 

# A NONLINEAR BOUNDARY VALUE PROBLEM ASSOCIATED WITH THE STATIC EQUILIBRIUM OF AN ELASTIC BEAM SUPPORTED BY SLIDING CLAMPS

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ABSTRACT. The fourth-order boundary value problem $\frac{d^4u}{dx^4} + f(x)u = e(x)$, $0 < x < \pi$; $u'(0) = u'(\pi) = u'''(0) = u'''(\pi) = 0$; where $f(x) \le 0$ for $0 \le x \le \pi$, describe the unstable static equilibrium of an elastic beam which is supported by sliding clamps at both ends. This paper concerns the nonlinear analogue of this boundary value problem, namely, $-\frac{d^4u}{dx^4} + g(x,u) = e(x)$, $0 < x < \pi$, $u'(0) = u'(\pi) = u'''(0) = u'''(\pi) = 0$, where $g(x,u)u \ge 0$ for a.e. $x$ in $[0,\pi]$ and all $u \in \mathbf{R}$ with $|u|$ sufficiently large. Some resonance and nonresonance conditions on the asymptotic behavior of $u^{-1}g(x,u)$, for $|u|$ sufficiently large, are studied for the existence of solutions of this nonlinear boundary value problem.

KEY WORDS AND PHRASES. elastic beam supported by sliding clamps, asymptotic conditions, resonance, nonresonance, $L^\infty$-resonance, Wirtinger's inequalities, coincidence degree theory.

AMS(MOS) SUBJECT CLASSIFICATION CODES. 34B10, 34B15, 73K05

## 1. INTRODUCTION

The static deformations of an elastic beam supported by sliding clamps at both ends are described by the following fourth-order two-point boundary value problem:

$$\frac{d^4u}{dx^4} + f(x)u = e(x), \quad 0 < x < \pi, \tag{1.1}$$

$$u'(0) = u'(\pi) = u'''(0) = u'''(\pi) = 0.$$

The static equilibrium of the elastic beam described by the boundary value problem (1.1) is said to be unstable if $f(x) \le 0$, for $0 < x < \pi$. This instability is caused by the fact that the term $f(x)u$ may interact with the eigenvalues, $\lambda = n^4$, $(n = 0,1,2,...)$, for the linear eigenvalue problem

$$\frac{d^4u}{dx^4} = \lambda u, \quad 0 < x < \pi, \tag{1.2}$$

$$u'(0) = u'(\pi) = u'''(0) = u'''(\pi) = 0,$$

when $f(x) \le 0$, $0 < x < \pi$.

The purpose of this paper is to study the following nonlinear analogue of the boundary value problem (1.1):

$$-\frac{d^4u}{dx^4} + g(x,u) = e(x), \quad 0 < x < \pi, \tag{1.3}$$

$$u'(0) = u'(\pi) = u'''(0) = u'''(\pi) = 0,$$

where the nonlinear function $g(x,u)$ is such that for some $\rho > 0$, $g(x,u)u \geq 0$ for $x \in [0,\pi]$, $u \in \mathbf{R}$ with $|u| \geq \rho$. More precisely, the purpose of this paper is to give non-resonance and resonance asymptotic conditions at infinity on $g(x,u)u^{-1}$ at the first two eigenvalues $\lambda = 0$ and $\lambda = 1$ of the linear eigenvalue problem (1.2).

The methods and results of this paper are motivated by the papers of Gupta and Mawhin ([1]) and Mawhin ([2]) (see also [3], [4]) for the second order boundary value problem

$$\frac{d^2u}{dx^2} + g(x,u) = e(x), \quad 0 < x < 2\pi,$$

$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0.$$

We present in Section 2 some lemmas giving *a priori* inequalities that are needed to apply degree-theoretic arguments to obtain existence of solutions for the problem (1.3). In Section 3, nonresonance conditions for the existence of solutions of (1.3) are studied, and in Section 4 we study the problem (1.3) when it is at resonance. We sharpen the theorem of Section 4 in Section 5 when (1.3) does not have any $L^\infty$-resonance at the second eigenvalue $\lambda = 1$ of the linear eigenvalue problem (1.2). Additionally, we present a necessary and sufficient condition that the righthand member $e$ in (1.3) needs to satisfy for the existence of a solution for (1.3) when, among other conditions, $g(x,u)$ is nondecreasing in $u$ for every $x$ in $[0,\pi]$.

In this paper, we use classical spaces $C[0,\pi]$, $C^k[0,\pi]$, $L^k[0,\pi]$, and $L^\infty[0,\pi]$ of continuous, $k$-times continuously differentiable, measurable real-valued functions whose $k$-th power of the absolute value is Lebesgue integrable or measurable functions that are essentially bounded on $[0,\pi]$. In addition, we use the Sobolev-spaces $H^k[0,\pi]$ ($k = 2,3 or 4$) defined by

$$H^k[0,\pi] = \{u : [0,\pi] \to \mathbf{R} \mid u^{(j)} \text{ abs. cont. on } [0,\pi],$$

$$j = 0,1,\ldots,k-1, \quad u^{(k)} \in L^2[0,\pi]\}$$

with the inner product defined by

$$(u,v)_{H^k} = \sum_{j=1}^{k} \frac{1}{\pi}\int_0^\pi u^{(j)}(x)v^{(j)}(x)\mathrm{d}x + \left[\frac{1}{\pi}\int_0^\pi u(x)\mathrm{d}x\right]\left[\frac{1}{\pi}\int_0^\pi v(x)\mathrm{d}x\right]$$

and the corresponding norm by $|\cdot|_{H^k}$. We define, for convenience, the norm in $L^k[0,\pi]$ by

$$|u|_{L^k} = \left[\frac{1}{\pi}\int_0^\pi |u(x)|^k \mathrm{d}x\right]^{\frac{1}{k}}.$$

We also use the Sobolev space $W^{4,1}[0,\pi]$ defined by

$$W^{4,1}[0,\pi] = \{u : [0,\pi] \to \mathbf{R} \mid u', u'', u''', \text{ abs. cont. on } [0,\pi]\}$$

with norm

$$|u|_{W^{4,1}} = \sum_{j=0}^{4} \int_0^\pi |u^j(t)|\, dt.$$

## 2. A PRIORI INEQUALITIES

For $u \in L^1[0,\pi]$, let us write

$$\bar{u} = \frac{1}{\pi}\int_0^\pi u(x)dx\ , \quad \tilde{u}(x) = u(x) - \bar{u}\ , \tag{2.1}$$

so that $\int_0^\pi \tilde{u}(x)dx = 0$. Let $\tilde{H}^2[0,\pi] = \{u \in H^2[0,\pi] \mid \bar{u} = 0\}$.

