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ON CODING THEOREM CONNECTED WITH 'USEFUL' ENTROPY OF ORDER - β

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ABSTRACT. Guiasu and Picard [1] introduced the mean length for 'useful' codes. They called this length as the 'useful' mean length. Longo [2] has proved a noiseless coding theorem for this 'useful' mean length. In this paper we will give two generalizations of 'useful' mean length. After then the noiseless coding theorems are proved using these two generalizations.

KEY WORDS AND PHRASES. Useful entropy, useful mean length, coding theorems.
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1. INTRODUCTION.

Belis and Guiasu [3] consider the following model for a finite random experiment (or information source) A:

$$A = \begin{bmatrix} x_1 & x_2 & \cdots\cdots & x_n \\ p_1 & p_2 & \cdots\cdots & p_n \\ u_1 & u_2 & \cdots\cdots & u_n \end{bmatrix} = \begin{bmatrix} X \\ P \\ U \end{bmatrix}, \tag{1.1}$$

where X is the alphabet, P the probability distribution and $U = (u_1, u_2, \ldots, u_n)$ $u_i > 0$ is the utility distribution. They introduced the measure

$$H(P,U) = -\sum_{i=1}^{n} u_i p_i \log p_i \tag{1.2}$$

about the scheme (1.1). They called it 'useful' information provided by a source letter. Guiasu and Picard [1] have considered the problem of encoding the letters output by the source (1.1) by means of a single letter prefix code, whose codewords $C_1, C_2, \ldots, C_n$ have lengths $\ell_1, \ldots, \ell_n$ satisfying the Kraft's [4] inequality

$$\sum_{i=1}^{n} D^{-\ell_i} \le 1, \tag{1.3}$$

where D is the size of the code alphabet. They defined the following quantity

$$L(u) = \frac{\sum_{i=1}^{n} \ell_i u_i p_i}{\sum_{i=1}^{n} u_i p_i} \tag{1.4}$$

and call it 'useful' mean length of the code. They also derived a lower bound for it.

In this communication two generalizations of (1.4) have been studied and then the bounds for these generalizations are obtained in terms of 'useful' entropy of type β, which is given by

$$H^{\beta}(P,U) = \frac{1}{(2^{1-\beta}-1)} \left[\sum_{i=1}^{n} u_i p_i \,(p_i^{\beta-1}-1)\right] \quad \beta > 0 \quad \beta \neq 1 \tag{1.5}$$

under the condition

$$\sum_{i=1}^{n} u_i D^{-\ell_i} \leq \sum_{i=1}^{n} u_i p_i, \tag{1.6}$$

which is the generalization of Kraft's inequality (1.4).

2. TWO GENERALIZATIONS OF 'USEFUL' MEAN LENGTH AND THE CODING THEOREMS.

Let us introduce the measure of length:

$$L_1^{\beta}(U) = \frac{1}{\log D(2^{1-\beta}-1)} \left[\left\{ \frac{\sum_{i=1}^{n} u_i p_i D^{\ell_i \frac{1-\beta}{\beta}}}{\sum_{i=1}^{n} u_i p_i} \right\}^{\beta} - 1 \right] \quad \beta \neq 1,\ \beta > 0. \tag{2.1}$$

It is easy to see that

$$\lim_{\beta \to 1} L_1^{\beta}(u) = L(U).$$

In the following theorem we obtain lower bound for (2.1) in terms of $H^{\beta}(P,U)$.

THEOREM 1. If $\ell_1, \ell_2, \ldots, \ell_n$ denote the lengths of a code satisfying (1.6) then

$$L_1^{\beta}(U) \geq H^{\beta}(P,U) \,/\, \bar{U} \log D, \ \beta \neq 1,\ \beta > 0 \tag{2.2}$$

where $\bar{U} = \sum_{i=1}^{n} u_i p_i$

with equality iff

$$D^{-\ell_i} = \frac{p_1^{\beta}}{\left(\sum_{i=1}^{n} u_i p_i^{\beta} \,/\, \sum_{i=1}^{n} u_i p_i \right)}. \tag{2.3}$$

PROOF. By Hölder's inequality

$$\left[\sum_{i=1}^{n} a_i^p\right]^{1/p} \left[\sum_{i=1}^{n} b_i^q\right]^{1/q} \leq \sum_{i=1}^{n} a_i b_i, \tag{2.4}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p < 1$ and $a_i,\ b_i > 0$.

