Internat. J. Math. & Math. Sci.
VOL. 12 NO. 4 (1989) 729-734

A VARIATIONAL FORMALISM FOR THE EIGENVALUES OF FOURTH ORDER BOUNDARY VALUE PROBLEMS

ADNAN A. HAJJ

Department of Mathematics
U.A.E. University, Al-Ain
United Arab Emirates

(Received January 6, 1987 and in revised form March 23, 1987)

ABSTRACT. This paper describes a variational approach for computing eigenvalues of a two point boundary value problem associated with coupled second order equations to which a fourth order linear differential equation is reduced. An attractive feature of this approach is the technique of enforcing the boundary conditions by the variational functional. Consequently, the expansion functions need not satisfy any of them.

KEY WORDS AND PHRASES. Variational principle, eigenvalue, functional, boundary condition.
1980 AMS SUBJECT CLASSIFICATION CODE. 34B05

1. INTRODUCTION.

In a number of papers finite difference methods have been used to solve the fourth order linear differential equation:

$$y^{(4)} + [\, p(x) - \lambda q(x) \,]\, y = 0 \tag{1.1}$$

subject to one of the following pairs of homogeneous boundary conditions:

$$y(a) = y'(a) = 0, \quad y(b) = y'(b) = 0 \tag{1.2a}$$

$$y(a) = y''(a) = 0, \quad y(b) = y''(b) = 0 \tag{1.2b}$$

$$y(a) = y'(a) = 0, \quad y''(b) = y'''(b) = 0 \tag{1.2c}$$

In (1.1), the functions $p(x), q(x) \in C[a,b]$ and they satisfy the conditions

$$p(x) \geqq 0\,,\ q(x) > 0\,,\ x \in [a,b]. \tag{1.3}$$

Such boundary value problems occur frequently in applied mathematics, modern physics and engineering,see [1,2,3,4].
Chawla and Katti [5] have developed a finite difference method of order 2 for computing approximate values of λ for a boundary value problem (1.1) - (1.2a). For the same problem, a fourth order method is developed by Chawla [6] which leads to a generalized seven - band symmetric matrix eigenvalue problem. More recently, Usmani [7] has presented finite differnce methods for (1.1) - (1.2b) and (1.1) - (1.2c) which lead to generlized five-band and seven-band symmetric matrix eigenvalue problem.

In the present paper we follow a different approach. We reduce the fourth order equation (1.1) to two coupled second order equations as follows:

Let
$$f(x) = y''(x) \tag{1.4}$$

The problem (1.1) can now be written as

$$f'' + [p(x) - \lambda q(x)]y = 0 \tag{1.5a}$$

$$y'' - f = 0 \tag{1.5b}$$

The associated boundary conditions (1.2) can be written in this case as:

$$y(a) = y'(a) = 0\,,\ y(b) = y'(b) = 0 \tag{1.6a}$$

$$y(a) = y(b) = 0\,,\ \ f(a) = f(b) = 0 \tag{1.6b}$$

$$y(a) = y'(a) = 0\,,\ f(b) = f'(b) = 0 \tag{1.6c}$$

In the next section we propose a variational priciple for the solution of (1.5) and (1.6) with the following attractive features:

I. The proposed functional is a general one in the sense that it solves (1.5) and any pair of associated boundary conditions (1.6).

II. The boundary conditions are enforced via suitable terms in the functional and hence the expansion (trial) functions need not satisfy any of them.

III. The variational technique employed leads to stable calculations and to a high convergence rate.

2. A FUNCTIONAL EMBODYING ALL THE BOUNDARY CONDITIONS.

In this section we produce the functional:

$$\begin{aligned}\lambda(u,v) = \frac{1}{\int_a^b qv^2\,dx}\Big\{ \int_a^b (-2u'v' + pv^2 - u^2)\,dx \\ + 2\alpha_1[v(b)u'(b) - v(a)u'(a)] \\ + 2\alpha_2[u(b)v'(b) - u(a)v'(a)] \\ + 2\alpha_3[u(b)v'(b) - u'(a)v(a)]\Big\}\end{aligned} \tag{2.1}$$

which incorporates the boundary conditions (1.6). The parameters α_1, α_2, and α_3 are set equal to either 1 or 0 depending on which pair of the boundary conditions (1.6) is taken with (1.5).

Theorem 1. The functional (2.1) is stationary at the solution of (1.5)-(1.6a), where for this pair of boundary conditions α_1 is set equal to 1; $\alpha_2 = \alpha_3 = 0$.