LEMMA 1. *Let $\Gamma \in L^1[0,\pi]$ be such that, for a.e. $x \in [0,\pi]$,*

$$\Gamma(x) \le 1 \tag{2.2}$$

*with strict inequality holding on a subset of $[0,\pi]$ of positive measure. Then there exists a $\delta = \delta(\Gamma) > 0$ such that for all $\tilde{u} \in \tilde{H}^2[0,\pi]$ with $\tilde{u}'(0) = \tilde{u}'(\pi) = 0$,*

$$B_\Gamma(\tilde{u}) = \frac{1}{\pi}\int_0^\pi [(\tilde{u}''(x))^2 - \Gamma(x)\tilde{u}^2(x)]dx \ge \delta |\tilde{u}|_{H^2}^2\ . \tag{2.3}$$

*PROOF.* Using (2.2), Wirtinger's inequality [5], and the method of expanding a function $\tilde{u} \in \tilde{H}^2[0,\pi]$ with $\tilde{u}'(0) = \tilde{u}'(\pi) = 0$ into a cosine-Fourier series, we see that, for all $\tilde{u} \in \tilde{H}^2[0,\pi]$ with $\tilde{u}'(0) = \tilde{u}'(\pi) = 0$,

$$B_\Gamma(\tilde{u}) \ge \frac{1}{\pi}\int_0^\pi [(\tilde{u}''(x))^2 - \tilde{u}^2(x)]dx \ge 0\ . \tag{2.4}$$

Moreover,

$$B_\Gamma(\tilde{u}) = 0\ , \tag{2.5}$$

if and only if

$$\tilde{u}(x) = A\cos x\ , \tag{2.6}$$

for some $A \in \mathbf{R}$. But then by (2.5),(2.6) we get

$$0 = B_\Gamma(\tilde{u}) = \frac{1}{\pi}\int_0^\pi [1 - \Gamma(x)]\tilde{u}^2(x)dx$$

$$= \frac{A^2}{\pi}\int_0^\pi [1 - \Gamma(x)]\cos^2 x dx\ ,$$

so that by our assumption (2.2) on $\Gamma$ we have $A = 0$ and hence $\tilde{u} = 0$.

Let us next assume that the conclusion of the lemma is false. Then there exists a sequence $\{\tilde{u}_n\}$, $\tilde{u}_n \in \tilde{H}^2[0,\pi]$ for every $n = 1,2,3,\ldots$ such that

$$B_\Gamma(\tilde{u}_n) \to 0 \ \text{ as } \ n \to \infty \tag{2.7}$$

$$|\tilde{u}_n|_{H^2} = 1\ , \ \text{ for every } n = 1,2,3,\ldots .$$

It follows from (2.7) and the compact embedding $H^2[0,\pi] \hookrightarrow C^1[0,\pi]$ that there exists a $\tilde{u} \in \tilde{H}^2[0,\pi]$ such that

$$\tilde{u}_n \to \tilde{u} \text{ weakly in } H^2[0,\pi]\ , \tag{2.8}$$

$$\tilde{u}_n \to \tilde{u} \text{ in } C^1[0,\pi]\ .$$

Now (2.8) implies that $\tilde{u}'(0) = \tilde{u}'(\pi) = 0$ and $|\tilde{u}|_{H^2} \le \liminf_{n\to\infty} |\tilde{u}_n|_{H^2}$. Hence,

$$0 \le B_\Gamma(\tilde{u}) \le \liminf_{n\to\infty} B_\Gamma(\tilde{u}_n) = 0\ . \tag{2.9}$$

It follows from (2.9) and the first part of this proof that $\tilde{u} = 0$. Also, (2.7)-(2.9) imply that

$$\frac{1}{2\pi}\int_0^{2\pi}[\tilde{u}''_n(x)]^2 dx = B_\Gamma(\tilde{u}_n) + \frac{1}{\pi}\int_0^{\pi}\Gamma(x)\tilde{u}_n^2(x)dx$$

$$\to \frac{1}{\pi}\int_0^{\pi}\Gamma(x)\tilde{u}^2(x)dx = \frac{1}{\pi}\int_0^{\pi}[\tilde{u}''(x)]^2 dx\ ,$$

so that $\tilde{u}_n \to \tilde{u}$ in $H^2[0,\pi]$ and $|\tilde{u}|_{H^2} = 1$. We have thus arrived at a contradiction. □

LEMMA 2. *Let* $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ *where* $\Gamma_\infty \in L^\infty[0,\pi]$, $\Gamma_1 \in L^1[0,\pi]$, *and* $\Gamma_0 \in L^1[0,\pi]$ *be such that* $\Gamma_0(x) \le 1$ *for a.e.* $x \in [0,\pi]$ *with strict inequality holding on a subset of* $[0,\pi]$ *of positive measure. Let* $\delta(\Gamma_0) > 0$ *be as given by Lemma 1. Then for every* $\tilde{u} \in \tilde{H}^2[0,\pi]$ *with* $\tilde{u}'(0) = \tilde{u}'(\pi) = 0$,

$$B_\Gamma(\tilde{u}) \ge \left[\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}\right]|\tilde{u}|_{H^2}^2\ . \tag{2.10}$$

*PROOF.* We have

$$B_\Gamma(\tilde{u}) = \frac{1}{\pi}\int_0^{\pi}[(\tilde{u}''(x))^2 - \Gamma_0(x)\tilde{u}^2(x)]dx$$

$$- \frac{1}{\pi}\int_0^{\pi}\Gamma_1(x)\tilde{u}^2(x)dx - \frac{1}{\pi}\int_0^{\pi}\Gamma_\infty(x)\tilde{u}^2(x)dx\ .$$

Using, now, the fact that $H^2[0,\pi] \subset C^1[0,\pi]$ and the inequalities (see [9])

$$|\tilde{u}|_{L^2} \le |\tilde{u}'|_{L^2} \le |\tilde{u}|_{H^2}\ ,\quad |\tilde{u}|_{L^\infty} \le \frac{\pi}{\sqrt{3}}|\tilde{u}'|_{L^2} \le \frac{\pi}{\sqrt{3}}|\tilde{u}|_{H^2} \tag{2.11}$$

for $\tilde{u} \in \tilde{H}^2[0,\pi]$ with $\tilde{u}'(0) = \tilde{u}'(\pi) = 0$, as well as Lemma 1, we get that

$$B_\Gamma(\tilde{u}) \ge \delta(\Gamma_0)|\tilde{u}|_{H^2}^2 - |\Gamma_1|_{L^1}\cdot|\tilde{u}|_{L^\infty}^2 - |\Gamma_\infty|_{L^\infty}\cdot|\tilde{u}|_{L^2}^2$$

$$\ge [\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}]|\tilde{u}|_{H^2}^2\ .$$

Remark 1. The best value for $\delta(0)$ is clearly $\frac{1}{2}$, so that $B_{\Gamma_1}(\tilde{u}) \ge (\frac{1}{2} - \pi^2|\Gamma_1|_{L^1})|\tilde{u}|_{H^2}^2$ for all $\tilde{u} \in \tilde{H}^2[0,\pi]$ with $\tilde{u}'(0) = \tilde{u}'(\pi) = 0$.