Put

$$P = \frac{(\beta-1)}{\beta},\ a_i = \left[u_i p_i \Big/ \sum_{i=1}^{n} u_i p_i\right]^{\frac{\beta}{\beta-1}} D^{-\ell_i}$$

$$q = 1-\beta,\ b_i \left[u_i p_i^{\beta} \Big/ \sum_{i=1}^{n} u_i p_i\right]^{1/(1-\beta)}$$

in (2.4), we get

$$\left[\sum_{i=1}^{n} u_i p_i D^{\ell_i(1-\beta)/\beta} \Big/ \sum_{i=1}^{n} u_i p_i\right]^{-\beta/(1-\beta)} \left[\sum_{i=1}^{n} u_i p_i^{\beta} \Big/ \sum_{i=1}^{n} u_i p_i\right]^{1/(1-\beta)}$$

$$\leq \sum_{i=1}^{n} u_i D^{-\ell_i} \Big/ \sum_{i=1}^{n} u_i p_i. \tag{2.5}$$

Using (1.6) in (2.5), we get

$$\sum_{i=1}^{n} u_i p_i D^{\ell_i(1-\beta)/\beta} \Big/ \sum_{i=1}^{n} u_i p_i]^{\beta/(\beta-1)} \leq \left[\sum_{i=1}^{n} u_i p_i^{\beta} \Big/ \sum_{i=1}^{n} u_i p_i\right]^{1/(\beta-1)}. \tag{2.6}$$

Let $0 < \beta < 1$. Raising both sides of (2.6) to the power $(\beta-1)$, we get

$$\left[\sum_{i=1}^{n} u_i p_i D^{\ell_i(1-\beta)/\beta} \Big/ \sum_{i=1}^{n} u_i p_i\right]^{\beta} \geq \left[\sum_{i=1}^{n} u_i p_i^{\beta} \Big/ \sum_{i=1}^{n} u_i p_i\right].$$

Since $2^{1-\beta} - 1 > 0$ for $0 < \beta < 1$, a simple manipulation proves (2.2) for $0 < \beta < 1$. The proof for $1 < \beta < \infty$ follows on the same lines. It is clear tht equality in (2.2) holds iff

$$D^{-l_i} = \frac{p_i^{\beta}}{\left(\sum_{i=1}^{n} u_i p_i^{\beta} \Big/ \sum_{i=1}^{n} u_i p_i\right)}, \tag{2.7}$$

which implies that

$$\ell_i = \log_D \left(\sum_{i=1}^{n} u_i p_i^{\beta} \Big/ \sum_{i=1}^{n} u_i p_i\right) - \log_D p_i^{\beta}. \tag{2.8}$$

Hence it is always possible to have a code satisfying

$$-\beta \log_D P_i + \log_D \left(\sum_{i=1}^{n} u_i p_i^{\beta} \Big/ \sum_{i=1}^{n} u_i p_i\right) \leq \ell_i <$$

$$< -\beta \log_D P_i + \log_D \left(\sum_{i=1}^{n} u_i p_i^{\beta} \Big/ \sum_{i=1}^{n} u_i p_i\right) + 1, \tag{2.9}$$

which is equivalent to

$$p_i^{-\beta}\left(\sum_{i=1}^{n} u_i p_i^{\beta} \Big/ \sum_{i=1}^{n} u_i p_i\right) \leq D^{\ell_i} < D p_i^{-\beta}\left(\sum_{i=1}^{n} u_i p_i^{\beta} \Big/ \sum_{i=1}^{n} u_i p_i\right). \tag{2.10}$$

PARTICULAR CASE. Let $u_i = 1$ for each i and $D = 2$, that is, the codes are binary, then (2.2) reduces to the result proved by Van der Lubbe [5].