Proof. Let

$$\begin{aligned}G[u,v,\lambda] = \int_a^b [-2u'v' + (p - \lambda q)v^2 - u^2]\,dx \\ + 2[v(b)u'(b) - v(a)u'(a)] = 0\end{aligned} \tag{2.2}$$

and let $v_1 = y + \delta y$, $\lambda_{v_1} = \lambda + \delta\lambda$, $u = f$. Then

$$\begin{aligned}G[u,v_1,\lambda_{v_1}] = \int_a^b [-2f'\delta y' + 2(p - \lambda q)y\delta y - 2\delta\lambda q y^2]\,dx \\ + 2[\delta y(b)f'(b) - \delta y(a)f'(a)] = 0\end{aligned} \tag{2.3}$$

But
$$\int_a^b -2f'\delta y'\,dx = \int_a^b 2f''\delta y\,dx - 2[\delta y(b)f'(b) - \delta y(a)f'(a)] \tag{2.4}$$

Upon substituting (2.4) in (2.3) and using (1.5a), we get

$$G[u,v_1,\lambda_{v_1}] = -2\delta\lambda \int_a^b qy^2\,dx = 0$$

Hence, $\delta\lambda = 0$ to $O\|\delta y\|$.

In an identical manner, it can be shown that $\delta\lambda = 0$ to $O\|\delta f\|$. Thence, the equation $G[u,v,\lambda] = 0$ does not change to the first order in δy and δf. This establishes the validity of $\lambda(u,v)$ as a functional for this problem.

Theorem2. The functional (2.1) is stationary at the solution of (1.5)-(1.6b), where for this pair of boundary conditions we set $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = 0$.

Theorem3. The functional (2.1) is stationary at the solution of (1.5)-(1.6c), where for this pair of boundary conditions we set $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = 1$.

The proof of theorem2 and Theorem3 Parallel that of theorem1 and, therefore, are omitted.

3. MATRIX SET-UP.

Let

$$f(x) \simeq f_N(x) = \sum_{i=1}^{N} a_i h_i(x), \quad y(x) \simeq y_N(x) = \sum_{i=1}^{N} b_i h_i(x), \quad x \in [a,b] \tag{3.1}$$

Inserting (3.1) into the functional (2.1) and finding the stationary value of the functional leads to the 2x2 block matrix equation

$$\begin{pmatrix} \Psi & \Omega \\ \Theta & \Psi^T \end{pmatrix} \begin{pmatrix} \underline{a} \\ \underline{b} \end{pmatrix} = \lambda \begin{pmatrix} 0 & \zeta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{a} \\ \underline{b} \end{pmatrix} \tag{3.2}$$

where $\Psi = D + S_1 + S_2 + S_3$ and where

$$D_{ij} = -\int_a^b h_i' h_j'\,dx \,, \quad \Omega_{ij} = \int_a^b p\,h_i h_j\,dx \,, \quad \Theta_{ij} = -\int_a^b h_i h_j\,dx \,, \quad \xi_{ij} = \int_a^b q\,h_i h_j\,dx \tag{3.3}$$

The matrices S_1, S_2 and S_3 are contributions from the boundary terms with elements

$$(S_1)_{ij} = \alpha_1[h_j(b)h_i'(b) - h_j(a)h_i'(a)]$$

$$(S_2)_{ij} = \alpha_2[h_i(b)h_j'(b) - h_i(a)h_j'(a)]$$

$$(S_3)_{ij} = \alpha_3[h_i(b)h_j'(b) - h_i'(a)h_j'(a)]$$

If, for example, we consider (1.5) with boundary conditions as given in (1.6a), then S_2 and S_3 will be null since in this case we set $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = 0$.

In order not to introduce artificial singularities in the matrix D and for stability reasons (see mikhlin [8]), we choose in (3.1):

$$h_{-2}(x) = 1 \,, \quad h_{-1}(x) = x \,, \quad h_i(x) = (1 - \sigma^2)\,T_i(\sigma) \,, \qquad i = 1,2,\ldots,N-3 \tag{3.4}$$

where for convenience, we number the basis functions from -2 to N-3 and where the $T_i(x)$ are chebyshev polynomials of the first kind; σ is a linear map of x onto [-1 , 1] .