LEMMA 3. *Let* $\gamma \in L^1[0,\pi]$, $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ *be as in Lemma 2 and* $\delta(\Gamma_0) > 0$ *be given by Lemma 1. Then for all measurable functions* $p(x)$ *on* $[0,\pi]$ *such that* $\bar{\gamma} \le \bar{p}$, $p(x) \le \Gamma(x)$ *for a.e.* $x \in [0,\pi]$ *and all* $u \in W^{4,1}[0,\pi]$ *with* $u'(0) = u'(\pi) = u'''(0) = u'''(\pi) = 0$, *we have*

$$\frac{1}{\pi}\int_0^{\pi}[\bar{u} - \tilde{u}(x)][-\tilde{u}^{(iv)}(x) + p(x)u(x)]dx \tag{2.12}$$

$$\ge \overline{\gamma u}^2 + \left[\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}\right]|\tilde{u}|_{H^2}^2\ .$$

*PROOF.* For $u \in W^{4,1}[0,\pi]$ with $u'(0) = u'(\pi) = u'''(0) = u'''(\pi) = 0$, we have (on integrating by parts and using Lemma 2) that

$$\frac{1}{\pi}\int_0^{\pi}[\bar{u} - \tilde{u}(x)][-u^{(iv)}(x) + p(x)u(x)]dx$$

$$\ge \bar{p}\cdot\bar{u}^2 + \frac{1}{\pi}\int_0^{\pi}[(\tilde{u}''(x))^2 - p(x)\tilde{u}^2(x)]dx$$

$$\geq \bar{\gamma}\cdot\bar{u}^2 + [\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}]|\tilde{u}|_{H^2}^2 . \quad \square$$

## 3. NONRESONANCE CONDITIONS FOR THE EXISTENCE OF SOLUTIONS

Let $g : [0,\pi]\times\mathbf{R} \to \mathbf{R}$ be a function satisfying Caratheodory conditions, namely,

(i) for each $u \in \mathbf{R}$, the function $x \in [0,\pi] \to g(x,u) \in \mathbf{R}$ is measurable on $[0,\pi]$;

(ii) for a.e. $x \in [0,\pi]$, the function $u \in \mathbf{R} \to g(x,u) \in \mathbf{R}$ is continuous on $\mathbf{R}$; and

(iii) for each $r > 0$, there exists a function $\alpha_r(x) \in L^1[0,\pi]$ such that $|g(x,u)| \leq \alpha_r(x)$ for a.e. $x \in [0,\pi]$ and all $u \in \mathbf{R}$ with $|u| \leq r$.

THEOREM 1. *Let $\gamma \in L^1[0,\pi]$ with $\bar{\gamma} > 0$ be given. Also let $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ with $\Gamma_1 \in L^1[0,\pi]$, $\Gamma_\infty \in L^\infty[0,\pi]$, $\Gamma_0$ measurable on $[0,\pi]$, $\Gamma_0(x) \leq 1$ with strict inequality holding on a subset of $[0,\pi]$ of positive measure, and $\frac{\pi^2}{3}|\Gamma_1|_{L^1} + |\Gamma_\infty|_{L^\infty} < \delta(\Gamma_0)$, where $\delta(\Gamma_0) > 0$, is given by Lemma 1. Assume that the inequalities*

$$\gamma(x) \leq \liminf_{|u|\to\infty} u^{-1}g(x,u) \leq \limsup_{|u|\to\infty} u^{-1}g(x,u) \leq \Gamma(x) , \tag{3.1}$$

*hold uniformly for a.e. $x \in [0,\pi]$.*

*Then, for every given $e(x) \in L^1[0,\pi]$ the boundary value problem (1.3) has at least one solution.*

*PROOF.* Let $\eta = \frac{1}{2}\min\{\bar{\gamma}, \delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}\} > 0$. Then, by (3.1) we can find an $r > 0$ such that for a.e. $x \in [0,\pi]$ and every $u \in \mathbf{R}$ with $|u| \geq r$ we have

$$\gamma(x) - \eta \leq g(x,u)u^{-1} \leq \Gamma(x) + \eta . \tag{3.2}$$

Next, define $\tilde{\gamma}$: $[0,\pi]\times\mathbf{R} \to \mathbf{R}$ by

$$\tilde{\gamma}(x,u) = \begin{cases} u^{-1}g(x,u) & \text{if } |u| \geq r \\ r^{-1}g(x,r) & \text{if } 0 < u < r \\ -r^{-1}g(x,-r) & \text{if } -r < u < 0 \\ \Gamma(x) & \text{if } u = 0 . \end{cases}$$

Note that $\tilde{\gamma}(x,u)u$ satisfies Caratheodory's conditions and, from (3.2),

$$\gamma(x) - \eta \leq \tilde{\gamma}(x,u) \leq \Gamma(x) + \eta \tag{3.3}$$

for a.e. $x \in [0,\pi]$ and all $u \in \mathbf{R}$. Now, define $h : [0,\pi]\times\mathbf{R} \to \mathbf{R}$ by

$$h(x,u) = g(x,u) - \tilde{\gamma}(x,u)u , \tag{3.4}$$

for $x \in [0,\pi]$, $u \in \mathbf{R}$. We then see that

$$\begin{aligned} |h(x,u)| &\leq \sup_{|u|\leq r} |g(x,u) - \tilde{\gamma}(x,u)u| \\ &\leq \alpha(x) , \end{aligned} \tag{3.5}$$

for $x \in [0,\pi]$, $u \in \mathbf{R}$, where $\alpha(x) \in L^1[0,\pi]$ depends on $\gamma$, $\Gamma$, and $\alpha_r$.

