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SUPERSETS FOR THE SPECTRUM OF ELEMENTS IN EXTENDED BANACH ALGEBRAS

MORTEZA SEDDIGHIN

Division of Natural Sciences
Lewis Clark College
Lewiston, ID 83501

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ABSTRACT. If A is a Banach Algebra with or without an identity, A can be always extended to a Banach algebra $\bar{A}$ with identity, where $\bar{A}$ is simply the direct sum of A and $\mathbb{C}$, the algebra of complex numbers. In this note we find supersets for the spectrum of elements of $\bar{A}$.

KEY WORDS AND PHRASES. Banach Algebra, spectrum of elements and quasi-singular elements.
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1. INTRODUCTION.

Let A be a Banach algebra. Then we know that the set $\bar{A} = \{(x,\alpha): x \quad A, \alpha \text{ complex}\}$ together with the operations $(x,\alpha) + (y,\beta) = (x + y, \alpha + \beta)$ and $(x,\alpha)(y,\beta) = (xy + \beta x + \alpha y, \alpha\beta)$, and norm $\|(x,\alpha)\| = \|x\| + |\alpha|$ is a Banach algebra, whose identity element is (0,1). Although this is usually done for algebras A without identity, to extend them to algebras with identity; we can also start with a Banach algebra A with identity (In this case the identity of A is no more an identity for $\bar{A}$).

2. MAIN RESULTS.

DEFINITION 2.1. An element x in a Banach algebra A is called quasi-regular if $xoy = yox = 0$ for some $y \in A$, where $xoy = x + y - xy$. xoy is called the circle operation. x is called quasi-singular if it is not quasi-regular. For an element x in A, the special radius of x is defined by

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} .$$

THEOREM 2.1. Let $\bar{A}$ be the extension of A, as above and let $\delta_{\bar{A}}((x,\alpha))$ denote the spectrum of (x,α) in A, then

$$\delta_{\bar{A}}((x,\alpha)) \subseteq \{\alpha\} \cup \delta_A(x + \alpha) ,$$

if A already has an identity, and

$$\delta_{\bar{A}}((x,\alpha)) \subseteq \{\lambda: |\lambda - \alpha| \leq r(x)\}$$

if A does not have an identity.

PROOF. First suppose A has an identity. Let λ be a complex number, then $(x,\alpha) - (0,\lambda) = (x, \alpha - \lambda)$. If $\lambda \neq \alpha$, then

$$(x, \alpha - \lambda)\ (y, \frac{1}{\alpha-\lambda}) = (xy + \frac{1}{\alpha-\lambda} x + (\alpha - \lambda) y, 1).$$

Now, if $\lambda \neq \alpha$ and $\lambda \notin \delta_A (x + \alpha)$, then the equation

$$xy + \frac{1}{\alpha-\lambda} x + (\alpha - \lambda) y = 0 \tag{2.1}$$

has a solution. To see this, write (2.1) as $(\alpha - \lambda)xy + x + (\alpha - \lambda)^2 y = 0$ or $(\alpha - \lambda)[x + \alpha - \lambda] y = -x$ or $y = \frac{1}{\alpha-\lambda}(x + \alpha - \lambda)^{-1}(-x)$. $(x + \alpha - \lambda)^{-1}$ exists since $\lambda \notin \delta_A (x + \alpha)$.

This implies $\sim \{\alpha\} \cap \sim\{\delta_A(x) + \alpha\} \subseteq \sim \delta_{\bar{A}}((x,\alpha))$, and, therefore, we have:

$$\delta_{\bar{A}}((x,\alpha)) \subseteq \{\alpha\} \cup \delta_A(x + \alpha).$$

Now suppose A does not have an identity and let $\lambda \neq \alpha$. If $\frac{1}{\alpha-\lambda} x$ is quasi-irregular, then there exists an element z in A such that:

$$\frac{1}{\alpha-\lambda} xz + \frac{1}{\alpha-\lambda} x + z = 0.$$

If we take $y = \frac{1}{\alpha-\lambda} z$, then we have:

$$xy + \frac{1}{\alpha-\lambda} x + (\alpha - \lambda) y = 0.$$

Hence, $\sim\{\alpha\} \cap \sim\{\lambda \mid \lambda \neq \alpha : \frac{1}{\alpha-\lambda} x \text{ is quasi-singular}\} \subseteq \sim \delta_{\bar{A}}((x,\alpha))$. For an element a in a Banach algebra, the inequality $r(a) < 1$ implies a is a quasi-regular with quasi-inverse $a = -\sum_{n=1}^{\infty} a^n$ (Rickart [1]). Hence, for an element $\frac{1}{\alpha-\lambda} x$ to be quasi-singular, it is necessary to have $r(\frac{1}{\alpha-\lambda} x) \geq 1$; that is $r(x) \geq |\lambda - \alpha|$

REFERENCES

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