In the following theorem, we will give an upper bound for $L_1^\beta(U)$ in terms of $H^\beta(P,U)$.

THEOREM 2. By properly choosing the lengths $\ell_1,\ell_2,\dots,\ell_n$ in the code of Theorem 1, $L_1^\beta(U)$ can be made to satisfy the following inequality:

$$L_1^\beta(U) < \frac{H^\beta(P,U)}{\bar{U}\log_D D} D^{1-\beta} + \frac{D^{1-\beta}-1}{(2^{1-\beta}-1)\log_D D} \quad \beta \neq 1,\ \beta > 0. \tag{2.11}$$

PROOF. From (2.10), we have

$$D^{\ell_i} < Dp_i^{-\beta}\left(\sum_{i=1}^n u_i p_i^\beta / \sum_{i=1}^n u_i p_i\right) \tag{2.12}$$

Let $0 < \beta < 1$. Raising both sides of (2.12) to the power $\frac{1-\beta}{\beta}$, we get

$$D^{\ell_i(1-\beta)/\beta} < p_i^{\beta-1} D^{(1-\beta)/\beta}\left(\sum_{i=1}^n u_i p_i^\beta / \sum_{i=1}^n u_i p_i\right)^{(1-\beta)/\beta} \tag{2.13}$$

Multiplying both sides of (2.13) by $u_i p_i / \sum_{i=1}^n u_i p_i$, summing over i and after then raising both sides to the power β, we get

$$\left[\sum_{i=1}^n u_i p_i D^{(1-\beta)\ell_i/\beta} / \sum_{i=1}^n u_i p_i\right] \leq D^{1-\beta}\left[\sum_{i=1}^n u_i p_i^\beta / \sum_{i=1}^n u_i p_i\right]. \tag{2.14}$$

Since for $0 < \beta < 1$, $2^{1-\beta}-1 > 0$, a simple manipulation proves the theorem for $0 < \beta < 1$. Let $1 < \beta < \infty$, the proof follows on same lines.

PARTICULAR CASE. Let $u_i = 1$ for each i and $D = 2$, that is, the codes are binary codes, then (2.11) reduces to the result proved by Van der Lubbe [5].

REMARK. When $\beta \to 1$, (2.2) and (2.11) give

$$\frac{H(P,U)}{\bar{U}\log D} \leq L(U) < \frac{H(P,U)}{\bar{U}\log D} + 1, \tag{2.15}$$

where $L(U)$ is the 'useful' mean length function (1.4), Longo [2] gave the lower and upper bounds on $L(U)$ as follows:

$$\frac{H(P,U) - \overline{u\log u} + \bar{u}\log\bar{u}}{\bar{u}\log D} \leq L(U) < \frac{H(P,U) - \overline{u\log u} + \bar{u}\log\bar{u}}{\bar{u}\log D} + 1, \tag{2.16}$$

where the bar means the value with respect to probability distribution $P = (p_1,\dots,p_n)$.

Since $x\log x$ is a convex U function, the inequality

$$\overline{u\log u} \geq \bar{u}\log\bar{u}$$

holds and therefore $H(P,U)$ does not seem to be as basic in (2.16) as in (2.15).

Now we will define another measure of length related to $H^\beta(P,U)$. We define the measure of length $L_2^\beta(U)$ by

$$L_2^{\beta}(U) = \frac{1}{(2^{1-\beta}-1)\log D}\left[\sum_{i=1}^{n} u_i p_i^{\beta} / \sum_{i=1}^{n} u_i p_i^{\beta} D^{\ell_i(\beta-1)} - 1\right] \quad \beta \neq 1,\ \beta > 0. \tag{2.17}$$

It is easy to see that

$$\lim_{\beta \to 1} L_2^{\beta}(U) = L(U).$$

In the following theorem we obtain the lower bound for $L_2^{\beta}(U)$ in terms of $H^{\beta}(\,,U)$.