The equation in (1.3) is equivalent to the equation

$$-\frac{d^4u}{dx^4} + \tilde{\gamma}(x,u(x))u(x) + h(x,u(x)) = e(x) ,$$

to which we apply coincidence degree theory [6,7] in a manner similar to the method used in Theorem 1 of [3]. Let $X = C[0,\pi]$, $Z = L^1[0,\pi]$, $\mathrm{dom}L = \{u \in W^{4,1}[0,\pi] \mid u'(0) = u'(\pi) = u'''(0) = u'''(\pi) = 0\}$.

$$L: \mathrm{dom}\, L \subset X \to Z\,,\ u \to -\frac{d^4u}{dx^4}$$

$$G: X \to Z\,,\ u \to \tilde{\gamma}(\cdot, u(\cdot))u(\cdot)$$

$$H: X \to Z\,,\ u \to h(\cdot, u(\cdot)) - e(\cdot)$$

$$A: X \to Z\,,\ u \to \tilde{\gamma}(\cdot, 0)u(\cdot) = \Gamma(\cdot)u(\cdot)\,.$$

It is easy to check that $G$, $H$, $A$ are well defined and $L$-compact on bounded subsets of $X$ and that $L$ is a linear Fredholm mapping of index zero. We consider the homotopy $\Phi$: $\mathrm{dom}\, L \times [0,\pi] \to Z$ defined by

$$\Phi(u,\lambda) \equiv Lu + (1-\lambda)Au + \lambda Gu + \lambda Hu\,,$$

for $u \in \mathrm{dom}\, L$, $\lambda \in [0,1]$. Now, in order to apply Theorem IV.5 of [7] (see also [8],[9]), it suffices to show that the set of possible solutions of the family of equations

$$-\frac{d^4u}{dx^4} + [(1-\lambda)\Gamma(x) + \lambda\tilde{\gamma}(x,u(x))]u(x) + \lambda h(x,u(x)) - \lambda e(x) = 0\,,\ \lambda \in (0,1) \tag{3.6}$$

$$u'(0) = u'(\pi) = u'''(0) = u'''(\pi) = 0\,,$$

is, *a priori* bounded in $C[0,\pi]$ independently of $\lambda \in (0,1)$. If $u$ is a solution of (3.6), then multiplying (3.6) by $\bar{u} - \tilde{u}$, integrating over $[0,\pi]$, and using (3.3),(3.5) together with Lemma 3 with $\Gamma_\infty$ replaced by $\Gamma_\infty + \eta$ and $\gamma$ replaced by $\gamma - \eta$, we get

$$\begin{aligned} 0 &= \frac{1}{\pi}\int_0^\pi (\bar{u} - \tilde{u}(x))\left\{-\frac{d^4u}{dx^4} + [(1-\lambda)\Gamma(x) + \tilde{\gamma}(x,u(x))]u(x) + \lambda h(x,u(x)) - \lambda l(x)\right\}dx \\ &\geq (\bar{\gamma} - \eta)\bar{u}^2 + \left[\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty} - \eta\right]|\tilde{u}|^2_{H^2} - (|\alpha|_{L^1} + |e|_{L^1})|\bar{u} - \tilde{u}|_{L^\infty} \\ &\geq \frac{\bar{\gamma}}{2}\bar{u}^2 + \frac{1}{2}\left[\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}\right]|\tilde{u}|^2_{H^2} - \beta|u|_{H^2} \end{aligned}$$

for some constant $\beta > 0$, independent of $\lambda \in [0,1]$. It then follows that $|u|_{H^2} \leq \beta/\eta$, which implies that

$$|u|_{C[0,\pi]} \leq C\,,$$

where $C$ is a constant independent of $\lambda \in [0,1]$, in view of the compact imbedding of $H^2[0,\pi] \subset C[0,1]$. □

COROLLARY 1. *Let $\gamma$ and $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ be as in Theorem 1 above. Then, for every given $e(x) \in L^1[0,\pi]$ the boundary value problem*

$$-\frac{d^4u}{dx^4} + \Gamma(x)u = e(x)\,, \quad 0 < x < \pi \tag{3.7}$$

$$u'(0) = u'(\pi) = u'''(0) = u'''(\pi) = 0\,,$$

*has exactly one solution.*

*PROOF.* The existence of a solution for (3.7) is immediate from Theorem 1 since $g(x,u) = \Gamma(x)u$ obviously satisfies all the conditions of Theorem 1.

Let $u_1(x)$, $u_2(x)$ be two solutions of (3.5). Setting $v(x) = u_1(x) - u_2(x)$, we get that

$$-\frac{d^4v}{dx^4}+\Gamma(x)v(x)=0\,,\quad 0<x<\pi\,, \tag{3.8}$$

$$v'(0)=v'(\pi)=v'''(0)=v'''(\pi)=0\,.$$

Multiplying the equation in (3.8) by $\bar{v}-\tilde{v}$, integrating by parts on $[0,\pi]$, and using Lemma 3, we get that

$$0=\int_0^\pi[\bar{v}-\tilde{v}(x)]\left\{-\frac{d^4v}{dx^4}+\Gamma(x)v(x)\right\}dx$$

$$\geq \overline{\gamma v}^2+[\delta(\Gamma_0)-\frac{\pi^2}{3}|\Gamma_1|_{L^1}-|\Gamma_\infty|_{L^\infty}]|\tilde{v}|^2_{H^2}\geq 0\,.$$

Thus $\bar{v}=0$ and $|\tilde{v}|_{H^2}=0$. Since, now,

$$|\tilde{v}|_{L^\infty}\leq\frac{\pi}{\sqrt{3}}|\tilde{v}'|_{L^2}\leq\frac{\pi}{\sqrt{3}}|\tilde{v}|_{H^2}=0\,,$$

we get $\tilde{v}=0$ and hence $v=\bar{v}+\tilde{v}=0$. □

## 4. RESONANCE CONDITIONS FOR THE EXISTENCE OF SOLUTIONS

Let $g:[0,\pi]\times\mathbf{R}\to\mathbf{R}$ be a function satisfying Caratheodory's conditions.