4. EFFICIENT CALCULATIONS.

To calculate the elements of the matrices D and Θ in (3.3), we need the following:

$$\beta_\ell^{(k)} = 2/\pi \int_{-1}^{1} \frac{(1 - x^2)^k\, T_\ell}{\sqrt{1 - x^2}}\,dx \tag{4.1}$$

From the relation:

$$(1 - x^2) = \tfrac{1}{2}(T_0 - T_2) \tag{4.2}$$

and the chebyshev polynomials orthogonality properties:

$$\int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}} T_n T_m \, dx = \begin{cases} 0 & \text{for } n \neq m \\ \pi & \text{for } n = m = 0 \\ \pi/2 & \text{for } n = m \neq 0 \end{cases} \tag{4.4}$$

we find that

$$\beta_\ell^{(1)} = \begin{cases} 1 & \text{if } \ell = 0 \\ -\frac{1}{2} & \text{if } \ell = 2 \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

Now $\beta_\ell^{(k)}$, $k \geqq 2$ can be related to $\beta_\ell^{(1)}$ by the following:

$$\beta_\ell^{(k)} = \tfrac{1}{4}[- \beta_{\ell+2}^{(k-1)} + \beta_\ell^{(k-1)} - \beta_{|\ell-2|}^{(k-1)}] \tag{4.4}$$

This follows from (4.1), (4.2) and the chebyshev relations

$$T_n T_m = \tfrac{1}{2}[T_{n+m} + T_{n-m}] \text{ and } T_{-n} = T_n \tag{4.5}$$

Similarly, the half-integers $\beta_\ell^{(s+\frac{1}{2})}$, can easily be related to $\beta_\ell^{(s)}$ by the following:

$$\beta_\ell^{(s+\frac{1}{2})} = 2/\pi [\beta_\ell^{(s)} - \sum_{m=1}^{\infty} \frac{1}{4m^2 - 1} \{ \beta_{2m+\ell}^{(s)} + \beta_{|2m - \ell|}^{(s)} \}] \tag{4.6}$$

which follows from (4.1), (4.5) and the well known chebyshev expansion:

$$(1-x^2)^{\frac{1}{2}} = 2/\pi \left(1 - \sum_{m=1}^{\infty} \frac{2}{4m^2 - 1} T_{2m} \right) \tag{4.7}$$

At this stage, the elements of the matrices Θ and D in (3.3) can br related to $\beta_\ell^{(s+\frac{1}{2})}$ as follows: Assume, without loss of generality, that $[a,b] = [-1,1]$, then

Theorem 3. The elements Θ_{ij} are given by

$$\begin{aligned} \Theta_{ij} &= -(\pi/4)[\beta_{i+j}^{(5/2)} + \beta_{|i-j|}^{(5/2)}], \quad i,j \geq 0 \\ \Theta_{ji} &= \Theta_{ij}, \quad \Theta_{-1,-1} = -2/3, \quad \Theta_{-1,-2} = 0, \quad \Theta_{-2,-2} = -2, \\ \Theta_{i,-1} &= -(\pi/4)[\beta_{i+1}^{(3/2)} + \beta_{|i-1|}^{(3/2)}], \quad i \geq 0 \\ \Theta_{i,-2} &= -(\pi/2)\beta_i^{(3/2)}, \quad i \geq 0. \end{aligned} \tag{4.8}$$

Proof. For $i,j \geq 0$, Θ_{ij} can be written in the form

$$\Theta_{ij} = \int_{-1}^{1} \frac{(1-x^2)^{5/2}}{(1-x^2)^{\frac{1}{2}}} T_i(x) T_j(x) \, dx \tag{4.9}$$

The result then follows from (4.1) and (4.5) and the orthogonality relations of the Chebyshev polynomials; and similarly for the first two rows and columns of Θ.
In the same manner, the elements of the matrix D in (3.3) can be related to $\beta_\ell^{(s+\frac{1}{2})}$ using the identity

$$\frac{d}{dx}[(1-x^2)T_i(x)] = \tfrac{1}{2}[(i-2)T_{i-1}(x) - (i+2)T_{i+1}(x)] \tag{4.10}$$

The elements of the matrices Ω and ξ require a slightly differet treatment due to the presence of the function p(x) in Ω_{ij} and q(x) in ξ_{ij}.

