Internat. J. Math. & Math. Sci.
VOL. 12 NO. 2 (1989) 409-412

RESEARCH NOTES

PERIODIC ORBITS OF MAPS WITH AN INFINITE NUMBER OF PARTITION POINTS

MANNY SCAROWSKY

and

ABRAHAM BOYARSKY

Department of Mathematics
Loyola Campus
Concordia University
Montreal, Canada H4B 1R6

(Received January 21, 1987 and in revised form November 5, 1987)

ABSTRACT. Let f: $[0,1] \to [0,1]$ be a piecewise linear map having an infinite number of partition points. Consider f restricted to the domain $D_N = \{\frac{a}{(m-1)p^N}, (a,p) = 1\}$, where p is a prime number. The main result establishes and explicit bound for the number of periodic orbits of $f|D_N$, namely $Ap^{\beta}N$, where A and β are constants.

KEY WORDS: Periodic Orbits, Transformation of an Interval.
1980 AMS SUBJECT CLASSIFICATION CODE. Primary 26A99, Secondary 58F22.

1. Introduction.

Since the results of Sarkovskii's famous paper [1] have become known, there has been a great deal of research done on the periodic points of continuous maps of an interval into itself. Recently, Holfbauer [2] has generalized some of these results to piecewise monotonic transformations.

In [3,4] maps with a countable number of monotonic segments were studied and conditions were given establishing the existence of absolute continuous invariant measures. The specification property [5], satisfied by many maps, guarantees that any invariant measure (in particular the absolu continuous ore) can be approached by the measures supported on periodic orbits. This motivates th study of periodic orbits for maps with an infinite number of partition points.

In this note we study a class of piecewise linear transformations with an infinite number of partition points. It is shown that if such a transformation is restricted to certain domains, then an explicit bound can be obtained for the number of periodic orbits in that domain. This has practical application in determining the distribution of long periodic orbits [6], and is, in turn, related to the study of computer orbits [7].

Let m be an integer > 1. We define the piecewise linear map f: $[0,1] \to [0,1]$ on the intervals

$I_h \equiv [\frac{h}{m-1}, \frac{h+1}{m-1}]$, $h = 0,1,2,\dots, m-2$, as follows: first we define f on I_0; we partition I_0 by $a_0^{(0)} = 0$, $a_i^{(0)} = \sum_{j=1}^{i} \frac{i}{m^j}$; note that $\lim_{i\to\infty} a_i^{(0)} = \frac{1}{m-1}$. We define $f(a_0^{(0)}) = 0$, $f(a_1^{(0)}) = 1$, $f(a_2^{(0)}) = 0, \dots$ Thus f is piecewise linear and continuous on I_0, with slopes $+m, -m^2, m^3, \dots\dots$, on $[0,\frac{1}{m}]$, $[\frac{1}{m}, \frac{1}{m}+\frac{1}{m^2}]$, $[\frac{1}{m}+\frac{1}{m^2}, \frac{1}{m}+\frac{1}{m^2}+\frac{1}{m^3}], \dots$. To complete the definition f, we define f to be periodic with period $\frac{1}{m-1}$. Thus, $f(x+\frac{h}{m-1}) = f(x)$, $h \in Z$. For $m = 2$, $I_0 = [0,1)$ and f has slopes $+2, -4, 8, -16, \dots$. For $m=3$, $I_0\ [0,\frac{1}{2})$ and $I_1 = [\frac{1}{2}, 1)$, where f has slopes $+3, -9, +27, \dots$ on $[0,\frac{1}{2})$, and then is repeated on $[\frac{1}{2}, 1]$. The partition points in I_h are defined by $a_0^{(h)} = \frac{h}{m-1}$, $a_i^{(h)} = \frac{h}{m-1} + \sum_{j=1}^{i} \frac{1}{m^j}$, $i = 1,2,\dots$. Thus, on $I_h^{i+1} \equiv [a_i^{(h)}, a_{i+1}^{(h)}$, $f(x)$ is linear with $f(x) = (-1)^i m^{i+1} x + d_i^{(h)}$. It is easy to see that $(m-1)d_i^{(h)} \in Z$, $i = 0, 1, 2, \dots$.

Let $D_N = \{\frac{a}{(m-1)p^N}, (a,p) = 1\}$, where p is a prime. We assume $(p,m) = (p,m-1) = 1$: since 2 divides m or $m-1$, it follows that $p>2$. It is easy to verify that $f: D_N \to D_N$, and thus all points in D_N are eventually periodic. Take for example $m = 2$, $p = 3$, $N = 2$. Then $D_N = \{\frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9}\}$. There is only one period: $\frac{2}{9} \to \frac{4}{9} \to \frac{8}{9} \to \frac{7}{9}$, of length 4, and there are two points which are not periodic. This is in contrast to the case when $[0,1]$ is partitioned into a finite number of intervals when all points of D_N are periodic. See [2].

2. Main Result.

Let m be fixed and consider $f|D_N$. The following Theorem gives an explicit upper bound on the number of periodic orbits possessed by $f|D_N$. Note that this is shown by obtaining an explicit lower bound for the number of points in any periodic orbit.

<u>Theorem</u>. For fixed m, the number of periodic orbits of f that are in D_N is bounded by

$$(p^{m'-1})\frac{p-1}{k(1)}(1 + N\frac{\ln p}{\ln m}),$$

where $k(1)$ is defined by: $p\,|m^{k(1)} \pm 1$, $k(1) > 0$ is minimal, and m' is defined by the condition $p^{m'}\,||\,m^{k(1)} \pm 1$.