THEOREM 2. *Let* $\Gamma\in L^1[0,2\pi]$ *be such that*

$$\limsup_{|u|\to\infty}\frac{g(x,u)}{u}\leq\Gamma(x) \tag{4.1}$$

*uniformly for a.e.* $x\in[0,\pi]$ *and* $\Gamma=\Gamma_0+\Gamma_1+\Gamma_\infty$ *where* $\Gamma_\infty\in L^\infty[0,\pi]$, $\Gamma_1\in L^1[0,\pi]$, *and* $\Gamma_0\in L^1[0,\pi]$ *are such that* $\Gamma_0(x)\leq 1$ *for a.e.* $x\in[0,\pi]$ *with strict inequality holding on a subset of* $[0,\pi]$ *of positive measure and* $|\Gamma_\infty|_{L^\infty}+\frac{\pi^2}{3}|\Gamma_1|_{L^1}<\delta(\Gamma_0)$, *where* $\delta(\Gamma_0)>0$ *is given by Lemma 3. Suppose, further, that there exist real numbers a, A, r, and R with* $a\leq A$ *and* $r<0<R$ *such that*

$$g(x,u)\geq A \tag{4.2}$$

*for a.e.* $x\in[0,\pi]$ *and all* $u\geq R$, *and*

$$g(x,u)\leq a \tag{4.3}$$

*for a.e.* $x\in[0,\pi]$ *and all* $u\leq r$.

*Then, for every given* $e(x)\in L^1[0,\pi]$ *with* $a\leq\bar{e}\leq A$, *the boundary value problem*

$$-\frac{d^4u}{dx^4}+g(x,u(x))=e(x)\,,\quad 0<\ <\pi\,, \tag{4.4}$$

$$u'(0)=u'(\pi)=u'''(0)=u'''(\pi)=0$$

*has at least one solution.*

*PROOF.* Define $g_1:[0,\pi]\times\mathbf{R}\to\mathbf{R}$ by $g_1(x,u)=g(x,u)-\frac{1}{2}(a+A)$ and $e_1\in L^1[0,\pi]$ by $e_1(x)=e(x)-\frac{1}{2}(a+A)$, so that for a.e. $x\in[0,\pi]$ we have, by using (4.2),(4.3), and the assumption $a\leq\bar{e}\leq A$, that

$$g_1(x,u)\geq\frac{1}{2}(A-a)\geq 0 \text{ if } u\geq R\,, \tag{4.5}$$

$$g_1(x,u)\leq\frac{1}{2}(a-A)\leq 0 \text{ if } u\leq r\,, \tag{4.6}$$

and

$$\frac{1}{2}(a-A) \le \bar{e}_1 \le \frac{1}{2}(A-a). \tag{4.7}$$

The equation in (4.4) is clearly equivalent to the equation

$$-\frac{d^4u}{dx^4} + g_1(x,u(x)) = e_1(x), \quad 0 < x < \pi. \tag{4.8}$$

Moreover, we have

$$\limsup_{|u| \to \infty} u^{-1} g_1(x,u) \le \Gamma(x)$$

uniformly for a.e. $x \in [0,\pi]$ and for $|u| \ge \max(R,-r)$, a.e. $x \in [0,\pi]$, $g_1(x,u)u \ge 0$. Hence, $\Gamma(x) \ge 0$ for a.e. $x \in [0,\pi]$.

Now let $\eta = \frac{1}{2}[\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}] > 0$. Then, there exists an $r_1 > 0$ such that for a.e. $x \in [0,\pi]$ and for all $u \in \mathbf{R}$, $|u| \ge r_1$, we have

$$0 \le u^{-1} g_1(x,u) \le \Gamma(x) + \eta. \tag{4.9}$$

Proceeding as in the proof of Theorem 1, we write the equation (4.8) in the equivalent form

$$-\frac{d^4u}{dx^4} + \tilde{\gamma}(x,u(x))u(x) + h(x,u(x)) = e_1(x), \tag{4.10}$$

where $0 \le \tilde{\gamma}(x,u) \le \Gamma(x) + \eta$, $|h(x,u)| \le \alpha(x)$, for a.e. $x \in [0,\pi]$, all $u \in \mathbf{R}$ and some $\alpha \in L^1[0,\pi]$. Once again degree arguments will ensure the existence of a solution for (4.4) if the set of all possible solutions of the family of equations

$$-\frac{d^4u}{dx^4} + [(1-\lambda)(\Gamma(x)+\eta) + \lambda\tilde{\gamma}(x,u(x))]u(x) \tag{4.11}$$

$$+ \lambda h(x,u(x)) = \lambda e_1(x), \quad \lambda \in (0,1)$$

$$u'(0) = u'(\pi) = u'''(0) = u'''(\pi) = 0,$$

is, *a priori*, bounded in $C[0,\pi]$ independently of $\lambda \in (0,1)$. If, now, $u(x)$ is a possible solution of (4.11) for some $\lambda \in (0,1)$, then integrating the equation in (4.11) over $[0,\pi]$ after multiplying it by $\bar{u} - \tilde{u}$, we get (using Lemma 3 with $\gamma = 0$, and $\Gamma_\infty$ replaced by $\Gamma_\infty + \eta$)

$$0 = \frac{1}{\pi}\int_0^\pi [\bar{u} - \tilde{u}(x)] \left\{ -\frac{d^4u}{dx^4} + [(1-\lambda)(\Gamma(x)+\eta) + \tilde{\gamma}(x,u(x))]u(x) + \lambda h(x,u(x)) - \lambda e_1(x) \right\} dx$$

$$\ge \left[ \delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty} - \eta \right] |\tilde{u}|_{H^2}^2 - (|\alpha|_{L^1} + |e_1|_{L^1})|\bar{u} - \tilde{u}|_{L^\infty}$$

$$\ge \eta |\tilde{u}|_{H^2}^2 - \beta(|\bar{u}| + |\tilde{u}|_{H^2})$$

for some constant $\beta > 0$, independent of $\lambda \in (0,1)$. Hence

$$|\tilde{u}|_{H^2}^2 \le (\beta/\eta)(|\bar{u}| + |\tilde{u}|_{H^2}). \tag{4.12}$$

Next, integrating the equation in (4.11) over $[0,\pi]$, we get

$$\frac{1}{\pi}(1-\lambda)\int_0^\pi (\Gamma(x)+\eta)u(x)dx + \frac{1}{\pi}\lambda\int_0^\pi [g_1(x,u(x)) - e_1(x)]dx = 0 . \tag{4.13}$$

If $u(x) \geq R$ for all $x \in [0,\pi]$, then (4.5) and (4.7) imply that $(1-\lambda)(\bar{\Gamma}+\eta)R \leq 0$, contradicting $\bar{\Gamma}+\eta \geq \eta > 0$. Similarly, $u(x) \leq r$ for all $x \in [0,\pi]$ leads to a contradiction. Thus, there must exist a $\tau \in [0,\pi]$ such that

$$r < u(\tau) < R .$$

It is easy to see from $u(x) = u(\tau) + \int_\tau^x u'(s)ds$ that

$$|\bar{u}| \leq \max(R, -r) + |\tilde{u}|_{H^2} . \tag{4.14}$$

The inequalities (4.12) and (4.14) now imply that

$$|\tilde{u}|_{H^2}^2 \leq (2\beta/\eta)|\tilde{u}|_{H^2} + (\beta/\eta)\max(R, -r) ,$$

so there exists a constant $\rho$, independent of $\lambda \in (0,1)$ such that

$$|\tilde{u}|_{H^2} \leq \rho . \tag{4.15}$$

Finally, (4.14) and (4.15) imply that there is a constant $C$ independent of $\lambda \in (0,1)$ such that

$$|u|_{H^2} \leq C ,$$

which implies that $|u|_{C[0,\pi]} \leq C_1$, for some constant $C_1$, independent of $\lambda \in (0,1)$. □

