) with f replaced by g. But f is defined by (2.14) and so

$$f = g \tag{2.36}$$

Hence (2.13)holds for his f and also $\mu_f = v$ by (2.35) and (2.36).

NOTE. This theorem implies that $\mu_f = v$, $f \in B$, $v \in G$ holds if and only if (2.13) and (2.14) hold. If $n = 1$, B is actually an algebra under the Hadamard product and the map $f \to \mu_f$ is an algebra isomorphism between B and G where in G the product is the convolution.

Let $n = 1$. If we consider $v \in G$ and assume v as a map is one-to-one from $\{(x-z)^{-1} / z \in U\} \subset D(T) \to C$, then the map f given by (2.14) is also univalent in U and hence by Bieberbach conjecture we have

$$|\hat{v}(n)| \leq n|\hat{v}(1)| \quad (n = 1,2,\ldots)$$

and this is stronger than the "a priori" estimate $|v(n)| = O(n^k)$ for $k > 0$. Hence the Fourier coefficients of certain distributions belonging to G also satisfy the Bieberbach conjecture.

In this context it is interesting to ask the following question. Can we characterize the set of all μ_f when f varies over the class of univalent functions or starlike functions or convex functions or functions with positive real part in the unit disc using properties of the associated distributions μ_f?

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