<u>Proof:</u> Let $x = \frac{a}{(m-1)p^N} \in D_N$ have period k. (Note that there is at least one periodic orbit in D_N.) Then $f^k(x) = x$. By the definition of f, $f^k(x) = \pm m^{i_x} . x + d_x$. We shall show that

$i_x < 1 + N\frac{\ln p}{\ln m}$. Note that this bound is independent of x. Let $\frac{a}{p^N} \in I_h^{i+1} = I_h^{i_x}$, $i > 0$. Then

$$\frac{h}{m-1} + \sum_{j=1}^{i} \frac{1}{m^j} < \frac{h}{m-1} + \frac{a}{(m-1)p^N} < \frac{h}{m-1} + \sum_{j=1}^{i+1} \frac{1}{m^j} < \frac{h+1}{m-1} .$$

Note that we cannot have equality, since $\frac{a}{p^N} \neq a_i^{(h)}$, $h = 0, 1, \ldots, m-2$, $i = 0,1,2, \ldots$. It is easy to see that $\frac{h+1}{m-1} - \frac{a}{(m-1)p^N} \geq \frac{1}{p^{N(m-1)}}$.

Thus

$$\frac{h}{m-1} + \sum_{j=1}^{i} \frac{1}{m^j} < \frac{h+1}{m-1} - \frac{1}{p^N(m-1)} ,$$

$$\sum_{j=1}^{i} \frac{1}{m^j} < \frac{1}{m-1} - \frac{1}{p^N(m-1)} \quad \frac{\frac{1}{m} - (\frac{1}{m})^{i+1}}{1 - \frac{1}{m}} < \frac{1}{m-1} - \frac{1}{p^N(m-1)} ,$$

and $i < N\frac{\ln p}{\ln m}$. Therefore $i_x \leq i + 1 < 1 + N\frac{\ln p}{\ln m}$. This is also true if $i=0$; then $i_x = 1$.

Now, $f^k(x) = \pm m^{k'} x + d_x = x$, where

$k' \leq k i_x < k(1 + N\frac{\ln p}{\ln m})$, and $(m-1)d_x \in Z$. Thus, since $p^N \mid m^{k'} \pm 1$,

$$k' \geq \begin{cases} k(1)p^{N-m'}, & N \geq m' \\ k(1), & N < m' \end{cases}$$

For more details refer to [1]. Thus $k > \frac{k(1)\,p^{N-m'}}{1 + N\frac{\ln p}{\ln m}}$, $N \geq m'$ and $k > \frac{k(1)}{1 + N\frac{\ln p}{\ln m}}$, $N < m'$.

(If p is constant, $k > \frac{cp^N}{N}$). It follows that the number of periodic orbits is less than

$$\frac{|D_N|}{\min_x k} = \frac{(p-1)p^{N-1}}{(k(1)p^{N-m'})/(1+N\frac{\ln p}{\ln m})} , N \geq m' \tag{2.1}$$

$$= (p^{m'-1})\frac{p-1}{k(1)}(1+N\frac{\ln p}{\ln m}), N \geq m'$$

If $N < m'$, the bound is $p^{N-1}\frac{(p-1)}{k(1)}(1+N\frac{\ln p}{\ln m})$

Q.E.D

Corollary . The bound is less than or equal to $c_1 p^{m'-1} \frac{p-1}{k(1)} N \ln p$, where $c_1 = 1 + \frac{1}{\ln 2}$.

Note 1 : Since $k(1) \geq \frac{\ln(p-1)}{\ln m}$, it is easy to see that this is less than or equal

$$c_1 \operatorname{Ln}_3 4 \, . \, p^{m'-1} \operatorname{Ln} m \, . \, N \, . \, (p-1) \, . \tag{2.2}$$

Q.E.D.

If $k(1) = \frac{p-1}{2}$, then the bound is $2c_1 p^{m'-1} N \ln p$. Note that usually $m' = 1$. See [8,9].

Note that (2.2) is bounded by $Ap^{\beta}N$.

Note 2: A similar result holds for the map g: [0,1] →[0,1/m] which is piecewise linear and continuous on [0,1/m-1] and is then extended as f is. The function g is defined on [0,1/m-1] tc have slope -m, +m, -m, ... until x = 1/m, then slope $-m, +m^2, -m^3, +m^4$, ... (if m is even). (If m is odd then one starts with slope +m, etc.)

ACKNOWLEDGEMENT. The Research of the second author was supported by NSERC Grant #A-9072.

References

[1] A. N. Sarkovskii, Coexistence of cycles of a continuous map of a line into itself. Ukr. Mat.Z. 16, 61-71, 1964.

[2] F. Holfbauer, Periodic points for piecewise monotonic transformations, Ergodic Theory and Dynamical Systems 5 #2, pp. 237-256, 1985.

[3] G. Pianigiani, First return map and invariant measures, Israel J. Math. Vol. 35. 1m (1980), pp. 32-47.

[4] P. Kasprowski, On invariant measures for piecewise convex transformations on the half line, Zeszyty Naukowe Universytetu Jagiellonskiego, DLXXX, 22, pp. 143-145, 1981.

[5] K. Sigmund, On dynamical systems witht he specification property, Trans. Amer. Math. Soc 190, (1974), pp. 285-299.

[6] M. Scarowsky & A. Boyarsky, Long Periodic Orbits of the Traingle Map, Proc. Amer. Math Soc., (1986), Vol. 97, #2, pp. 247-254.

[7] A. Boyarsky, Computer Orbits, Mathematics and Computers with Appls. V.12A, No. 10, pp. 1057-1064, 1986.

[8] J. Brillhart, J. Tonascia and T. Weinberger, On the Fermat Quotient, In Computers in Number Theory, ed. A.O.L. Atkin and B.J. Birch, Academic Press, N.Y., 1969.

[9] R. Ribenboim, 1093, The Mathematical Intellingencer 5(2), (1983), 28-34.