Remark 2. We say that the boundary value problem (4.4) has "no $L^\infty$-resonance" at the second eigenvalue $\lambda = 1$, of the linear eigenvalue problem $\frac{d^4u}{dx^4} = \lambda u$, $u'(0) = u'(\pi) = u'''(0) = u'''(\pi) = 0$, if $\Gamma_0 = \Gamma_\infty = 0$ in Theorem 2. In the case of no $L^\infty$-resonance, Theorem 2 implies the existence of a solution for the boundary value problem (4.4) if $|\Gamma_1|_{L^1} < \frac{3}{2\pi^2}$. We develop this result further in Section 5.

## 5. RESONANCE CONDITION WHEN NO $L^\infty$-RESONANCE EXISTS

We need the following lemma for a sharper resonance condition which gives the existence of a solution for the boundary value problem (4.4) when there is no $L^\infty$-resonance.

LEMMA 4. *Let $e \in L^1[0,\pi]$, $\Gamma \in L^1[0,\pi]$ with $\bar{\Gamma} \geq 0$. Then every possible solution $u(x)$ of the linear boundary value problem*

$$-\frac{d^4u}{dx^4} + p(x)u(x) = e(x) , \quad 0 < x < \pi , \tag{5.1}$$

$$u'(0) = u'(\pi) = u'''(0) = u'''(\pi) = 0$$

*with $p \in L^1[0,\pi]$ such that*

$$\bar{p} \leq \bar{\Gamma} , \quad 0 \leq p(x) \tag{5.2}$$

*for a.e. $x \in [0,\pi]$, satisfies the inequality*

$$\left(1 - \frac{\pi^2}{4}\bar{\Gamma}\right)\left|\frac{d^4u}{dx^4}\right|_{L^1}^2 \leq 2|e|_{L^1}\left|\frac{d^4u}{dx^4}\right|_{L^1} + \bar{\Gamma}|e|_{L^1}\cdot|u|_{L^\infty} + 3|e|_{L^1}^2 \tag{5.3}$$

*PROOF.* Let $p \in L^1[0,\pi]$ be as in the lemma, and let $u(x)$ be a solution of (5.1). Then, on multiplying the equation in (5.1) by $\frac{1}{\pi}u(x)$ and integrating over $[0,\pi]$, we get

$$-\frac{1}{\pi}\int_0^{\pi}(u''(x))^2dx + \frac{1}{\pi}\int_0^{\pi}p(x)u^2(x)dx = \frac{1}{\pi}\int_0^{\pi}e(x)u(x)dx\,. \tag{5.4}$$

Since $\bar{p} \le \bar{\Gamma}$, we have, by using Schwarz's inequality,

$$\left[\frac{1}{\pi}\int_0^{\pi}|p(x)u(x)|dx\right]^2 \le \left[\frac{1}{\pi}\int_0^{\pi}p(x)dx\right]\left[\frac{1}{\pi}\int_0^{\pi}p(x)u^2(x)dx\right] \tag{5.5}$$

$$\le \bar{\Gamma}\left[\frac{1}{\pi}\int_0^{\pi}p(x)u^2(x)dx\right],$$

and hence, using the equation in (5.1),

$$\left[\frac{1}{\pi}\int_0^{\pi}\left|e(x)+\frac{d^4u}{dx^4}\right|dx\right]^2 \le \bar{\Gamma}\left[\frac{1}{\pi}\int_0^{\pi}p(x)u^2(x)dx\right]. \tag{5.6}$$

Since $u'''(0) = u'''(\pi) = 0$, we have

$$u'''(x) = \int_0^x \frac{d^4u}{dx^4}(s)ds = -\int_x^{\pi}\frac{d^4u}{dx^4}(s)ds\,,$$

so that

$$2|u'''(x)| \le \int_0^{\pi}\left|\frac{d^4u}{dx^4}\right|dx\,.$$

Hence,

$$\frac{1}{\pi}\int_0^{\pi}|u'''(x)|^2dx \le \frac{\pi^2}{4}\left[\frac{1}{\pi}\int_0^{\pi}\left|\frac{d^4u}{dx^4}\right|dx\right]^2. \tag{5.7}$$

Now, we get from (5.4), (5.6), and (5.7) that

$$\frac{1}{\bar{\Gamma}}\left[\frac{1}{\pi}\int_0^{\pi}\left|e(x)+\frac{d^4u}{dx^4}\right|dx\right]^2 + \frac{1}{\pi}\int_0^{\pi}(u'''(x))^2dx$$

$$\le \frac{1}{\pi}\int_0^{\pi}(u''(x))^2dx + \frac{1}{\pi}\int_0^{\pi}e(x)u(x)dx + \frac{\pi^2}{4}\left[\frac{1}{\pi}\int_0^{\pi}\left|\frac{d^4u}{dx^4}\right|dx\right]^2.$$

So,

$$-\frac{\pi^2}{4}\left|\frac{d^4u}{dx^4}\right|_{L^1}^2 + \frac{1}{\bar{\Gamma}}\left|e(x)+\frac{d^4u}{dx^4}\right|_{L^1}^2 \le |u''|_{L^2}^2 - |u'''|_{L^2}^2 + |e|_{L^1}\cdot|u|_{L^\infty}$$

$$\le |e|_{L^1}\cdot|u|_{L^\infty}\,,$$

which then gives that

$$-\frac{\pi^2}{4}\bar{\Gamma}\left|\frac{d^4u}{dx^4}\right|_{L^1}^2 + \left|e(x)+\frac{d^4u}{dx^4}\right|_{L^1}^2 \le \bar{\Gamma}|e|_{L^1}\cdot|u|_{L^\infty}.$$