THEOREM 3. If $\ell_1, \ell_2, \ldots, \ell_n$ denote the lengths of code satisfying (1.6), then

$$L_2^{\beta}(U) \geq \frac{H^{\beta}(P,U)}{\overline{U}\log D} \qquad \beta \neq 1,\ \beta > 0, \tag{2.18}$$

where $\overline{U} = \sum_{i=1}^{n} u_i p_i$

with equality if and only if

$$p_i = D^{-\ell_i}. \tag{2.19}$$

PROOF. Let $0 < \beta < 1$. By using Hölder's inequality and (1.6) it easily follows that

$$\sum_{i=1}^{n} u_i p_i^{\beta} D^{\ell_i(\beta-1)} \leq \sum_{i=1}^{n} u_i p_i. \tag{2.20}$$

Obviously (2.20) implies

$$\left[\left(\sum_{i=1}^{n} u_i p_i^{\beta} / \sum_{i=1}^{n} u_i p_i^{\beta} D^{\ell_i(\beta-1)}\right) - 1\right] \geq \left[\left(\sum_{i=1}^{n} u_i p_i^{\beta} / \sum_{i=1}^{n} u_i p_i\right) - 1\right] \quad 0 < \beta < 1. \tag{2.21}$$

Since $(2^{1-\beta}-1) > 0$ whenever $0 < \beta < 1$, a simple manipulation proves (2.18). The proof for $1 < \beta < \infty$ follows on the same lines. It is clear that the equality in (2.18) is true if and only if

$$D^{-\ell_i} = p_i \tag{2.22}$$

which implies that

$$\ell_i = \log_D (1/p_i). \tag{2.23}$$

Thus it is always possible to have a code word satisfying the requirement

$$\log_D \frac{1}{p_i} \leq \ell_i < \log_D \frac{1}{p_i} + 1, \tag{2.24}$$

which is equivalent to

$$\frac{1}{p_i} \leq D^{\ell_i} < \frac{D}{p_i}. \tag{2.25}$$

PARTICULAR CASE. Let $u_i = 1$ for each i and $D = 2$, then (2.18) reduces to the result proved by Nath and Mittal [6].

Next we obtain a result giving the upper bound to the 'useful' mean length $L_2^{\beta}(U)$.

THEOREM 4. By properly choosing the lengths $\ell_1, \ell_2, \ldots, \ell_n$ in the code of Theorem 3, $L_2^{\beta}(U)$ can be made to satisfy the following

$$L_2^{\beta}(U) < \frac{D^{-\beta}\, H^{\beta}(P,U)}{\log D} + \frac{D^{-\beta} - 1}{(2^{1-\beta}-1)\log D} \qquad \beta \neq 1,\ 0 < \beta < 1. \tag{2.26}$$

PROOF. From (2.25), we have

$$p_i\, D^{\ell_i} < D.$$

Consequently

$$p_i^{\beta}\, D^{(\beta-1)\ell_i} < D^{\beta}\, D^{-\ell_i} \qquad \beta > 0,\ \beta \neq 1. \tag{2.27}$$

Multiplying both sides by u_i and then summing over i and using (1.6) we get

$$\sum_{i=1}^{n} u_i p_i^{\beta}\, D^{(\beta-1)\ell_i} < D^{\beta} \sum_{i=1}^{n} u_i p_i. \tag{2.28}$$

Obviously (2.28) implies that

$$[(\sum_{i=1}^{n} u_i p_i^{\beta} / \sum_{i=1}^{n} u_i p_i^{\beta}\, D^{(\beta-1)\ell_i}) - 1] > D^{-\beta}(\sum_{i=1}^{n} u_i p_i^{\beta} / \sum_{i=1}^{n} u_i p_i) - 1. \tag{2.29}$$

Since $2^{1-\beta} - 1 < 0$ for $0 < \beta < 1$, (2.29) implies (2.26).

PARTICULAR CASE. Let $u_i = 1$ for each i and $D = 2$, then (2.26) reduces to the result proved by Nath and Mittal [6].

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