Here we require the expansion

$$(1-x^2)^k\, p(x) = \sum_{\gamma=0}^{\infty}{}' \alpha_\gamma^{(k)} T_\gamma(x) \tag{4.11}$$

and similarly for the function q(x). We assume that Fast Fourier Transform techniques are used to approximate $\alpha_\gamma^{(0)}$, $\gamma = 0,1,\ldots,N-1$ via the scheme

$$\alpha_\gamma^{(0)} = (2/\pi)\int_{-1}^{1}[p(x)T_\gamma(x)/(1-x^2)^{\frac{1}{2}}]\,dx \tag{4.12}$$

$$\simeq (2/n)\sum_{m=0}^{n}{}'' p(\cos\frac{m\pi}{n})\cos(\frac{m\gamma\pi}{n}),\quad n \geq N \tag{4.13}$$

and that we approximate $\alpha_\gamma^{(0)} \simeq 0$, $\gamma \geq N$ whenever these coefficients appear. It is not difficult to show that the coefficients $\alpha_\gamma^{(h)}$ and $\alpha_\gamma^{(h+\frac{1}{2})}$, $h \geq 1$ are related to $\alpha_\gamma^{(0)}$ by relations identical to (4.4) and (4.6). The elements of the matrix Ω are related to $\alpha_\gamma^{(h+\frac{1}{2})}$ (and similarly for the elements of ξ in terms of the expansion coefficients of q(x)), in a manner that parallels that of theorem 3 and details are omitted.

5. THE STURM - LIOVILLE PROBLEM.

To test the formalism itroduced in this paper numerically, we take the second order version of problem (1.1) by considering the solution of the following regular Sturm-Liovville problem:

$$[r(x)y']' + (\lambda p(x) - q(x))y = 0 \qquad x \in [a,b] \tag{5.1}$$

subject to

$$y(a) = y(b) = 0 \tag{5.1a}$$

An appropriate functional for this problem is :

$$\lambda(w) = N(w)/M(w) \tag{5.2}$$

where,

$$N(w) = \int_a^b (w'r(x)w' + q(x)w^2)\,dx + 2[w(b)r(b)w'(b) - w(a)r(a)w'(a)]$$

$$M(w) = \int_a^b p(x)w^2\,dx$$

The proof of the validity of this functional parallels that of theorem1 and, therefor, is omitted. Next, let

$$y(x) \simeq y_N(x) = \sum_{i=1}^{N} z_i h_i(x), \qquad x \in [a,b] \tag{5.3}$$

Substituting $y_N(x)$ for w(x) in (5.2) and finding the stationary value of the functional leads to the symmetric matrix eigenvalue problem:

$$(\,H + \lambda B\,) = 0 \tag{5.4}$$

where, $B_{ij} = \int_a^b h_i p(x) h_i\,dx$, $H = R + S$. The elements of the matrices R and S are given by:

$$R_{ij} = \int_a^b [h_i' r(x) h_j' + h_i q(x) h_j]\,dx$$

$$S_{ij} = -[h_i(b)r(b)h_j'(b) - h_i(a)r(a)h_j'(a)]$$

To apply the technique presented in this paper, we consider the numerical solution of problem (5.1) where we choose $r(x) = 1$, $p(x) = \frac{1}{2}$, $q(x) = 0$ and $[a,b] = [0,\pi]$. In this case problem (5.1) has a theoretical eigenvalue $\lambda = 2$. With basis (3.4), we obtain from a one dimensional program an excellent approximation to the eigenvalue using inverse iterations. Searching for the eigenvalue closest to 1 and using a zero starting value with the number of expansion functions $N = 7$, it took the program only three iterations to produce an approximated eigenvalue with an error of order 10^{-8}. This shows that the variational principle derived here gives an attractive extension of the global variational method to the eigenvalue problems. The technique avoids the need to search for trial functions that must satisf the boundary conditions since searching for such trial functions has proven, in many cases, to be technically complicated.

REFERENCES

1. MIKHLIN, S.G. Variational Methods in Mathematical Physics, Pergamon Press, 1964.
2. FOX, L. The numerical solution of two-point boundary value problems in ordinary differential equations, Oxford University Press, Oxford, 1957.
3. HILDEBRAND, F.B. Advance Calculus for Applications, Prentice-Hall Inc., Inglewood Cliff, N.J., 1964.
4. REISS, E.L. CALLEGARI, A.J. and AHLUWALIA, D.S. Ordinary Differential Equations with Applications, Holt, Rinhart and Winston, New york, 1976.
5. CHAWLA, M.M. and KATTI, C.P. A new symmetric five-diagonal finite difference method for computing eigenvalues of fourth-order two-point boundary value problem, J. Comput. and Appl. Math., 8 (1982), 135-136.
6. CHAWLA, M.M. A new fourth-order finite difference metho for computing eigenvalues of fourth-order two-point boundary value problem, IMA Journal of Numerical Analysis, 3 (1983), 291-293.
7. USMANI, R.A. Finit difference methods for computing eigenvalues of fourth order boundary value problems, Internat. J. Math. & Math. Sci. 9 (1986), 137-143.
8. MIKHLIN, S.G. The Numerical Performance of Variational Methods, Noordhoff, Amsterdam, 1971.