Finally,

$$\left[1-\frac{\pi^2}{4}\bar{\Gamma}\right]\left|\frac{d^4u}{dx^4}\right|_{L^1}^2 = \left|\frac{d^4u}{dx^4}+e(x)-e(x)\right|_{L^1}^2 - \frac{\pi^2}{4}\bar{\Gamma}\left|\frac{d^4u}{dx^4}\right|_{L^1}^2$$

$$\le \left|\frac{d^4u}{dx^4}+e(x)\right|_{L^1}^2 + 2|e|_{L^1}\cdot\left|\frac{d^4u}{dx^4}+e(x)\right|_{L^1} + |e|_{L^1}^2 - \frac{\pi^2}{4}\bar{\Gamma}\left|\frac{d^4u}{dx^4}\right|_{L^1}^2$$

$$\le 2|e|_{L^1}\cdot\left|\frac{d^4u}{dx^4}\right|_{L^1} + \bar{\Gamma}|e|_{L^1}\cdot|u|_{L^\infty} + 3|e|_{L^1}^2. \quad \square$$

THEOREM 3. *Let $g:[0,\pi]\times\mathbf{R}\to\mathbf{R}$ be a function satisfying Casatheodory's conditions. Assume that there exists a $\Gamma\in L^1[0,\pi]$ such that*

$$\limsup_{|u|\to\infty} u^{-1}g(x,u)\le\Gamma(x)$$

*uniformly for a.e. $x\in[0,\pi]$ and that $\bar{\Gamma}<\frac{4}{\pi^2}$. Suppose further that there exist real numbers $a$, $A$, $r$, $R$ with $a\le A$ and $r<0<R$ such that for a.e. $x\in[0,\pi]$, $g(x,u)\ge A$ when $u\ge R$ and $g(x,u)\le a$ when $u\le r$. Then the boundary value problem*

$$-\frac{d^4u}{dx^4}+g(x,u(x))=e(x), \quad 0<x<\pi, \tag{5.8}$$

$$u'(0)=u'(\pi)=u'''(0)=u'''(\pi)=0,$$

*has at least one solution for each given $e\in L^1[0,\pi]$ with $a\le\bar{e}\le A$.*

*PROOF.* We first define $g_1$ and $e_1$ as in the proof of Theorem 2 so that the equation in (5.8) can be written as

$$-\frac{d^4u}{dx^4}+g_1(x,u(x))=e_1(x), \tag{5.9}$$

with $g_1(x,u)\ge0$ where $u\ge R$ and $g_1(x,u)\le0$ when $u\le r$ for a.e. $x\in[0,\pi]$ and $\limsup u^{-1}g_1(x,u)\le\Gamma(x)$ uniformly for a.e. $x\in[0,\pi]$. Consequently, for a.e. $x\in[0,\pi]$, $\Gamma(x)\ge0$. Let $\eta=\frac{1}{2}\left[\frac{4}{\pi^2}-\bar{\Gamma}\right]>0$, so that $\bar{\Gamma}+\eta<\frac{4}{\pi^2}$ and let $r_1>0$ be such that

$$0\le u^{-1}g_1(x,u)\le\Gamma(x)+\eta \tag{5.10}$$

for a.e. $x\in[0,\pi]$, $|u|\ge r_1$. proceeding as in the proof of Theorem 1, we can write (5.9) in the form

$$-\frac{d^4u}{dx^4}+\tilde{\gamma}(x,u(x))u(x)+h(x,u(x))=e_1(x), \tag{5.11}$$

where $0\le\tilde{\gamma}(x,u)\le\Gamma(x)+\eta$, $|h(x,u)|\le\alpha(x)$ for a.e. $x\in[0,\pi]$ and all $u\in\mathbf{R}$ and some $\alpha\in L^1[0,\pi]$. The same degree arguments will imply the existence of a solution for (5.8) if the set of possible solutions of the family of equations

$$-\frac{d^4u}{dx^4}+[(1-\lambda)(\Gamma(x)+\eta)+\tilde{\gamma}(x,u(x))]u(x)=-\lambda h(x,u(x))+\lambda e_1(x)\,,\quad \lambda\in(0,1)\,, \tag{5.12}$$

$$u'(0)=u'(\pi)=u'''(0)=u'''(\pi)=0\,,$$

is, *a priori*, bounded in $C[0,\pi]$ independently of $\lambda \in (0,1)$. Let $u(x)$ be a solution of (5.12) for some $\lambda \in (0,1)$. Since

$$0\le(1-\lambda)(\Gamma(x)+\eta)+\lambda\tilde{\gamma}(x,u(x))\le\Gamma(x)+\eta$$

for a.e. $x \in [0,\pi]$, with $\bar{\Gamma}+\eta<\frac{4}{\pi^2}$, and since

$$|e_1-h(\cdot,u(\cdot))|_{L^1}\le|e_1|_{L^1}+|\alpha|_{L^1}\,,$$

it follows from Lemma 4 that

$$\left[1-\frac{\pi^2}{4}(\bar{\Gamma}+\eta)\right]\left|\frac{d^4u}{dx^4}\right|_{L^1}^2\le 2(|e_1|_{L^1}+|\alpha|_{L^1})\left|\frac{d^4u}{dx^4}\right|_{L^1} \tag{5.13}$$

$$+(\bar{\Gamma}+\eta)(|e_1|_{L^1}+|\alpha|_{L^1})|u|_{L^\infty}+3(|e_1|_{L^1}+|\alpha|_{L^1})^2\,.$$

Also, we see as in the proof of Theorem 2 that there exists a $\tau \in [0,\pi]$ such that

$$r<u(\tau)<R\,. \tag{5.14}$$

Next, since it is easy to obtain the solution $u$, with $\bar{u}=0$, of the linear problem $\frac{d^4u}{dx^4}=y$, $u'(0)=u'(\pi)=u'''(0)=u'''(\pi)=0$, for any given $y \in L^1[0,\pi]$ with $\bar{y}=0$, we see that there exist constants $\delta_1>0$, $\delta_2>0$ such that

$$|\tilde{u}|_{L^\infty}\le\delta_1\left|\frac{d^4u}{dx^4}\right|_{L^1}\,, \tag{5.15}$$

and

$$|u'|_{L^\infty}\le\delta_2\left|\frac{d^4u}{dx^4}\right|_{L^1}\,. \tag{5.16}$$

Using (5.15) in (5.13), we find that

$$\left[1-\frac{\pi^2}{4}(\bar{\Gamma}+\eta)\right]\left|\frac{d^4u}{dx^4}\right|_{L^1}^2\le(|e_1|_{L^1}+|\alpha|_{L^1})(2+\delta_1(\bar{\Gamma}+\eta))\left|\frac{d^4u}{dx^4}\right|_{L^1} \tag{5.17}$$

$$+(\bar{\Gamma}+\eta)(|e_1|_{L^1}+|\alpha|_{L^1})|\bar{u}|+3(|e_1|_{L^1}+|\alpha|_{L^1})^2\,.$$

Also, it follows from (5.14),(5.16) that

$$|u(x)|=\left|u(\tau)+\int_\tau^x u'(s)ds\right|\le\max(-r,R)+\pi|u'|_{L^\infty}\le\max(-r,R)+\pi\delta_2\left|\frac{d^4u}{dx^4}\right|_{L^1}$$

so that

$$|\bar{u}| \leq \max(-r,R) + \pi\delta_2 \left|\frac{d^4u}{dx^4}\right|_{L^1} . \tag{5.18}$$

Finally, it follows from (5.15), (5.17), and (5.18) that there exists a constant $\rho$, independent of $\lambda \in (0,1)$ such that

$$|u|_{L^\infty} \leq \rho . \ \square$$

Remark 3. If there is no $L^\infty$-resonance, (i.e., $\Gamma_0 = \Gamma_\infty = 0$), Theorem 3 improves the condition on $\bar{\Gamma}$ to $\bar{\Gamma} < \frac{4}{\pi^2}$, when compared to Theorem 2, where $\bar{\Gamma}$ would be required to be such that $\bar{\Gamma} < \frac{3}{2\pi^2}$.

Remark 4. If $p(x)$ in Lemma 4 satisfies in addition that for a given $\eta > 0$, $p(x) \geq \eta > 0$ for a.e. $x \in [0,\pi]$ and $\bar{\Gamma} < \frac{4}{\pi^2}$, then it follows easily from inequality (5.3) that the boundary value problem (5.1) has at most one solution.

We need the following theorem of Mawhin (Theorem 1, [2]) which we state here as a proposition.

PROPOSITION 1. *Let $X$ and $Z$ be normed vector-spaces such that $C[0,\pi] \subset X \subset Z \subset L^1[0,\pi]$. Let $L : \mathrm{dom}(L) \subset X \to Z$ be a linear Fredholm operator of index zero such that $D(L) \subset C[0,\pi]$, $\ker L = \{u \in D(L) \mid u$ is constant on $D\}$, $\mathrm{Im}L = \{v \in Z \mid \int_0^\pi v(x)dx = 0\}$. Let $g : [0,\pi]\times\mathbf{R} \to \mathbf{R}$ be a function such that for a.e. $x \in [0,\pi]$, $g(x,\cdot)$ is non-decreasing and that the corresponding Nemytskii operator $N : X \to Z$, defined by $(Nu)(x) = g(x,u(x))$, $x \in [0,\pi]$ is L-compact. Further, suppose that the canonical injection $J : X \to Z$ is L-compact and $h \in Z$ be given.*

*Let, now, there exist a positive measurable function $a : [0,\pi] \to \mathbf{R}$ such that $\ker(L + A) = \{0\}$, where $A : X \to Z$ is defined by $Au(x) = a(x)u(x)$, and there exists a real number $R_1 > 0$ and a $\delta$, $0 \leq \delta < 1$ such that*

$$Lu + (1-\lambda)Au + \lambda Nu = \lambda h , \quad \lambda \in (0,1)$$

*implies*

$$|\tilde{u}|_{L^\infty} \leq R_1 + \delta|\bar{u}| .$$

*Then the equation*

$$Lu + Nu = h ,$$

*has at least one solution if and only if $\bar{h} \in \mathrm{Im}\bar{g}$ where $\bar{g} : \mathbf{R} \to \mathbf{R}$ is defined by*

$$\bar{g}(v) = \frac{1}{\pi}\int_0^\pi g(x,v)dx .$$

THEOREM 4. *Let $g : [0,\pi]\times\mathbf{R} \to \mathbf{R}$ be a function satisfying Caratheodory's conditions and suppose that for a.e. $x \in [0,\pi]$, $g(x,u)$ is nondecreasing in $u$. Let $\Gamma \in L^1[0,\pi]$ be as in Theorem 2 or 3 and*

$$\limsup_{|u|\to\infty} u^{-1}g(x,u) \leq \Gamma(x)$$

*uniformly for a.e. $x \in [0,\pi]$.*

*Then, for $e \in L^1[0,\pi]$, given the boundary value problem*

$$-\frac{d^4u}{dx^4} + g(x,u(x)) = e(x) ,$$

$$u'(0) = u'(\pi) = u'''(0) = u'''(\pi) = 0 ,$$

*has at least one solution if and only if*

$$\frac{1}{\pi}\int_0^{\pi} e(x)dx \in \operatorname{Im}\left[\frac{1}{\pi}\int_0^{\pi} g(x,\cdot)dx\right].$$

In case $\Gamma$ satisfies the conditions of Theorem 2, Theorem 4 follows from Proposition 1 in view of (4.12) with

$$L : \operatorname{dom}L \subset C[0,\pi] \to L^1[0,\pi]$$

by

$$\operatorname{dom}L = \{u \in W^{4,1}[0,\pi] \mid u'(0) = u'(\pi) = u'''(0) = u'''(\pi) = 0\}$$

and

$$Lu = -\frac{d^4u}{dx^4} \text{ for } u \in \operatorname{dom}L .$$

and $A : C[0,\pi] \to L^1[0,\pi]$ defined by $(Au)(x) = (\Gamma(x) + \eta)u(x)$, where $\eta = \frac{1}{2}\left[\delta(\Gamma_0) - \pi^2/3\,|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}\right]$. We note that $\ker(L + A) = \{0\}$ by Corollary 1.

And in case $\Gamma$ satisfies the conditions of Theorem 3, Theorem 4 again follows from Proposition 1 in view of (5.15),(5.17) with $L$ and $A$ as in the above paragraph except now $\eta = \frac{1}{2}\left[\frac{4}{\pi^2} - \bar{\Gamma}\right]$ and Remark 4 implies $\ker(L + A) = \{0\}$ in this case.

Example. It is easy to see that the boundary value problem

$$-\frac{d^4u}{dx^4} + u(x) = \cos x\,, \quad 0 < x < \pi\,,$$

$$u'(0) = u'(\pi) = u'''(0) = u'''(\pi) = 0\,,$$

has no solution, even though

$$0 = \frac{1}{\pi}\int_0^{\pi} \cos x\,dx \in \mathbf{R} = \operatorname{Im}(\text{identity})\,.$$

(Note here $g(x,u) = u$ so that $\frac{1}{\pi}\int_0^{\pi} g(x,u)dx = u$ for $u \in \mathbf{R}$.)

This example points out the necessity of some conditions on $\Gamma$ in Theorem 4.

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